Matrices

A matrix is a rectangular grid of numbers arranged into *rows* and *columns*.

An $m \times n$ matrix **A** has m rows and n columns and is written

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a_{11}} & \cdots & \boldsymbol{a_{1n}} \\ \boldsymbol{a_{21}} & \dots & \boldsymbol{a_{2n}} \\ \boldsymbol{a_{31}} & \dots & \boldsymbol{a_{3n}} \\ \vdots & \dots & \vdots \\ \boldsymbol{a_{m1}} & \dots & \boldsymbol{a_{mn}} \end{bmatrix}$$

the numbers m and n are called the dimensions of **A.**

where the element a_{ij} , located in the i^{th} row and the j^{th} column, is a *scalar* quantity; a numerical constant, or a single valued expression.

The *diagonal elements* of a matrix are those elements where the row and column index are the same. For example, the *diagonal elements* of the 3×3 matrix **A** are a_{11}, a_{22}, and a_{33}. The other elements are *non-diagonal* or *off-diagonal* elements.

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The sum of the diagonal elements of **A** is called the **trace** of the matrix, denoted by **tr(A).**

$$\boldsymbol{tr(A) = a_{11} + a_{22} + a_{33}}$$

A matrix having either a single row ($m = 1$) or a single column ($n = 1$) is defined to be a *vector* because it is often used to define the coordinates of a point in a multi-dimensional space. (In this note the convention has been adopted of representing a vector by a lower case "bold-face" letter such as **x**, and a general matrix by a "bold-face" upper case letter such as **A**.)

A vector having a single row, for example

$$\boldsymbol{x = [x_{11}\ \ x_{12} \dots\ \ x_{1n}]}$$

is defined to be a *row vector*, while a vector having a single column is defined to be a *column vector*

$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{y_{11}} \\ \boldsymbol{y_{21}} \\ \vdots \\ \boldsymbol{y_{m1}} \end{bmatrix}$$

Special Types of Matrices

1-Square Matrices

Matrices with the same number of *rows* as *columns* ($m=n$) are said to be *square* and of order n (or m).

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 & 5 \\ i & 0 & -2 \\ -7 & 1 & 8 \end{bmatrix}$$

Here **A** is a square matrix of order 3 (sometimes it is written as $\mathbf{A}_3$).

2- **Diagonal Matrix**

If all *non-diagonal* (*off-diagonal*) elements in a matrix are zero, then the matrix is a diagonal matrix.

$$\boldsymbol{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -9 \end{bmatrix}$$

3- **Identity Matrix**

The *square identity matrix* **I**, is a special matrix where all of its elements equal to zero except those on the main diagonal (where $i = j$) which have a value of one:

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

4- **Null Matrix**

The null matrix **0**, which has the value of zero for all of its elements:

$$\mathbf{0} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

5- Upper Triangular matrix:

A square matrix said to be an *Upper triangular matrix* if

$$a_{ij} = 0 \; for \; all \; i > j$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

6- **Lower Triangular Matrix**:

A square matrix said to be a *Lower triangular matrix* if

$$a_{ij} = 0 \; for \; all \; i < j$$

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

7- **Symmetric Matrix**:

A square matrix $\mathbf{A} = (a_{ij})_{nxn}$ said to be a *symmetric matrix* if $a_{ij} = a_{ji}$ for all i and j.

$$A = \begin{bmatrix} 8 & -2 & 6 \\ -2 & 7 & 3 \\ 6 & 3 & 1 \end{bmatrix}$$

8- **Skew- Symmetric Matrix**:

A square matrix $\mathbf{A} = (a_{ij})_{nxn}$ said to be a *skew-symmetric matrix* if $a_{ij} = -a_{ji}$ for all i and j.

$$A = \begin{bmatrix} 8 & -2 & 6 \\ 2 & 7 & 3 \\ -6 & -3 & 1 \end{bmatrix}$$

Transposition of a Matrix

Consider a matrix **A** with dimensions $m \times n$. The transpose of **A** (denoted $\mathbf{A^T}$) is the $n \times m$ matrix where the columns are formed from the rows of **A**. In other words, $\mathbf{A^T}_{ij} = \mathbf{A}_{ji}$. This "flips" the matrix diagonally

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 5 & 6 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 3 & 6 \end{bmatrix}$$

There are two important observations concerning matrix transposition:

1. For a matrix **A** of any dimension $\mathbf{(A^T)^T = A}$.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 5 & 6 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 3 & 6 \end{bmatrix}, A^{T^T} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 5 & 6 \end{bmatrix}$$

2. For any diagonal matrix **D,** (including the identity matrix **I**), $\mathbf{D}^{\mathrm{T}} = \mathbf{D}$.

$$\boldsymbol{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -9 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -9 \end{bmatrix}$$

3. $(A + B)^T = A^T + B^T$
4. $(A - B)^T = A^T - B^T$

Matrix Algebra

- **Equality of two matrices**

  Two matrices **A** and **B** are said to be equal if
  - (i) They are of same dimensions.
  - (ii) Their corresponding elements are equal.

- **Addition and Subtraction of Matrices**

It is possible to add two matrices together, but only if they have the same dimensions. If **A** and **B** are both $(m \times n)$, then the sum

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

Where **C** is also $(m \times n)$ and is defined to have each element the sum of the corresponding elements of **A** and **B**, thus

$$c_{ij} = a_{ij} + b_{ij}$$

Matrix addition is both associative, that is

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$

and commutative

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

Example:

If **A** and **B** are matrices given by

$$A = \begin{bmatrix} 3 & 2 \\ 4 & -9 \\ -2 & -3 \end{bmatrix}, B = \begin{bmatrix} -3 & 5 \\ 8 & -1 \\ 2 & 8 \end{bmatrix}$$

Find **A** + **B**