$$C = \begin{bmatrix} -1\times1+2\times0+0\times7 & -1\times3+2\times-1+0\times5 \\ 4\times1+1\times0+-2\times7 & 4\times3+1\times-1+-2\times5 \end{bmatrix} = \begin{bmatrix} -1 & -5 \\ -10 & 1 \end{bmatrix}$$

## General Properties for Matrix Multiplication

1. Matrix multiplication is associative, that is

$$\mathbf{A\ (BC)=(AB)\ C}$$

2. Matrix multiplication is distributive

$$(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC}+\boldsymbol{BC}$$

3. Matrix multiplication is not commutative in general

$$\mathbf{AB} \neq \mathbf{BA}$$

   in fact, unless the two matrices are square, reversing the order in the product will cause the matrices to be *nonconformal*. The order of the terms in the product is therefore very important. In the product **C = AB, A** is said to *pre-multiply* **B**, while **B** is said to **post-multiply A**.

4. If $\mathbf{A}^2 = \mathbf{I}$, then this does not imply that **A = +I or A = -I.**
5. **AB = 0** does not imply that either **A = 0** or that **B = 0.**
6. Even if $\mathbf{A} \neq \mathbf{0}$ and (**AB = AC**) it does not imply that **B =C.**
7. **AI = IA.**
8. $\mathbf{A}^0 = \mathbf{I}$.
9. **0A= 0.**

3/ # Determinants

## The Determinant of a Matrix

Every *square* matrix can be associated with a real number called its *determinant.*

**Definition of the Determinant of a $2 \times 2$ Matrix**

The **determinant** of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is given by

$$\det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}.$$

**EXAMPLE 1** The Determinant of a Matrix of Order 2

Find the determinant of each matrix.

(a) $A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ (c) $C = \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix}$

SOLUTION (a) $|A| = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) = 4 + 3 = 7$

(b) $|B| = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 2(2) - 4(1) = 4 - 4 = 0$

(c) $|C| = \begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} = 0(4) - 2(3) = 0 - 6 = -6$

REMARK:

The determinant of a matrix of order 1 is defined simply as the entry of the matrix. For instance, if $A = [-2]$, then

$$\det(A) = -2.$$

## The Determinant of a Matrix of Order 3

There is an alternative method commonly used for evaluating the determinant of a $3 \times 3$ matrix $A$. To apply this method, copy the first and second columns of $A$ to form fourth and fifth columns. The determinant of $A$ is then obtained by adding (or subtracting) the products of the six diagonals, as shown in the following diagram.

*Subtract these three products.*

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

*Add these three products.*

Try confirming that the determinant of $A$ is

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}.$$

### EXAMPLE 5 The Determinant of a Matrix of Order 3

Find the determinant of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix}.$$

SOLUTION Begin by recopying the first two columns and then computing the six diagonal products as follows.

$$\begin{array}{ccccc} & & -4 & 0 & 6 \\ 0 & 2 & 1 & 0 & 2 \\ 3 & -1 & 2 & 3 & -1 \\ 4 & -4 & 1 & 4 & -4 \\ & & 0 & 16 & -12 \end{array}$$

← Subtract these products.

← Add these products.

Now, by adding the lower three products and subtracting the upper three products, you can find the determinant of $A$ to be $|A| = 0 + 16 + (-12) - (-4) - 0 - 6 = 2$.