

EXAMPLE 2 Find $\partial f / \partial y$ as a function of $f(x, y) = y \sin xy$.

Solution We treat x as a constant and f as a product of y and $\sin xy$:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y \sin xy) = y \frac{\partial}{\partial y} \sin xy + (\sin xy) \frac{\partial}{\partial y}(y) \\ &= (y \cos xy) \frac{\partial}{\partial y}(xy) + \sin xy = xy \cos xy + \sin xy.\end{aligned}$$

EXAMPLE 3 Find f_x and f_y as functions of

$$f(x, y) = \frac{2y}{y + \cos x}.$$

Solution We treat f as a quotient. With y held constant, we get

$$\begin{aligned}f_x &= \frac{\partial}{\partial x} \left(\frac{2y}{y + \cos x} \right) = \frac{(y + \cos x) \frac{\partial}{\partial x}(2y) - 2y \frac{\partial}{\partial x}(y + \cos x)}{(y + \cos x)^2} \\ &= \frac{(y + \cos x)(0) - 2y(-\sin x)}{(y + \cos x)^2} = \frac{2y \sin x}{(y + \cos x)^2}.\end{aligned}$$

With x held constant, we get

$$\begin{aligned}f_y &= \frac{\partial}{\partial y} \left(\frac{2y}{y + \cos x} \right) = \frac{(y + \cos x) \frac{\partial}{\partial y}(2y) - 2y \frac{\partial}{\partial y}(y + \cos x)}{(y + \cos x)^2} \\ &= \frac{(y + \cos x)(2) - 2y(1)}{(y + \cos x)^2} = \frac{2 \cos x}{(y + \cos x)^2}.\end{aligned}$$

3.6 Implicit differentiation

It works for partial derivatives the way it works for ordinary derivatives, as the next example illustrates.

EXAMPLE 4 Find $\partial z / \partial x$ if the equation

$$yz - \ln z = x + y$$

defines z as a function of the two independent variables x and y and the partial derivative exists.

Solution We differentiate both sides of the equation with respect to x , holding y constant and treating z as a differentiable function of x :

$$\begin{aligned}\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x} \ln z &= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x} \\ y \frac{\partial z}{\partial x} - \frac{1}{z} \frac{\partial z}{\partial x} &= 1 + 0 \quad \text{With } y \text{ constant, } \frac{\partial}{\partial x}(yz) = y \frac{\partial z}{\partial x} \\ \left(y - \frac{1}{z}\right) \frac{\partial z}{\partial x} &= 1 \\ \frac{\partial z}{\partial x} &= \frac{z}{yz - 1}.\end{aligned}$$

3.7 Functions of More Than Two Variables

EXAMPLE 6 If x , y , and z are independent variables and

$$f(x, y, z) = x \sin(y + 3z),$$

then

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{\partial}{\partial z} [x \sin(y + 3z)] = x \frac{\partial}{\partial z} \sin(y + 3z) \\ &= x \cos(y + 3z) \frac{\partial}{\partial z} (y + 3z) = 3x \cos(y + 3z).\end{aligned}$$

Other examples of evaluating partial derivatives

- (i) $z = \ln(x^2 - y)$. Then $\frac{\partial z}{\partial x} = \frac{2x}{x^2 - y}$ and $\frac{\partial z}{\partial y} = \frac{-1}{x^2 - y}$.
- (ii) $z = x \cos y + ye^x$. Then $\frac{\partial z}{\partial x} = \cos y + ye^x$ and $\frac{\partial z}{\partial y} = -x \sin y + e^x$.
- (iii) $z = y \sin xy$. Then $\frac{\partial z}{\partial x} = y(y \cos xy) = y^2 \cos xy$ and $\frac{\partial z}{\partial y} = yx \cos xy + \sin xy$.
- (iv) If $x^2 + y^2 + z^2 = 1$ find the rate at which z is changing with respect to y at the point $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$.

Solution. We have $z = (1 - x^2 - y^2)^{1/2}$. We want $\frac{\partial z}{\partial y}$ when

$$(x, y) = (\frac{2}{3}, \frac{1}{3}).$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}(1 - x^2 - y^2)^{-1/2}(-2y) = -\frac{y}{(1 - x^2 - y^2)^{1/2}}$$

Putting in $(x, y) = (\frac{2}{3}, \frac{1}{3})$ gives

$$\frac{\partial z}{\partial y} = -\frac{1/3}{(1 - (2/3)^2 - (1/3)^2)^{1/2}} = -\frac{1}{2}.$$

3.8 Second-Order Partial Derivatives

When we differentiate a function $f(x, y)$ twice, we produce its second-order derivatives. These derivatives are usually denoted by

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}, \quad \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy}, \\ \frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx}, \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy}. \end{aligned}$$

The defining equations are

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right),$$

and so on. Notice the order in which the mixed partial derivatives are taken:

$$\frac{\partial^2 f}{\partial x \partial y} \quad \text{Differentiate first with respect to } y, \text{ then with respect to } x.$$

$$f_{yx} = (f_y)_x \quad \text{Means the same thing.}$$

Example

If $f(x, y) = x \cos y + ye^x$, find the second-order derivatives

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}.$$

Solution The first step is to calculate both first partial derivatives.

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x} (x \cos y + ye^x) & \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y} (x \cos y + ye^x) \\ &= \cos y + ye^x & &= -x \sin y + e^x \end{aligned}$$

Now we find both partial derivatives of each first partial:

$$\begin{aligned} \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = -\sin y + e^x & \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = -\sin y + e^x \\ \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = ye^x. & \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = -x \cos y. \end{aligned}$$

Example

Let $z = 4x^2 - 8xy^4 + 7y^5 - 3$. Find all the first and second order partial derivatives of z . the first and second order partial derivatives of the function $z = \sin xy$.

$$\begin{aligned} \frac{\partial z}{\partial x} &= y \cos xy \\ \frac{\partial z}{\partial y} &= x \cos xy \\ \frac{\partial^2 z}{\partial x^2} &= -y^2 \sin xy \\ \frac{\partial^2 z}{\partial y^2} &= -x^2 \sin xy \\ \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} (x \cos xy) = x(-y \sin xy) + \cos xy = -xy \sin xy + \cos xy \\ \frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} (y \cos xy) = y(-x \sin xy) + \cos xy = -xy \sin xy + \cos xy \end{aligned}$$