3 Partial Derivatives

Functions of 2 or more variables

Functions which have more than one variable arise very commonly. Simple examples are

- formula for the area of a triangle $A = \frac{1}{2}bh$ is a function of the two variables, base b and height h
- formula for electrical resistors in parallel:
  $$R = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^{-1}$$
  is a function of three variables R_1, R_2 and R_3, the resistances of the individual resistors.

You should be used to the notation $y = f(x)$ for a function of one variable, and that the graph of $y = f(x)$ is a curve. For the functions of two variables the notation simply becomes

$$z = f(x,y)$$

where the two **independent** variables are x and y, while z is the **dependent** variable. The graph of something like $z = f(x,y)$ is a **surface** in three-dimensional space. Such graphs are usually quite difficult to draw by hand.

3.5 Partial Derivatives of a Function of Two Variables

If y is a function of x then $\frac{dy}{dx}$ is the **derivative** meaning the gradient (slope of the graph) or the rate of change with respect to x.

Since $z = f(x,y)$ is a function of two variables, if we want to differentiate we have to decide whether we are differentiating with respect to x or with respect to y (the answers are different). A special notation is used. We use the symbol ∂ instead of d and introduce the **partial derivatives** of z, which are:

- $\frac{\partial z}{\partial x}$ is read as "partial derivative of z (or f) with respect to x", and means differentiate with respect to x holding y constant
- $\frac{\partial z}{\partial y}$ -means differentiate with respect to y holding x constant

Another common notation is the subscript notation: z_x means $\frac{\partial z}{\partial x}$; z_y means $\frac{\partial z}{\partial y}$

Note that we cannot use the dash ′ symbol for partial differentiation because it would not be clear what we are differentiating with respect to.

Example 1

Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ when $z = x^2 + 3xy + y - 1$.

Solution

To find $\frac{\partial z}{\partial x}$; treat y as a constant and differentiate with respect to x. We have $z = x^2 + 3xy + y - 1$ so

$$\frac{\partial z}{\partial x} = 2x + 3y$$

Similarly

$$\frac{\partial z}{\partial y} = 3x + 1$$

EXAMPLE 2 Find $\partial f/\partial y$ as a function if $f(x, y) = y \sin xy$.

Solution We treat x as a constant and f as a product of y and $\sin xy$:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y \sin xy) = y\frac{\partial}{\partial y}\sin xy + (\sin xy)\frac{\partial}{\partial y}(y)\\ &= (y\cos xy)\frac{\partial}{\partial y}(xy) + \sin xy = xy\cos xy + \sin xy.\end{aligned}$$

EXAMPLE 3 Find f_x and f_y as functions if

$$f(x, y) = \frac{2y}{y + \cos x}.$$

Solution We treat f as a quotient. With y held constant, we get

$$\begin{aligned}f_x = \frac{\partial}{\partial x}\left(\frac{2y}{y + \cos x}\right) &= \frac{(y + \cos x)\frac{\partial}{\partial x}(2y) - 2y\frac{\partial}{\partial x}(y + \cos x)}{(y + \cos x)^2}\\ &= \frac{(y + \cos x)(0) - 2y(-\sin x)}{(y + \cos x)^2} = \frac{2y\sin x}{(y + \cos x)^2}.\end{aligned}$$

With x held constant, we get

$$\begin{aligned}f_y = \frac{\partial}{\partial y}\left(\frac{2y}{y + \cos x}\right) &= \frac{(y + \cos x)\frac{\partial}{\partial y}(2y) - 2y\frac{\partial}{dy}(y + \cos x)}{(y + \cos x)^2}\\ &= \frac{(y + \cos x)(2) - 2y(1)}{(y + \cos x)^2} = \frac{2\cos x}{(y + \cos x)^2}.\end{aligned}$$

3.6 Implicit differentiation

It works for partial derivatives the way it works for ordinary derivatives, as the next example illustrates.

EXAMPLE 4 Find $\partial z/\partial x$ if the equation

$$yz - \ln z = x + y$$

defines z as a function of the two independent variables x and y and the partial derivative exists.