Chapter 2

Normed Linear Space

Definition 2.1.
Let $L(F)$ be a linear space over a field F. A mapping $\|.\| : L \to \mathbb{R}$ is called **norm** if the following conditions hold

(1) $\|x\| \geq 0 \quad \forall x \in L.$ (Positivity)

(2) $\|x\| = 0$ if and only if $x = \mathbf{0_L}$.

(3) $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in L.$ (Triangle Inequality)

(4) $\|\alpha x\| = |\alpha| \, \|x\| \quad \forall x \in L, \, \forall \alpha \in F.$

$(L, \| \, \|)$ is called **normed linear space**.

Remark 2.2.

From now on, the field F is either $\mathbb{R}$ or C.

Theorem 2.3.

Let $(L, \| \, \|)$ be a normed linear space. Then, for each $x, y \in L$

(1) $\|\mathbf{0_L}\| = 0$.

(2) $\|x\| = \|-x\|$.

(3) $\|x - y\| = \|y - x\|$.

(4) $\left|\ \|x\| - \|y\|\ \right| \leq \|x - y\|$. (Reverse Triangle Inequality)

(5) $\left|\ \|x\| - \|y\|\ \right| \leq \|x + y\|$. (Reverse Triangle Inequality)

(6) Every subspace of a normed space is itself normed space with respect to the same norm.

Proof. (1) $\|\mathbf{0_L}\| = \|0\mathbf{0_L}\|$ (see Theorem (1.3)(1))

$= 0\,\|\mathbf{0_L}\| = 0.$

(2) $\|-x\| = |-1|\,\|x\| = \|x\| \quad \forall x \in L.$

(3) $\|x - y\| = \|-(y - x)\| = \|y - x\|$ (by part (2)).

(4) We must prove $-\|x - y\| \leq \|x\| - \|y\| \leq \|x - y\|$

$\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$ (by Definition 2.1(3)).

Hence, $\|x\| - \|y\| \leq \|x - y\|$ (I)

Similarly, $\|y\| = \|y - x + x\| \leq \|y - x\| + \|x\|$ (by Definition 2.1(3)).

Hence, $\|y\| - \|x\| \leq \|x - y\|$ (II)

Hence, by (I) and (II), we get $\|x - y\| \geq |\|x\| - \|y\|| \quad \forall x, y \in L.$

(5) We must prove $-\|x + y\| \leq \|x\| - \|y\| \leq \|x + y\|$

$\|x\| = \|x + y - y\| \leq \|x + y\| + \|-y\|$ (by Definition 2.1(3)).

Hence, $\|x\| - \|y\| \leq \|x + y\|$ (III)

Similarly, $\|y\| = \|y + x - x\| \leq \|y + x\| + \|-x\|$ (by Definition 2.1(3)).

Hence, $\|y\| - \|x\| \leq \|x + y\|$

$\|x\| - \|y\| \geq -\|x + y\|$ (IV)

Hence, by (III) and (IV), we get $-\|x + y\| \leq \|x\| - \|y\| \leq \|x + y\|$ $\forall x, y \in L.$

□

2.1 Examples of Normed Linear Space

Example 2.4.

Let $L = \mathbb{R}$ be a linear space over $\mathbb{R}$ with $\|\ \| : L \to \mathbb{R}$ such that $\|x\| = |x|$.

Show that $(\mathbb{R}, \|\ \|)$ is a normed space.

Solution: We show that

(1) $\|x\| = |x| \geq 0 \quad \forall x \in \mathbb{R}$; hence $\|x\| \geq 0$.

(2) Let $x \in \mathbb{R}$, $\|x\| = 0 \iff |x| = 0 \iff x = 0$.

(3) $\forall x \in \mathbb{R}, \forall \alpha \in \mathbb{R}$,

$\|\alpha x\| = |\alpha x| = |\alpha|\,|x| = |\alpha|\,\|x\|$.

(4) $\|x + y\| = |x + y| \leq |x| + |y| = \|x\| + \|y\| \quad \forall x, y \in \mathbb{R}$.

Example 2.5.

Let $L = C$ be a complex linear space over C with $\|\ \| : C \to \mathbb{R}$ such that $\|z\| = |z| = \sqrt{a^2 + b^2} \quad \forall z = a + ib$. Show that $(C, \|\ \|)$ is a normed space.

Solution: We show that

(1) $\|z\| = |z| = \sqrt{a^2 + b^2} \geq 0 \quad \forall z = a + ib \in C$; hence $\|z\| \geq 0$.

(2) Let $z = a + ib \in C$

$\|z\| = |z| = \sqrt{a^2 + b^2} = 0 \iff a^2 + b^2 = 0 \iff a^2 = b^2 = 0 \iff a = b = 0 \iff z = 0 + 0i = 0$.

(3) Let $z, w \in C$

$$\begin{aligned}
\|z + w\|^2 = |z + w|^2 &= (z + w)(\overline{z + w}) \qquad (|z|^2 = z\overline{z}) \\
&= (z + w)(\overline{z} + \overline{w}) \\
&= z\overline{z} + w\overline{w} + w\overline{z} + \overline{w}z \\
&= z\overline{z} + w\overline{w} + \underbrace{w\overline{z} + \overline{w\overline{z}}} \qquad (\overline{w}z = \overline{\overline{\overline{w}z}} = \overline{w\overline{z}}\ )
\end{aligned}$$

$$= z\overline{z} + w\overline{w} + 2Re\ w\overline{z} \qquad (z + \overline{z} = 2Rez)$$

$$\leq z\overline{z} + w\overline{w} + 2\,|w|\,|z| \qquad (Re\ w\overline{z} \leq |w|\,|z|)$$

$$= |z|^2 + |w|^2 + 2\,|w|\,|z| = \|z\|^2 + \|w\|^2 + 2\,\|w\|\,\|z\|$$

$$= (\|z\| + \|w\|)^2.$$

Thus, $\|z+w\|^2 \leq (\|z\| + \|w\|)^2$, hence, $\|z+w\| \leq \|z\| + \|w\|$.

(4) Let $z \in C, \alpha \in C$, $\|\alpha z\| = |\alpha z| = |\alpha(a+ib)|$

$$= \sqrt{(\alpha a)^2 + (\alpha b)^2} = \sqrt{\alpha^2(a^2+b^2)}$$

$$= \sqrt{\alpha^2}\sqrt{a^2+b^2} = |\alpha|\,|z| = |\alpha|\,\|z\|.$$

<u>**As an application to Example 2.5:**</u> Let $z = 2+3i, w = 1-i$, then

$\|z+w\| = \|(2+1)+(3i-i)\| = \|3+2i\| = \sqrt{3^2+2^2} = \sqrt{13}$.

$\|5z\| = \|10+15i\| = \sqrt{10^2+15^2} = \sqrt{325} = 5\sqrt{13}$.

$5\,\|z\| = 5\sqrt{2^2+3^2} = 5\sqrt{13}$.

Example 2.6.

Show that the linear space $C^b(\mathbb{R})$ is a normed space under the norm

$\|f\| = \sup\{|f(x)| : x \in \mathbb{R}\}, \quad \forall f \in C^b(\mathbb{R})$.

(1) Since $|f(x)| \geq 0 \ \ \forall x \in \mathbb{R}$. Then, $\|f\| = \sup|f(x)| \geq 0$. Hence, $\|f\| \geq 0$.

(2) $\|f\| = 0 \iff \sup\{|f(x)| : x \in \mathbb{R}\} = 0$

$$\iff |f(x)| = 0 \ \ \forall x \in \mathbb{R}$$

$$\iff f(x) = 0 \ \ \forall x \in \mathbb{R} \iff f = \hat{0} \text{ (zero mapping)}$$

(3) Let $f, g \in C^b(\mathbb{R})$. Then

$\|f+g\| = \sup\{|f(x)+g(x)| : x \in \mathbb{R}\}$

$$\leq \sup\{|f(x)| + |g(x)| : x \in \mathbb{R}\}$$

$$\leq \sup\{|f(x)| : x \in \mathbb{R}\} + \sup\{|g(x)| : x \in \mathbb{R}\} = \|f\| + \|g\|.$$

Hence, $\|f + g\| \leq \|f\| + \|g\|$.

(4) Let $f \in C^b(\mathbb{R}), \alpha \in \mathbb{R}$. Then

$\|\alpha f\| = \sup\{|(\alpha f)(x)| : x \in \mathbb{R}\}$

$= \sup\{|\alpha|\,|f(x)| : x \in \mathbb{R}\}$

$= |\alpha| \sup\{|f(x)| : x \in \mathbb{R}\}$ (By Theorem 2.7 below where $A = |f(x)|$ and $\beta = |\alpha|$)

$= |\alpha|\,\|f\|$.

Theorem 2.7.

If A is a bounded above set and $\beta > 0$, then βA is bounded above and $\sup(\beta A) = \beta \sup(A)$.

As an application to Example 2.6: Let $f, g \in C^b(\mathbb{R})$ such that $f(x) = sin(x)$ and $g(x) = 2cos(x) + 1$. Hence,

$\|f\| = \sup\{|sin(x)| : x \in \mathbb{R}\} = 1$ (since $|sin(x)| \leq 1, \ \forall x \in \mathbb{R}$).

$\|g\| = \sup\{|2cos(x) + 1| : x \in \mathbb{R}\}$.

But $|2cos(x) + 1| \leq 2\,|cos(x)| + 1$

$\leq 2(1) + 1 = 3$. (since $|cos(x)| \leq 1, \ \forall x \in \mathbb{R}$).

So, $\|g\| = 3$.

Example 2.8.

The linear space $C^b[a, b]$ of all real valued continuous functions on $[a, b]$ is a normed space under the norm defined in Example 2.6. (**H.W.**)

Example 2.9.

The linear space $C[0,1]$ of all real valued continuous functions on $[0,1]$ is a normed space with the norm defined as

$\|f\| = \int_0^1 |f(x)| \; dx \;\; \forall f \in C[0,1]$.

solution: (1) Since $|f(x)| \geq 0, \;\; \forall x \in [0,1]$, then $\int_0^1 |f(x)| \; dx \geq 0$. Thus, $\|f\| \geq 0$.

(2) $\|f\| = 0 \iff \int_0^1 |f(x)| \; dx = 0$

$$\iff |f(x)| = 0 \;\; \forall x \in [0,1]$$

$$\iff f(x) = 0 \;\; \forall x \in [0,1]$$

$$\iff f = \hat{0} \text{ (zero mapping).}$$

(3) Let $f, g \in C[0,1]$. Then

$\|f+g\| = \int_0^1 |f(x)+g(x)| \, dx$

$\leq \int_0^1 (|f(x)| + |g(x)|) \; dx$

$= \int_0^1 |f(x)| \; dx + \int_0^1 |g(x)| \; dx = \|f\| + \|g\|$

(4) Let $f \in C[0,1], \alpha \in \mathbb{R}$. Then

$\|\alpha f\| = \int_0^1 |\alpha f(x)| \; dx = \int_0^1 |\alpha| \, |f(x)| \; dx = |\alpha| \int_0^1 |f(x)| \; dx = |\alpha| \, \|f\|$.

As an application to Example 2.9: Let $f \in C[0,1]$ such that $f(x) = x^3$ and $g(x) = -x^2$. Find $\|f\|$, $\|g\|$ and $\|f+g\|$.

$\|f\| = \int_0^1 |f(x)| \; dx = \int_0^1 \left|x^3\right| \; dx = \int_0^1 x^3 \; dx = \frac{1}{4}$.

$\|g\| = \int_0^1 |g(x)| \; dx = \int_0^1 \left|-x^2\right| \; dx = \int_0^1 x^2 \; dx = \frac{1}{3}$.

$$\|f+g\| = \int_0^1 |(f+g)(x)| \; dx = \int_0^1 \left| \underbrace{x^3 - x^2}_{\leq 0} \right| \; dx$$

$$= \int_0^1 (x^2 - x^3) \; dx = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

Example 2.10.

Consider the linear space F^n over F ($F = \mathbb{R}$ or C). Define $\| \| : F^n \to \mathbb{R}$ by $\|x\| = \max\{|x_1|, ..., |x_n|\}$ $\forall x = (x_1, ..., x_n) \in F^n$. Then $(F^n, \| \|)$ is a normed space.

solution: (1) For any $x = (x_1, ..., x_n) \in F^n$, $|x_i| \geq 0$, $\forall i = 1, ..., n$.

Then $\max\{|x_1|, ..., |x_n|\} \geq 0$, then $\|x\| \geq 0$.

(2) $\|x\| = 0$, where $x = (x_1, ..., x_n) \in F^n$

$$\iff \max\{|x_1|, ..., |x_n|\} = 0$$

$$\iff |x_1| = ... = |x_n| = 0 \iff x_1 = ... = x_n = 0$$

$$\iff x = (x_1, ..., x_n) = (0, ..., 0) = \mathbf{0}_{F^n}$$

(3) Let $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in F^n$

$x + y = (x_+ y_1, x_2 + y_2, ..., x_n + y_n)$

$$\|x + y\| = \max\{|x_1 + y_1|, ..., |x_n + y_n|\}$$

$$\leq \max\{|x_1| + |y_1|, ..., |x_n| + |y_n|\}$$

$$\leq \max\{|x_1|, ..., |x_n|\} + \max\{|y_1|, ..., |y_n|\} = \|x\| + \|y\|$$

(4) Let $x = (x_1, ..., x_n) \in F^n$ and $\alpha \in F$

$$\|\alpha x\| = \max\{|\alpha x_1|, ..., |\alpha x_n|\}$$

$$= \max\{|\alpha| |x_1|, ..., |\alpha| |x_n|\} = |\alpha| \max\{|x_1|, ..., |x_n|\} = |\alpha| \|x\|$$

As an application to Example 2.10: Consider the linear space $\mathbb{R}^3$ over $\mathbb{R}$. Let $x = (x_1, x_2, x_3) = (1, 2, -5), y = (y_1, y_2, y_3) = (0, -7, 3)$. Then ,

(1) $\|x\| = \max\{|1|, |2|, |-5|\} = 5$ and

$\|y\| = \max\{|0|, |-7|, |3|\} = 7$.

$\|x+2y\| = \max\{|1|, |-12|, |11|\} = 12$

(2) Find $\|2x-y\|$, $\|2x+3y\|$, $\|3x\|$

(3) Show that

$$\max\{|x_1|+|y_1|, |x_2|+|y_2|, |x_3|+|y_3|\} \leq \max\{|x_1|, |x_2|, |x_3|\}+\max\{|y_1|, |y_2|, |y_3|\}.$$

Exercise 2.11.

(1) Let $L = C^2$ be a linear space over $F = C$. Define $\| \| : C^2 \to \mathbb{R}$ such that $\|x\| = a\,|x_1| + b\,|x_2|,\ \forall x = (x_1, x_2) \in C^2$ and $a, b > 0$. Show that $\| \|$ is a norm on C^2. (**H.W.**)

(2) Consider the linear space $\mathbb{R}^2$. Let $\|x\| = \min\{|x_1|, |x_2|\}, \quad \forall x = (x_1, x_2) \in \mathbb{R}^2$. Show that $\| \|$ is not a norm on $\mathbb{R}^2$.

solution: Let $x = (0, -3) \in \mathbb{R}^2$

$\|x\| = \min\{|0|, |-3|\} = \min\{0, 3\} = 0$

Since $X \neq \mathbf{0}_{\mathbb{R}^2}$, but $\|x\| = 0$. Condition (2) of the definition of the norm is not valid. Hence, $\| \|$ is not a norm on $\mathbb{R}^2$.

(3) Consider the linear space $\mathbb{R}^2$. Let $\|x\| = |x_1|^2 + |x_2|^2, \quad \forall x = (x_1, x_2) \in \mathbb{R}^2$. Show that $\| \|$ does not satisfies condition (4).

solution: Let $x = (1, 3), \alpha = 2$

$|\alpha|\,\|x\| = 2(|x_1|^2 + |x_2|^2) = 2(1^2 + 3^2) = 20$

$\|\alpha x\| = \|2(1, 3)\| = \|(2, 6)\| = 2^2 + 6^2 = 40$

Thus, $|\alpha|\,\|x\| = 20 \neq \|\alpha x\| = 40$.

Some Important Inequalities

To give more examples about normed space, it is important to present some inequalities.

If $l^p = \{x = (x_1, x_2, ...) : x_i \in \mathbb{R}(or\ C) \quad \text{and} \quad \sum_{i=1}^{\infty} |x_i|^p < \infty\}$ be a set of sequence space (see Example 1.6). Let $x = (x_1, x_2, ...) \in l^p$, $y = (y_1, y_2, ...) \in l^q$. Then

(1) Holder's Inequality

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \Big[\sum_{i=1}^{\infty} |x_i|^p\Big]^{\frac{1}{p}} \Big[\sum_{i=1}^{\infty} |y_i|^q\Big]^{\frac{1}{q}},$$

where $p > 1, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(2) Cauchy Schwarz's Inequality

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \Big[\sum_{i=1}^{\infty} |x_i|^2\Big]^{\frac{1}{2}} \Big[\sum_{i=1}^{\infty} |y_i|^2\Big]^{\frac{1}{2}},$$

Note that Cauchy Schwarz's inequality is a special case of Holder's inequality where $p = q = 2$.

(3) Minkowski's Inequality

If $p \geq 1$

$$\Big[\sum_{i=1}^{\infty} |x_i + y_i|^p\Big]^{\frac{1}{p}} \leq \Big[\sum_{i=1}^{\infty} |x_i|^p\Big]^{\frac{1}{p}} + \Big[\sum_{i=1}^{\infty} |y_i|^p\Big]^{\frac{1}{p}},$$

Remark 2.12.

The three inequalities above hold for the linear spaces $L = \mathbb{R}^n$ and $L = C^n$.

Example 2.13.

Let $L = \mathbb{R}^2$ be a linear space over $\mathbb{R}$. If $x = (-1, 2), y = (0, 5) \in \mathbb{R}^2$.

(1) Verify Cauchy Shwarz inequality ($p = q = 2$).

(2) Verify Minkowski's inequality ($p = 3$).

Now we can give the following examples

Example 2.14.

(1) Show that the linear space $\mathbb{R}^n$ over $\mathbb{R}$ (or C^n over C) is a normed space with $\|x\|_2 = \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}}$ $\forall x \in \mathbb{R}^n$ or $C^n, x = (x_1, ..., x_n)$.

(2) Show that the linear space $\mathbb{R}^n$ over $\mathbb{R}$ (or C^n over C) is a normed space with $\|x\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{\frac{1}{p}}$ $\forall x \in \mathbb{R}^n$ or $C^n, x = (x_1, ..., x_n)$ and $1 \leq p < +\infty$. (**H.W.**)

(3) Show that $(l^p, \| \|_p)$ is a normed space where $\|x\|_p = \left[\sum_{i=1}^{+\infty} |x_i|^p\right]^{\frac{1}{p}}$ $\forall x = (x_1, x_2, ...) \in l^p$ and $1 \leq p < +\infty$.

Solution (1): Let $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{R}^n$ (or C^n) and $\alpha \in \mathbb{R}$ (or C).

(1) Since $|x_i| \geq 0, \quad \forall i = 1, ..., n$. Then, $\left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} \geq 0$; that is $\|x\|_2 \geq 0$.

(2) $\|x\|_2 = 0 \iff \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} = 0 \iff \sum_{i=1}^{n} |x_i|^2 = 0$

$$\iff |x_i|^2 = 0, \ \ \forall i = 1, ..., n$$

$$\iff x_i = 0, \ \ \forall i = 1, ..., n$$

$$\iff x = (x_1, ..., x_n) = \mathbf{0}_{\mathbb{R}^n}$$

(3) $\|x + y\|_2 = \|(x_1 + y_1, ..., x_n + y_n)\|_2$

$= \left[\sum_{i=1}^{n} |x_i + y_i|^2\right]^{\frac{1}{2}} \leq \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} + \left[\sum_{i=1}^{n} |y_i|^2\right]^{\frac{1}{2}}$ (Minkowski's Inequality)

$$= \|x\|_2 + \|y\|_2$$

(4) $\|\alpha x\|_2 = \|(\alpha x_1, ..., \alpha x_n)\|_2 = \left[\sum_{i=1}^{n} |\alpha x_i|^2\right]^{\frac{1}{2}}$

$$= \left[\sum_{i=1}^{n} |\alpha|^2 |x_i|^2\right]^{\frac{1}{2}}$$

$$= |\alpha| \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} = |\alpha| \, \|x\|_2.$$

Solution (3): Let $x = (x_1, x_2, ...), y = (y_1, y_2, ...) \in l^p$ and $\alpha \in \mathbb{R}$ (or C).

(1) Since $|x_i| \geq 0, \ \ \forall i \in N$. Then, $\left[\sum_{i=1}^{\infty} |x_i|^p\right]^{\frac{1}{p}} \geq 0$; that is $\|x\|_p \geq 0$.

(2) $\|x\|_p = 0 \iff \left[\sum_{i=1}^{\infty} |x_i|^p\right]^{\frac{1}{p}} = 0 \iff \sum_{i=1}^{\infty} |x_i|^p = 0$

$$\iff |x_i|^2 = 0, \ \ \forall i \in N$$

$$\iff x_i = 0, \ \ \forall i \in N$$

$$\iff x = (0, 0, ...)$$

(3) $\|x + y\|_p = \|(x_1 + y_1, ..., x_n + y_n)\|_p$

$= \left[\sum_{i=1}^{\infty} |x_i + y_i|^p\right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^{\infty} |x_i|^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^{\infty} |y_i|^p\right]^{\frac{1}{p}}$ (Minkowski's Inequality)

$$= \|x\|_p + \|y\|_p$$

(4) $\|\alpha x\|_p = \|(\alpha x_1, ...)\|_p = \left[\sum_{i=1}^{\infty} |\alpha x_i|^p\right]^{\frac{1}{p}}$

$$= \left[\sum_{i=1}^{\infty} |\alpha|^p |x_i|^p\right]^{\frac{1}{p}}$$

$$= |\alpha| \left[\sum_{i=1}^{\infty} |x_i|^p\right]^{\frac{1}{p}} = |\alpha| \, \|x\|_p$$

<u>As an application to Example 2.14</u>(1):

(1) Let $(\mathbb{R}^3, \| \, \|_2)$ be a normed space and $x = (x_1, x_2, x_3) = (1, -2, 4)$. Then, find $\|x\|_2$.

(2) Let $(C^2, \| \, \|_2)$ be a normed space and $x = (x_1, x_2) = (1 + i, -2i)$. Then, find $\|x\|_2$.

2.2 Product of Normed Spaces

Definition 2.15.
Let $(L, \| \|_L), (L', \| \|_L{}')$ be normed linear spaces over a field F. Let

$L \times L' = \{(x, y) : x \in L, y \in L'\}$ be the Cartesian product of L and L'.

Define $+$ on $L \times L'$ by

$(x_1, y_1) + (x_2, y_2) = (\underbrace{x_1 + x_2}_{\text{sum on L}}, \underbrace{y_1 + y_2}_{\text{sum on } L'}), \ \forall (x_1, y_1) + (x_2, y_2) \in L \times L'.$

Define a scalar multiplication

$\alpha.(x, y) = (\alpha x, \alpha y), \ \forall (x, y) \in L \times L', \forall \alpha \in F.$

Proposition 2.16.

Show that $(L \times L', +, .)$ is a linear space over F. (**H. W.**)

Remark 2.17.

The product linear space defined above can be made a normed space by different ways as we show in the following example.

Example 2.18.

Define $\| \| : L \times L' \to \mathbb{R}$ such that

(1) $\|(x, y)\|_1 = \|x\|_L + \|y\|_{L'}$

(2) $\|(x, y)\|_2 = \max\{\|x\|_L, \|y\|_{L'}\}$

(3) $\|(x, y)\|_3 = \min\{\|x\|_L, \|y\|_{L'}\}$ (**H. W.**)

Show that $(L \times L', \| \|_1), (L \times L', \| \|_2)$ are normed spaces.

Is $(L \times L', \| \|_3)$ is normed space?

Solution (1): To show $(L \times L', \| \|_1)$ is a normed space,

(i) Since $\|x\|_L \geq 0$ and $\|y\|_{L'} \geq 0 \;\; \forall x \in L, \forall y \in L'$, then

$$\|x\|_L + \|y\|_{L'} = \|(x,y)\|_1 \geq 0.$$

(ii) $\|(x,y)\|_1 = 0 \iff \|x\|_L + \|y\|_{L'} = 0$

$$\iff \|x\|_L = \|y\|_{L'} = 0$$

$$\iff x = y = 0 \quad ((L, \| \, \|_L), (L', \| \, \|_{L'}) \text{ are normed spaces})$$

$$\iff (x,y) = (0,0)$$

(iii) For each $(x_1,y_1),(x_2,y_2) \in L \times L'$

$$\begin{aligned}
\|(x_1,y_1)+(x_2,y_2)\|_1 &= \|(x_1+x_2, y_1+y_2)\|_1 \\
&= \|x_1+x_2\|_L + \|y_1+y_2\|_{L'} \\
&\leq \|x_1\|_L + \|x_2\|_L + \|y_1\|_{L'} + \|y_2\|_{L'} \\
&= (\|x_1\|_L + \|y_1\|_{L'}) + (\|x_2\|_L + \|y_2\|_{L'}) \\
&= \|(x_1,y_1)\|_1 + \|(x_2,y_2)\|_1
\end{aligned}$$

(iv) For each $(x,y) \in L \times L'$ and for each $\alpha \in F$

$$\|\alpha(x,y)\|_1 = \|(\alpha x, \alpha y)\|_1 = \|\alpha x\|_L + \|\alpha y\|_{L'}$$

$$= |\alpha| \, \|x\|_L + |\alpha| \, \|y\|_{L'} = |\alpha| \, (\|x\|_L + \|y\|_{L'}) = |\alpha| \, \|(x,y)\|_1$$

Solution (2): Now, we show that $\|(x,y)\|_2 = \max\{\|x\|_L, \|y\|_{L'}\}$ is a norm on $L \times L'$

(i) Since $\|x\|_L \geq 0$ and $\|y\|_{L'} \geq 0 \;\; \forall x \in L, \forall y \in L'$, then

$\max\{\|x\|_L, \|y\|_{L'}\} = \|(x,y)\|_2 \geq 0.$

(ii) $\|(x,y)\|_2 = 0 \iff \max\{\|x\|_L, \|y\|_{L'}\} = 0$

$$\iff \|x\|_L = \|y\|_{L'} = 0$$

$$\iff x = y = 0 \quad ((L, \|\ \|_L), (L', \|\ \|_{L'}) \text{ are normed spaces})$$

$$\iff (x, y) = (0, 0)$$

(iii) For each $(x_1, y_1), (x_2, y_2) \in L \times L'$

$$\begin{aligned}
\|(x_1, y_1) + (x_2, y_2)\|_2 &= \|(x_1 + x_2, y_1 + y_2)\|_2 \\
&= \max\{\|x_1 + x_2\|_L, \|y_1 + y_2\|_{L'}\} \\
&\leq \max\{\|x_1\|_L + \|x_2\|_L, \|y_1\|_{L'} + \|y_2\|_{L'}\} \\
&\leq \max\{\|x_1\|_L, \|y_1\|_{L'}\} + \max\{\|x_2\|_L, \|y_2\|_{L'}\} \\
&= \|(x_1, y_1)\|_2 + \|(x_2, y_2)\|_2
\end{aligned}$$

(iv) For each $(x, y) \in L \times L'$ and for each $\alpha \in F$

$$\begin{aligned}
\|\alpha(x, y)\|_2 = \|(\alpha x, \alpha y)\|_2 &= \max\{\|\alpha x\|_L, \|\alpha y\|_{L'}\} \\
&= \max\{|\alpha|\,\|x\|_L, |\alpha|\,\|y\|_{L'}\} \\
&= |\alpha| \max\{\|x\|_L, \|y\|_{L'}\} = |\alpha|\,\|(x, y)\|_2
\end{aligned}$$

As an application to Example 2.18: Let $L = (\mathbb{R}, |\ |)$ and $L' = (\mathbb{R}^2, \|\ \|_2)$ where $\|x\|_2 = \left[\sum_{i=1}^{2} |x_i|^2\right]^{\frac{1}{2}}$. If $x = 3 \in L = \mathbb{R}$ and $y = (1, -2) \in L' = \mathbb{R}^2$. Find $\|(x, y)\|_1$ and $\|(x, y)\|_2$

Solution: $\|(x, y)\|_1 = \|(3, (1, -2))\|_1 = \|3\|_{\mathbb{R}} + \|(1, -2)\|_{\mathbb{R}^2}$

$$= |3| + \left[\sum_{i=1}^{2} |y_i|^2\right]^{\frac{1}{2}}$$

$$= 3 + \left[|1|^2 + |-2|^2\right]^{\frac{1}{2}} = 3 + \sqrt{5}.$$

Find $\|(x, y)\|_2$ (**H.W.**)

2.3 Normed space and Metric space

Definition 2.19.

Let X be a non empty set and $d : X \times X \to \mathbb{R}$ be a mapping. Then d is called metric if

(1) $d(x, y) \geq 0 \quad \forall x, y \in X$

(2) $d(x, y) = 0 \iff x = y \quad \forall x, y \in X$

(3) $d(x, y) = d(y, x) \quad \forall x, y \in X$

(4) $d(x, y) \leq d(x, z) + d(z, y) \quad \forall x, y, z \in X$.

(X, d) is called **metric space**

Theorem 2.20.

Let $(L, \| \|)$ be a normed linear space. Let $d : L \times L \to \mathbb{R}$ defined by $d(x, y) = \|x - y\| \quad \forall x, y \in L$. Prove that (L, d) is a metric space. (i.e., every normed space is a metric space). The metric d is called metric **induced** by the norm.

Proof. To prove (L, d) is a metric space.

(i) By definition of norm, $\|x - y\| \geq 0 \quad \forall x, y \in L$. Hence, $d(x, y) = \|x - y\| \geq 0$

(ii) $d(x, y) = \|x - y\| = \|y - x\| = d(y, x)$

(iii) $d(x, y) = 0 \iff \|x - y\| = 0 \iff x - y = 0 \iff x = y$

(iv) $d(x, y) = \|x - y\| = \|x - z + z - y\| \leq \|x - z\| + \|z - y\| = d(x, z) + d(y, z)$ □

Lemma 2.21.

Let d be a metric induced by a normed space $(L, \| \, \|)$ (i.e., $d(x,y) = \|x-y\|$). Then d satisfies the following:

(i) $d(x+a, y+a) = d(x,y) \quad \forall x, y, a \in L$.

(ii) $d(\alpha x, \alpha y) = |\alpha| \, d(x,y) \quad \forall x, y \in L, \ \forall a \in F$.

Proof. (1) $d(x+a, y+a) = \|x+a-(y+a)\| = d(x,y) \quad \forall x, y, a \in L$

(2) $d(\alpha x, \alpha y) = \|\alpha x - \alpha y\| = \|\alpha(x-y)\| = |\alpha| \, \|x-y\| = |\alpha| \, d(x,y)$. □

Remark 2.22.

Not every metric space is a normed space as we show in the next example

Example 2.23.

Let d be the discrete metric on a space X. Then d can't be obtained from a norm on L (i.e., $(L, \| \, \|)$, where

$$d(x,y) = \begin{cases} 0 & \text{if } \ x = y \\ 1 & \ x \neq y \ . \end{cases}$$

Solution: Suppose d induced by a norm on L. Then, by previous Lemma,

$d(\alpha x, \alpha y) = |\alpha| \, d(x,y) \ \forall x, y \in X$ and $\forall \alpha \in F$.

Let $x, y \in L$ such that $x \neq y$. Then $\alpha x \neq \alpha y$ such that $d(x,y) = 1, d(\alpha x, \alpha y) = 1 \qquad (1)$

But $|\alpha| \, d(x,y) = |\alpha| \qquad (2)$

Hence, $d(\alpha x, \alpha y) = 1 \neq |\alpha| = |\alpha| \, d(x,y)$ for any $\alpha \neq \pm 1$. Thus, d can not be induced by a normed space.

Example 2.24.

Let $d(x, y) = |x| + |y| \quad \forall x, y \in \mathbb{R}$. Then, d is a metric on $\mathbb{R}$ (check!). However, d is not induced by a normed space. To show this, let $x = 1, y = 3, a = 2 \in \mathbb{R}$.

$d(x, y) = d(1, 3) = |1| + |3| = 4$

On the other hand, $d(x + a, y + a) = d(3, 5) = |3| + |5| = 8$

Thus, $d(x, y) \neq d(x + a, y + a)$. By Lemma 2.21, d is not induced by a norm.

2.4 Generalizations of Some Concepts from Metric Space

In what follow, we give generalizations of some known concepts from metric space such as open (closed) ball, open (closed) set, interior set, closure of a set, convergent sequence, Cauchy sequence, and bounded sequence.

Definition 2.25.
Let $(L, \| \|)$ be a normed linear space. Let $x_0 \in L, r \in \mathbb{R}, r > 0$. Then the set

$$B_r(x_0) = \{x \in L : \|x - x_0\| < r\}$$

is called an **open ball** with center x_0 and radius r. Similarly,

$$\bar{B}_r(x_0) = \{x \in L : \|x - x_0\| \leq r\}$$

is called an **closed ball** with center x_0 and radius r.

Definition 2.26.

Let $(L, \| \ \|)$ be a normed space and $A \subseteq L$. Then A is said to be

- **open set** if $\forall x \in A, \exists r > 0$ such that $B_r(x) \subseteq A$.
- **closed set** if $A^c = L \setminus A$ is open set

Remark 2.27.

Let $(L, \| \ \|)$ be a normed space. Then

(1) L, ϕ are closed and open.

(2) The union of any family of open sets is open

(3) The union of finite family of closed sets is closed

(4) The intersection of finite family of open sets is open

(5) The intersection of any family of closed sets is closed.

Theorem 2.28.

Any finite subset of a normed space is closed.

Proof. Let L be a normed space and $A \subseteq L$.

If $A = \phi$, then A is closed (by Remark 2.27(1))

If $A = \{x\}$ to prove A is closed (i.e., to prove $L \setminus A$ is open)

Let $y \in L \setminus A = L \setminus \{x\}$ so that $y \neq x$. Put $\|x - y\| = r > 0$. Thus, $x \notin B_r(y)$ and hence $B_r(y) \subseteq A^c = L \setminus \{x\}$. Thus, A^c is open and thus A is closed.

If $A = \{x_1, .., x_n\}, n \in Z_+, n > 1$ then $A = \cup_{i=1}^{n} \{x_i\}$. By Remark 2.27(3), A is closed $\square$

Exercise 2.29.

Let $(L, \| \|)$ be a normed space. Prove that

(i) The set $A_1 = \{x \in L : \|x\| \leq 1\}$ is closed

(ii) The set $A_2 = \{x \in L : \|x\| < 1\}$ is open

(iii) The set $C = \{x \in L : \|x\| = 1\}$ is closed

Solution:

(i) $A_1 = \{x \in L : \|x\| \leq 1\} = \overline{B}_1(0)$.

So, A_1 is a closed set (by Definition 2.25)

(ii) $A_2 = \{x \in L : \|x\| < 1\} = B_1(0)$.

So, A_1 is an open set (by Definition 2.25)

(iii) $C = \{x \in L : \|x\| = 1\}$

$L \setminus C = \{x \in L : \|x\| < 1\} \cup \{x \in L : \|x\| > 1\}$

Let $C_1 = \{x \in L : \|x\| < 1\}$ is open set

Let $C_2 = \{x \in L : \|x\| > 1\}$

So, $L \setminus C_2 = \{x \in L : \|x\| \leq 1\}$ which is closed set. Hence, C_2 is an open set.

Thus, $L \setminus C = C_1 \cup C_2$ is an open set (by Remark 2.27(2)).

Definition 2.30.

Let L be a normed space and $A \subseteq L$. A point $x \in L$ is called **limit point** of A if for each open set G containing x, we have $(G \cap A) \setminus \{x\} \neq \phi$.

The set of all limit points of A is denoted by A' and is called **derived set**.

The closure of A is denoted by $\overline{A}$ and is defined as $\overline{A} = A \cup A'$.

Proposition 2.31.

Let L be a normed linear space and $A \subseteq L$. Then $x \in \overline{A}$ if and only if $\forall r > 0, \exists y \in A, \|x - y\| < r$.

Proof. ($\Rightarrow$) Let $x \in \overline{A} = A \cup A'$

If $x \in A'$ then for each open set G, $x \in G, (G \cap A) \setminus \{x\} \neq \phi$.

Since $B_r(x)$ is an open set then $\forall r > 0$, we have $B_r(x) \cap A \setminus \{x\} \neq \phi$. Thus, $\exists y \in B_r(x) \cap A, y \neq x \implies \|y - x\| < r$ **(I)**

If $x \in A$ then $\exists y = x$ such that $\|y - x\| = 0 < r$ **(II)**

From **(I)** and **(II)**, we get the required result.

($\Leftarrow$) If for each $r > 0, \exists y \in A$ such that $\|y - x\| < r$; that is $\forall r > 0, \exists y \in A, y \in B_r(x)$

$\implies \forall r > 0, (B_r(x) \cap A) \setminus \{x\} \neq \phi \implies x \in A'$. Thus, $x \in \overline{A}$. □

2.5 Convergence in Normed Space

Definition 2.32.
Let $\langle x_n \rangle$ be a sequence in a normed space $(L, \| \|)$. Then $\langle x_n \rangle$ is said to be **convergent** in L if $\exists x \in L$ such that $\forall \epsilon > 0, \exists k \in Z_+$ such that $\|x_n - x\| < \epsilon, \ \forall n > k$

We write $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty}(x_n) = x$; that is

$\|x_n - x\| \to 0 \iff x_n \to x$.

$\langle x_n \rangle$ is **divergent** if it is not convergent.

Theorem 2.33.

If $\langle x_n \rangle$ is a convergent sequence in $(L, \| \|)$, then its limit is unique. i.e., If $\langle x_n \rangle \to x$ and $\langle x_n \rangle \to y$ then $x = y$.

Proof. Let $\epsilon > 0$. Since $\langle x_n \rangle \to x$ and $\langle x_n \rangle \to y$, then $\exists k_1, k_2 \in Z_+$ such that

$\|x_n - x\| < \frac{\epsilon}{2}, \ \forall n > k_1$ and $\|x_n - y\| < \frac{\epsilon}{2}, \ \forall n > k_2$

Let $k = \max\{k_1, k_2\}$, so $\forall n > k$

$$\|x - y\| = \|x_n - y - x_n + x\| = \|(x_n - y) - (x_n - x)\|$$

$$\leq \|x_n - y\| + \|x_n - x\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\implies \|x - y\| < \epsilon, \ \forall \epsilon > 0$. Thus, $\|x - y\| = 0$, so $x = y$. □

Theorem 2.34.

Let $A \subseteq L$ where L is a normed space, let $x \in L$. Then

$x \in \overline{A} \iff \exists \langle x_n \rangle$ a sequence in A such that $\langle x_n \rangle \to x$.

Proof. ($\Rightarrow$) Let $x \in \overline{A} = A \cup A'$

If $x \in A$ then the sequence $\langle x, x, x, ... \rangle \to x$ **(I)**

If $x \notin A, i.e., x \in A'$ then for each open set G, $x \in G, (G \cap A) \setminus \{x\} \neq \phi$.

Since $B_r(x)$ is an open set then $\forall r > 0$, we have $B_r(x) \cap A \setminus \{x\} \neq \phi$. Set $0 < r = \frac{1}{n} \in Z_+$. Then $\forall n \in Z_+, (B_{\frac{1}{n}}(x) \cap A) \setminus \{x\} \neq \phi$

Let $x_n \in B_{\frac{1}{n}}(x) \cap A$, s.t $x_n \neq x$, hence, $\|x_n - x\| < \frac{1}{n}, \ \forall n \in Z_+$ **(*)**

Thus, $\exists \langle x_n \rangle \in A$ such that $\|x_n - x\| < \frac{1}{n}, \ \forall n \in Z_+$.

To show $\langle x_n \rangle \to x$; that is $\|x_n - x\| < \epsilon, \ \forall \epsilon > 0$

Let $\epsilon > 0$ so by Archmedian theorem $\exists k \in Z_+$ such that $\frac{1}{k} < \epsilon$

Hence, $\forall n > k, \frac{1}{n} < \frac{1}{k} < \epsilon$

From **(*)**, $\forall n > k, \|x_n - x\| < \frac{1}{n} < \frac{1}{k} < \epsilon$. Thus, $x_n \to x$ **(II)**

From **(I)** and **(II)**, we get the required result.

($\Leftarrow$) If $\exists \langle x_n \rangle$ a sequence in A such that $\langle x_n \rangle \to x$. To prove $x \in \overline{A} = A \cup A'$

If $x \in A$ then $x \in \overline{A}$

If $x \notin A$. Let G be an open set in L such that $x \in G$. Then $\exists r > 0$ such that $B_r(x) \subseteq G$. Since $r > 0$ and $x_n \to x, \exists k \in Z_+$ such that $\|x_n - x\| < r, \quad \forall n > k$.

This implies, $x_n \in B_r(x) \quad \forall n > k$ and since $x_n \in A \;\; \forall n \in Z_+$. Then $(B_r(x) \cap A) \setminus \{x\} \neq \phi$. Since $B_r(x) \subseteq G$, then $(G \cap A) \setminus \{x\} \neq \phi$. So, $x \in A'$, and therefore $x \in \overline{A}$. □

Theorem 2.35.

Let $\langle x_n \rangle, \langle y_n \rangle$ be two sequences in normed space $(L, \| \, \|)$ such that $x_n \to x$ and $y_n \to y$. Then

(1) $\langle x_n \rangle \pm \langle y_n \rangle \to x \pm y$

(2) $\lambda \langle x_n \rangle \to \lambda x \quad$ for any scalar λ

(3) $\|\langle x_n \rangle\| \to \|x\|$

Proof. **(1)** Since $x_n \to x$, then

for each $\epsilon > 0, \exists k_1 \in Z_+$ such that $\|x_n - x\| < \dfrac{\epsilon}{2}, \;\; \forall n > k_1$

Also since $y_n \to y$, then

for each $\epsilon > 0, \exists k_2 \in Z_+$ such that $\|y_n - y\| < \dfrac{\epsilon}{2}, \;\; \forall n > k_2$

Let $k = \max\{k_1, k_2\}$. Then, for each $n > k$

$\|x_n - x\| < \dfrac{\epsilon}{2}$ and $\|y_n - y\| < \dfrac{\epsilon}{2}$ **(I)**

Now, for each $n > k$,

$$\|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\| \leq \|x_n - x\| + \|y_n - y\|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad (\text{from } \mathbf{(I)})$$

Thus, $x_n + y_n \to x + y$ as required.

(2) Let $\epsilon > 0$. Since $x_n \to x, \exists k \in Z_+$ s.t $\|x_n - x\| < \frac{\epsilon}{|\lambda|}, \quad \forall n > k$ **(II)**

But $\|\lambda x_n - \lambda x\| = |\lambda| \underbrace{\|x_n - x\| < \frac{\epsilon}{|\lambda|}}_{\text{using } \textbf{(II)}} |\lambda| = \epsilon$

Thus, $\lambda\langle x_n\rangle \to \lambda x$

(3) Let $\epsilon > 0$. Since $x_n \to x, \exists k \in Z_+$ s.t $\|x_n - x\| < \epsilon, \quad \forall n > k$ **(III)**

But $| \, \|x_n\| - \|x\| \, | \le \underbrace{\|x_n - x\| < \epsilon}_{\text{using } \textbf{(III)}} \quad \forall n > k$. Hence, $\|x_n\| \to \|x\|$. □

Definition 2.36.
Let $\langle x_n\rangle$ be a sequence in a normed space $(L, \| \, \|)$. Then $\langle x_n\rangle$ is said to be **Cauchy sequence** if $\forall \epsilon > 0, \exists k \in Z_+$ s.t $\|x_n - x_m\| < \epsilon$, $\forall n, m > k$.

Theorem 2.37.

Every convergent sequence in a normed space $(L, \| \, \|)$ is a Cauchy sequence.

Proof. Let $\langle x_n\rangle$ be a convergent sequence in L. Then $\exists x \in L$ such that $x_n \to x$ and so $\forall \epsilon > 0, \exists k \in Z_+$ such that $\|x_n - x\| < \frac{\epsilon}{2} \quad \forall n > k$ **(I)**

Now, for $n, m > k$,

$\|x_n - x_m\| = \|(x_n - x) + (x - x_m)\| \le \underbrace{\|x_n - x\| + \|x_m - x\| < \frac{\epsilon}{2} + \frac{\epsilon}{2}}_{\text{using } \textbf{(I)}} = \epsilon.$

Thus, $\langle x_n\rangle$ is a Cauchy sequence. □

Definition 2.38.
Let $\langle x_n\rangle$ be a sequence in a normed space $(L, \| \, \|)$. Then $\langle x_n\rangle$ is said to be **bounded sequence** if $\exists k \in \mathbb{R}, k > 0$ such that $\|x_n\| \le k, \quad \forall n \in Z_+$.

Theorem 2.39.

Every Cauchy sequence $\langle x_n\rangle$ in a normed space $(L, \| \, \|)$ is bounded.

Proof. Let $\epsilon = 1$. Since $\langle x_n\rangle$ is a Cauchy sequence, $\exists k \in Z_+$ such that $\|x_n - x_m\| < 1, \quad \forall n, m > k$. Hence, $\|x_n - x_{k+1}\| < 1, \quad \forall n > k$ (by considering $m = k + 1$) **(I)**

By Theorem 2.3(4), we have $|\, \|x_n\| - \|x_{k+1}\| \,| \leq \underbrace{\|x_n - x_{k+1}\| < 1}_{\text{using } \mathbf{(I)}} \quad \forall n > k$

Thus, $\|x_n\| - \|x_{k+1}\| < 1 \quad \forall n > k$

Then, $\|x_n\| < 1 + \|x_{k+1}\| \quad \forall n > k$

Let $M = \max\{\|x_1\|, \|x_2\|, ..., \|x_k\|, 1 + \|x_{k+1}\|\}$

Hence, $\|x_n\| \leq M \quad \forall n \in Z_+$. So, $\langle x_n \rangle$ is bounded. □

Corollary 2.40.

Every convergent sequence in a normed space $(L, \| \|)$ is bounded.

Proof. From Theorem 2.37, Every convergent sequence in a normed space $(L, \| \|)$ is Cauchy, and from Theorem 2.39, every Cauchy sequence in a normed space $(L, \| \|)$ is bounded. □

2.6 Convexity in Normed Linear Space

Definition 2.41. *(revisit)*
A subset A of a linear space L is said to be **convex** if $\forall x, y \in A, \lambda \in [0, 1]$ then $\lambda x + (1 - \lambda)y \in A$.

Example 2.42.

Let $A = (1, 3) \subset \mathbb{R}$. Is A convex set?

Solution: Let $x, y \in A, \lambda \in [0, 1]$

Since $1 < x < 3 \implies 1\lambda < \lambda x < 3\lambda$ (I)

Since $1 < y < 3 \implies 1(1 - \lambda) < (1 - \lambda)y < 3(1 - \lambda)$ (II)

By summing up (I) and (II)

$$\lambda + (1 - \lambda) < \lambda x + (1 - \lambda)y < 3\lambda + 3(1 - \lambda)$$

$$1 < \lambda x + (1-\lambda)y << 3$$

Thus, $\lambda x + (1-\lambda)y \in A$. Hence, A is convex set.

Proposition 2.43.
Let L linear space. Then

(1) Every subspace of L is convex

(2) If $A, B \subset L$ are convex sets then $A \cap B$ is convex (**H.W.**)

(3) If $A, B \subset L$ are convex sets then $A + B$ is convex

Proof. **(1)** Let L be a linear space over a field $F = \mathbb{R}$ or C, let A be a subspace of L. Hence, by Theorem 1.13, $\forall x, y \in A$ and $\forall \alpha, \beta \in F$ we have $\alpha x + \beta y \in A$.

Take $\alpha = \lambda \in [0,1]$ and $\beta = 1 - \lambda$. Hence, $\alpha x + \beta y = \lambda x + (1-\lambda)y \in A$.

Thus, A is a convex set.

(3) Let $a_1 + b_1, a_2 + b_2 \in A + B$, then $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

To prove $\lambda(a_1 + b_1) + (1-\lambda)(a_2 + b_2) \in A + B, \ \ \forall \lambda \in [0,1]$.

Since A convex and $a_1, a_2 \in A \implies \lambda a_1 + (1-\lambda)a_2 \in A \quad \forall \lambda \in [0,1]$ (I)

Since B convex and $b_1, b_2 \in B \implies \lambda b_1 + (1-\lambda)b_2 \in B \quad \forall \lambda \in [0,1]$ (II)

By summing up (I) and (II) we get

$\lambda a_1 + (1-\lambda)a_2 + \lambda b_1 + (1-\lambda)b_2 \in A + B$

i.e., $\lambda(a_1 + b_1) + (1-\lambda)(a_2 + b_2) \in A + B$. Thus, $A + B$ is a convex set. $\square$

Remark 2.44.
The union of two convex sets is not necessary convex. For example, let $A = (3,7) \cup (7,12)$. Then A is not convex. To show this, take $x = 6, y = 8, \lambda = \dfrac{1}{2}$ then $\lambda x + (1-\lambda)y = \dfrac{1}{2}(6) + \dfrac{1}{2}(8) = 7 \notin A \cup B$.

Proposition 2.45.

Let $(L, \| \|)$ be a normed linear space, let $x_0 \in L$. Then $B_r(x_0)$ and $\overline{B}_r(x_0)$ are convex sets.

Proof. To prove $B_r(x_0)$ is a convex set. Let $x, y \in B_r(x_0)$, and let $\lambda \in [0,1]$. Then,

$\|x - x_0\| < r$ and $\|y - x_0\| < r$ (I)

We must prove $\lambda x + (1-\lambda)y \in B_r(x_0)$; that is we must prove

$\|\lambda x + (1-\lambda)y - x_0\| < r$

$\|\lambda x + (1-\lambda)y - x_0\| = \|\lambda x + \lambda \mathbf{x_0} - \lambda \mathbf{x_0} + (1-\lambda)y - x_0\|$ (adding and subtracting λx_0)

$$= \|\lambda(x - x_0) + (1-\lambda)(y - x_0)\|$$

$$\leq |\lambda| \, \|(x - x_0)\| + |1-\lambda| \, \|y - x_0\| < \lambda r + (1-\lambda)r = r$$

(by (I) and since $\lambda > 0$ then $|\lambda| = \lambda, |1-\lambda| = 1-\lambda$)

Thus, $\lambda x + (1-\lambda)y \in B_r(x_0)$ and hence $B_r(x_0)$ is convex. Similarly, $\overline{B}_r(x_0)$ is a convex set. $\square$

Proposition 2.46.

Let $(L, \| \|)$ be a normed linear space and $A \subseteq L$ and convex then $\overline{A}$ is a convex set.

Proof. Let $x, y \in \overline{A}$ and $\lambda \in [0,1]$. To prove $\lambda x + (1-\lambda)y \in \overline{A}$

Let $r > 0$. Since $x, y \in \overline{A}$ then by Proposition 2.31, $\exists a, b \in A$ such that

$\|x - a\| < r$ and $\|y - b\| < r$ (I)

Since A is convex then $\lambda a + (1-\lambda)b \in A$

Now, $\big\|\lambda x + (1-\lambda)y - \big(\lambda a + (1-\lambda)b\big)\big\| = \|\lambda(x-a) + (1-\lambda)(y-b)\|$

$$\leq \lambda \|x-a\| + (1-\lambda)\|y-b\|$$

$$< \lambda r + (1-\lambda) r \qquad \text{(from (I))}$$

$$= r$$

Thus, $\left\| \big(\lambda x + (1-\lambda)y\big) - \big(\underbrace{\lambda a + (1-\lambda)b}_{\in A}\big) \right\| < r$

Thus, from Proposition 2.31, $\lambda x + (1-\lambda)y \in \overline{A}$. □

Remark 2.47.
The converse of the above proposition is not true. For example, let $A = [1,2) \cup (2,5] \subset (\mathbb{R}, |\ |)$ then $\overline{A} = [1,5]$ is a convex set. But A is not convex, since if $x = 1, y = 3, \lambda = \dfrac{1}{2}$ then $\lambda x + (1-\lambda)y = \frac{1}{2} + \frac{1}{2}(3) = 2 \notin A$.

2.7 Continuity in Normed Linear Space

Definition 2.48.
Let $(L, \|\ \|_L), (L', \|\ \|_L{}')$ be normed linear spaces. A mapping $f : L \to L'$ is called **continuous** at $x_0 \in L$ if for each $\epsilon > 0, \exists \delta > 0$ (depend on x_0) such that

$$\forall x \in L, \ \text{ if } \ \|x - x_0\|_L < \delta \ \text{ then } \ \|f(x) - f(x_0)\|_L{}' < \epsilon.$$

i.e., $\forall x \in L,$ if $x \in B_\delta(x_0)$ then $f(x) \in B_\epsilon(f(x_0))$

Theorem 2.49.
Let $(L, \|\ \|_L), (L', \|\ \|_L{}')$ be normed linear spaces. A mapping $f : L \to L'$ is continuous at $x_0 \in L$ if and only if $\forall \langle x_n \rangle \in L$ with $x_n \to x_0$ implies that $f(x_n) \to f(x_0)$.

Proof. ($\Rightarrow$) Let f be a contiunuous mapping at x_0 and let $\langle x_n \rangle$ be a sequence in L such that $x_n \to x_0$. To prove $f(x_n) \to f(x_0)$.

Let $\epsilon > 0$, then $\exists \delta > 0$ such that $\forall x \in L$

if $\|x - x_0\|_L < \delta$ then $\|f(x) - f(x_0)\|_L{}' < \epsilon$ (From continuity of f at x_0).

Since $x_n \to x_0$ and $\delta > 0, \exists k \in Z_+$ such that $\|x_n - x_0\|_L < \delta, \quad \forall n > k$.

Hence, $\|f(x_n) - f(x_0)\|_L{}' < \epsilon, \quad \forall n > k$; that is $f(x_n) \to f(x_0)$.

($\Leftarrow$) Suppose that $x_n \to x_0$ implies that $f(x_n) \to f(x_0)$. To prove f is continuous at x_0.

Assume that f is not continuous at x_0, so $\exists \epsilon > 0$ such that $\forall \delta > 0, \exists x \in L$ and

$\|x - x_0\|_L < \delta$ but $\|f(x) - f(x_0)\|_L{}' \geq \epsilon$.

Now, $\forall n \in Z_+, \dfrac{1}{n} > 0$, then $\exists x_n \in L$ such that

$\|x_n - x_0\|_L < \dfrac{1}{n}$ but $\|f(x_n) - f(x_0)\|_L{}' \geq \epsilon$. This means $x_n \to x_0$ but $f(x_n) \nrightarrow f(x_0)$ in L' which is a contradiction. Thus, f is continuous at x_0. □

Theorem 2.50.

Let $(L, \| \|)$ be a normed space and let $f : (L, \| \|) \to (\mathbb{R}, | |)$ such that $f(x) = \|x\| \quad \forall x \in L$. Then f is continuous at x_0.

Proof. Let $x_n \to x_0$ in L. Then $\forall \epsilon > 0, \exists k \in Z_+$ such that

$\|x_n - x_0\| < \epsilon \quad \forall n > k \qquad$ **(I)**

But $| \, \|x_n\| - \|x_0\| \, | \leq \|x_n - x_0\| \quad \forall n > k$

$\Longrightarrow | \, \|x_n\| - \|x_0\| \, | < \epsilon \quad \forall n > k \quad$ (Using **(I)**)

$\Longrightarrow |f(x_n) - f(x_0)| < \epsilon \quad \forall n > k \quad$ (Using (since $f(x) = \|x\|$))

$f(x_n) \to f(x_0)$; that is f is continuous at x_0. □

Remark 2.51.

Let $(L_1, \| \|_1), (L_2, \| \|_2)$ and $(L_3, \| \|_3)$ be normed spaces and let $f : L_1 \times L_2 \to L_3$ be a mapping. Then f is continuous at $(x_0, y_0) \in L_1 \times L_2$ if and only if $\forall \langle (x_n, y_n) \rangle \in L_1 \times L_2$ and $\langle (x_n, y_n) \rangle \to (x_0, y_0)$ then $f(x_n, y_n) \to f(x_0, y_0)$.

Theorem 2.52.

Let $(L, \| \|$ be a normed space over a field F. Then

(1) The mapping $f : L \times L \to L$ such that $f(x, y) = x + y \quad \forall x, y \in L$ is continuous at any point in $L \times L$.

(2) The mapping $g : F \times L \to L$ such that $g(\lambda, x) = \lambda x \quad \forall x \in L, \forall \lambda \in F$ is continuous at any point in $F \times L$.

Proof. **(1)** Let (x_0, y_0) be an arbitrary point in $L \times L$ and $(x_n, y_n) \to (x_0, y_0)$. Then, $x_n \to x_0$ and $y_n \to y_0$ such that

$\|x_n - x_0\| \to 0$ and $\|y_n - y_0\| \to 0$ as $n \to +\infty$.

We must prove $f(x_n, y_n) \to f(x_0, y_0)$. i.e., $\|f(x_n, y_n) - f(x_0, y_0)\| \to 0$

Now, $\|f(x_n, y_n) - f(x_0, y_0)\| = \|(x_n + y_n) - (x_0 + y_0)\|$

$$= \|(x_n - x_0) + (y_n - y_0)\|$$

$$\leq \|x_n - x_0\| + \|y_n - y_0\|$$

Thus, $\|f(x_n, y_n) - f(x_0, y_0)\| \to 0$ as $n \to +\infty$; that is f is continuous at (x_0, y_0). Since (x_0, y_0) is arbitrary, f is continuous at $L \times L$.

(2) Let (λ_0, x_0) be an arbitrary point in $F \times L$ and $(\lambda_n, x_n) \to (\lambda_0, x_0)$. Then, $\lambda_n \to \lambda_0$ and $x_n \to x_0$.

Hence, $|\lambda_n - \lambda_0| \to 0$, $\|x_n - x_0\| \to 0$ as $n \to +\infty$.

We must prove $g(\lambda_n, x_n) \to g(\lambda_0, x_0)$. i.e., $\|g(\lambda_n, x_n) - g(\lambda_0, x_0)\| \to 0$

$$
\begin{aligned}
\|g(\lambda_n, x_n) - g(\lambda_0, x_0)\| &= \|\lambda_n x_n - \lambda_0 x_0\| \\
&= \|\lambda_n x_n - \mathbf{\lambda_n x_0} + \mathbf{\lambda_n x_0} - \lambda_0 x_0\| \\
&= \|\lambda_n (x_n - x_0) + (\lambda_n - \lambda_0) x_0\| \\
&\leq |\lambda_n| \, \|x_n - x_0\| + |\lambda_n - \lambda_0| \, \|x_0\|
\end{aligned}
$$

But $\|x_n - x_0\| \to 0$ and $|\lambda_n - \lambda_0| \to 0$ so that

$\|g(\lambda_n, x_n) - g(\lambda_0, x_0)\| \to 0$ as $n \to \infty$; that is $g(\lambda_n, x_n) \to g(\lambda_0, x_0)$. Thus, g is continuous at (λ_0, x_0). □

Theorem 2.53.

Let $(L, \| \|_L), (L', \| \|_L{}')$ be normed spaces and let $f : L \to L'$ be a linear transformation. If f is continuous at 0 then f is continuous at any point.

Proof. Let $x_0 \in L$ be an arbitrary point and let $x_n \to x_0$.

To prove $f(x_n) \to f(x_0)$ (using Theorem 2.49).

Since $x_n \to x_0$, then $x_n - x_0 \to 0$

But f is continuous at 0, thus $f(x_n - x_0) \to f(0)$

Since f is a linear transformation, then $f(x_n) - f(x_0) \to f(0) = 0$

It follows that $f(x_n) \to f(x_0)$. □

Remark 2.54.

The condition f is a linear transformation in the above theorem is necessary condition. For example: consider the normed space $(\mathbb{R}, | |)$. Let f is defined as

$$f(x) = \begin{cases} x & \text{if } x \leq 1 \\ x+1 & \text{if } x > 1\ . \end{cases}$$

It is clear that f is continuous at 0 and discontinuous at 1.

Also f is not linear transformation because if $x = 5, y = 6$ and $\alpha = \beta = 1$

$f(\alpha x + \beta y) = f(5+6) = f(11) = 11 + 1 = 12$

and $\alpha f(x) + \beta f(y) = f(5) + f(6) = (5+1) + (6+1) = 13$

Hence $f(\alpha x + \beta y) \neq \alpha f(x) + \beta f(y)$.

Theorem 2.55.

Let $(L, \| \|_L), (L', \| \|_L{}')$ be normed spaces and let $f : L \to L'$ be a linear transformation. If f is continuous at a point $x_1 \in L$ then f is continuous at each point.

Proof. Let $x_1 \in L$ and assume that f is continuous at x_1. Let $x_2 \in L$ be any point. To prove that f is continuous at x_2. Let $x_n \to x_2$ in L. Then, $x_n - x_2 \to 0$ and hence $x_n - x_2 + x_1 \to x_1$. Since f is continuous at x_1 then $f(x_n - x_2 + x_1) \to f(x_1)$.

Since f is a linear transformation, then $f(x_n) - f(x_2) + f(x_1) \to f(x_1)$.

Hence, $f(x_n) - f(x_2) \to 0$, and thus, $f(x_n) \to f(x_2)$.

Therefore, f is continuous at x_2. Thus, f can not be continuous at some points and discontinuos at some points. □

Example 2.56.

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \,. \end{cases}$$

Show that f is not continuous at $(0,0)$.

Solution: Let $x_n = \dfrac{1}{n}$ and $y_n = \dfrac{-1}{n}$ $\forall n \in N$.

Then, $x_n \to 0$ and $y_n \to 0$. Thus, $(x_n, y_n) \to (0,0)$. But

$$f(x_n, y_n) = \frac{\frac{1}{n}(\frac{-1}{n})}{\frac{1}{n^2} + \frac{1}{n^2}} = \frac{\frac{-1}{n^2}}{\frac{2}{n^2}} = \frac{-1}{2}$$

Hence, $f(x_n, y_n) \to \frac{-1}{2}$ but $f(0,0) = (0,0)$. Thus, $f(x_n, y_n) \nrightarrow f(0,0)$. Thus, f is not continuous at $(0,0)$.

2.8 Boundedness in Normed Linear Space

Definition 2.57. ***Bounded Set***
Let $(L, \| \|_L)$ be a normed space and let $A \subset L$. A is called a **bounded set** if there exists $k > 0$ such that $\|x\| \leq k \quad \forall x \in A$.

Example 2.58.

Consider $(\mathbb{R}, | |)$ and let $A = [-1, 1)$. Since $|x| \leq 1$, then A is bounded.

Example 2.59.

Consider $(\mathbb{R}^2, \| \|)$ be a normed space such that

$\|x\| = \left[\sum_{i=1}^{2} |x_i|^2 \right]^{\frac{1}{2}}$ be the Eucledian norm, for each $x = (x_1, x_2) \in \mathbb{R}^2$.

Let $A = \{(x_1, x_2) \in \mathbb{R}^2 : -1 \leq x_1 \leq 1, \; x_2 \geq 0\}$. Then, A is unbounded.

Theorem 2.60.

Let $(L, \| \|_L)$ be a normed space and let $A \subseteq L$. Then the following statements are equivalent.

(1) A is bounded.

(2) If $\langle x_n \rangle$ is a sequence in A and $\langle \alpha_n \rangle$ is a sequence in F such that $\alpha_n \to 0$ then $\alpha_n x_n \to 0$.

Proof. (1)⇒(2) Since A is bounded, $\exists k > 0$ such that $\|x_n\| \leq k \quad \forall x_n \in A$. Since $\alpha_n \to 0$ as $n \to \infty$, then $|\alpha_n| \to 0$. Hence,

$\|\alpha_n x_n - 0\| = \|\alpha_n x_n\| = |\alpha_n| \, \|x_n\| \leq |\alpha_n| \, k \quad$ (since $\|x_n\| \leq k$)

But $|\alpha_n| \to 0$, thus $|\alpha_n| \, k \to 0$. Therefore, $\|\alpha_n x_n - 0\| \to 0$ and hence $\alpha_n x_n \to \mathbf{0}_X$.

(2)⇒(1) Suppose A is not bounded. Then, $\forall k \in Z_+, \exists x_k \in A$ such that $\|x_k\| > k$.

Put $\alpha_k = \dfrac{1}{k}$. Hence, $\alpha_k \to 0$. But

$\|\alpha_k x_k\| = \left\|\frac{1}{k} x_k\right\| = \dfrac{1}{k} \|x_k\| > \dfrac{1}{k}.k = 1$

Then, $\|\alpha_k x_k\| > 1$, thus $\alpha_k x_k \nrightarrow 0$ which contradicts (2). □

Definition 2.61. ***Bounded Mapping***
Let $(L, \| \|_L), (L', \| \|_L{}')$ be two normed space and $f : L \to L'$ be a linear transformation. f is called **bounded mapping** if for each $A \subseteq L$ bounded then $f(A) = \{f(a) : a \in A\}$ is bounded set in L'.

Example 2.62.

Let Consider $(\mathbb{R}, | \, |)$ and $(\mathbb{R}^2, \| \, \|)$ be a normed space such that

$\|x\| = \left[\sum_{i=1}^{2} |x_i|^2\right]^{\frac{1}{2}} = [|x_1|^2 + |x_2|^2]^{\frac{1}{2}} = [x_1^2 + x_2^2]^{\frac{1}{2}} \quad \forall (x_1, x_2) \in \mathbb{R}^2.$

Define $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x_1, x_2) = x_1 + x_2 \quad \forall (x_1, x_2) \in \mathbb{R}^2$. Show that f is a linear transformation (**H.W.**). Let $A \subseteq \mathbb{R}^2$ and A is bounded. Show that $f(A)$ is bounded.

Solution: Let $A \subseteq \mathbb{R}^2$ and A is bounded to prove $f(A) = \{f(x_1, x_2) : (x_1, x_2) \in A\}$ is bounded.

Note that $\forall (x_1, x_2) \in A \implies f(x_1, x_2) = x_1 + x_2 \in f(A)$

$|fx_1, x_2| = |x_1 + x_2| \leq |x_1| + |x_2| \qquad \textbf{(I)}$

Since A is bounded then $\exists k > 0$ such that $\|(x_1, x_2)\| \leq k \quad \forall (x_1, x_2) \in A$

$\implies (x_1^2 + x_2^2)^{\frac{1}{2}} \leq k \implies x_1^2 + x_2^2 \leq k^2$

Since $x_1^2 \leq x_1^2 + x_2^2 \leq k^2$, then $x_1^2 \leq k^2 \implies |x_1| \leq k \qquad \textbf{(II)}$

Similarly, $x_2^2 \leq x_1^2 + x_2^2 \leq k^2$, then $x_2^2 \leq k^2 \implies |x_2| \leq k \qquad \textbf{(III)}$

Substitute **(II)** and **(III)** in **(I)**

$$|f(x_1, x_2)| = |x_1 + x_2| \leq \underbrace{|x_1| + |x_2| \leq k + k}_{\text{by } \textbf{(II) and (III)}} = 2k$$

i.e., $|f(x_1, x_2)| \leq 2k$. Thus, $f(A)$ is bounded, and hence, f is bounded.

Theorem 2.63.

Let $(L, \|\ \|_L), (L', \|\ \|_L{}')$ be normed spaces and $f : L \to L'$ be a linear transformation. Then f is bounded if and only if $\exists k > 0$ such that $\|f(x)\|_L{}' \leq k \|x\|_L \qquad \forall x \in L$.

Proof. ($\Rightarrow$) If f is bounded and let $A = \{x \in L : \|x\|_L \leq 1\}$.

It is clear A is bounded, and hence, $f(A)$ is bounded in L' (by definition of bnd function).

Thus, $\exists k > 0$ such that $\|f(x)\|_L{}' \leq k \qquad \forall x \in A \qquad \textbf{(I)}$

(1) If $x = \mathbf{0_L}$ then $f(\mathbf{0_L}) = \mathbf{0'_L}$, and thus, $\|f(\mathbf{0_L})\| = 0 \leq k \|\mathbf{0_L}\| = 0$.

(2) If $x \neq \mathbf{0_L}$, put $y = \dfrac{x}{\|x\|}$ such that $\|y\| = \left\|\dfrac{x}{\|x\|}\right\| = \dfrac{1}{\|x\|}.\|x\| = 1$.

Hence, $y \in A$. Thus, $\|f(y)\| \leq k$ **(II)**

$\|f(y)\| = \left\|f(\frac{x}{\|x\|})\right\| = \left\|\frac{1}{\|x\|}f(x)\right\| = \frac{1}{\|x\|}\|f(x)\|$

By **(II)**, $\|f(y)\| \leq k$, thus $\frac{1}{\|x\|}\|f(x)\| \leq k$. i.e., $\|f(x)\| \leq k.\|x\|$ as required.

($\Leftarrow$) Let A be a bounded set. Then, $\exists k_1 > 0$ such that $\|x\| \leq k_1 \quad \forall x \in A$

Since $\|f(x)\| \leq k\|x\| \quad \forall x \in X$, hence $\|f(x)\| \leq k\|x\| \quad \forall x \in A$. Then we get $\|f(x)\| \leq kk_1 \quad \forall x \in A$. Thus, $\|f(x)\| \leq k_2 \quad \forall x \in A$ where $k_2 = kk_1$; that is, $f(A)$ is a bounded set. □

Theorem 2.64.

Let $(L, \|\ \|_L), (L', \|\ \|_L{}')$ be normed spaces and $f : L \to L'$ be a linear transformation. Then f is bounded if and only if f is continuous.

Proof. ($\Leftarrow$) Suppose that f is continuous and not bounded,

hence $\forall n \in Z_+, \exists x_n \in L$ such that $\|f(x_n)\|_L{}' > n\|x_n\|_L$.

Let $y_n = \frac{x_n}{n\|x_n\|}$. Then, $\|f(y_n)\| = \left\|f(\frac{x_n}{n\|x_n\|})\right\| = \frac{\|f(x_n)\|}{n\|x_n\|} > \frac{n\|x_n\|}{n\|x_n\|} = 1$

Thus, $\|f(y_n) - f(0)\| = \|f(y_n)\| > 1$, i.e., $f(y_n) \nrightarrow f(0)$ **(I)**

but $\|y_n\| = \left\|\frac{x_n}{n\|x_n\|}\right\| = \frac{\|x_n\|}{n\|x_n\|} = \frac{1}{n}$

as $n \to \infty$, we get $\|y_n\| \to 0$, and hence, $y_n \to \mathbf{0}_L$.

It follows that $f(y_n) \to \underbrace{f(\mathbf{0}_L) = \mathbf{0}'_L}_{\text{By Theorem 1.19(i)}}$ (Since f is a linear transformation)

This contradicts **(I)**, thus, f is bounded.

($\Rightarrow$) Assume that f is bounded to prove f is continuous for all $x \in L$. Let $x_0 \in L$ and $\epsilon > 0$, to find $\delta > 0$ such that

$\forall x \in L, \|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$.

$\|f(x) - f(x_0)\| = \|f(x - x_0)\|$ (f is linear transformation)

Since f is bounded, then $\exists k > 0$ s.t. $\|f(x)\| \leq k\,\|x\| \quad \forall x \in L$ **(I)**

$$\text{Hence, } \|f(x) - f(x_0)\| = \underbrace{\|f(x - x_0)\| \leq k\,\|x - x_0\|}_{\text{By } \mathbf{(I)}}$$

$$< k\delta \qquad (\text{Since } \|x - x_0\| < \delta)$$

$$= k.\tfrac{\epsilon}{k} \qquad (\text{By choosing } \delta = \tfrac{\epsilon}{k} = \epsilon)$$

Thus, $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$.

Hence, f is continuous at $x_0 \in L$. Since x_0 is an arbitrary, then f is cont. $\forall x \in L$. □

Theorem 2.65.

Let $(L, \|\ \|_L), (L', \|\ \|_L{}')$ be normed spaces and $f : L \to L'$ be a linear transformation. If L is a finite dimensional space then f is bounded (hence, continuous).

Example 2.66.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x, y) = x + y \quad \forall (x, y) \in \mathbb{R}^2$.

f is a linear transformation function (check!)

and $dim(\mathbb{R}^2) = 2$. Hence, f is bounded (hence, continuous).

2.9 Bounded Linear Transformation

Definition 2.67.
Let $(L, \|\ \|_L), (L', \|\ \|_L{}')$ be normed spaces over a field F. The set of all bounded linear transformation mappings from L to L' is defined as

$B(L, L') = \{T : T : L \to L' \text{ is a linear bounded (hence, cont.) trans.}\}$

Theorem 2.68.

Prove that $B(L, L')$ is a linear subspace (over a field F) of the space of linear transformation mappings with respect to usual addition and usual scalar multiplication.

Proof. Let $\alpha, \beta \in F$ and $T_1, T_2 \in B(L, L')$. To prove $\alpha T_1 + \beta T_2 \in B(L, L')$ Since T_1, T_2 are linear transformations, then by Theorem 1.30, $\alpha T_1, \beta T_2$ are linear trans.

Now, $\alpha T_1, \beta T_2$ are linear trans., by Theorem 1.30,$\alpha T_1 + \beta T_2$ is linear transformation.

Next, we show $\alpha T_1 + \beta T_2$ is bounded.

Since T_1, T_2 are bounded, then $\exists k_1, k_2 > 0$ such that $\forall x \in L$ we have

$\|T_1(x)\|'_L \leq k_1 \|x\|_L$ and $\|T_2(x)\|'_L \leq k_2 \|x\|_L$ **(I)**

Then, $\|(\alpha T_1 + \beta T_2)(x)\|'_L = \|(\alpha T_1)(x) + (\beta T_2)(x)\|'_L$

$$= \|\alpha . T_1(x) + \beta . T_2(x)\|'_L \qquad \text{(Definition of scalar multiplication)}$$

$$\leq \|\alpha . T_1(x)\|'_L + \|\beta . T_2(x)\|'_L$$

$$= |\alpha| \|T_1(x)\|'_L + |\beta| \|T_2(x)\|'_L$$

$$\leq |\alpha| k_1 \|x\|_L + |\beta| k_2 \|x\|_L$$

$$= (|\alpha| k_1 + |\beta| k_2) \|x\|_L = k \|x\|_L \qquad (k = |\alpha| k_1 + |\beta| k_2)$$

Hence, $\alpha T_1 + \beta T_2$ is bounded.

Since $\alpha T_1 + \beta T_2$ is bounded and linear transformation, then $\alpha T_1 + \beta T_2 \in B(L, L')$. □

Theorem 2.69.

Let $(L, \| \|_L), (L', \| \|_L{}')$ be normed space. Prove that $B(L, L')$ is a normed space such that $\forall T \in B(L, L')$ we have

$\|T\| = \sup\{\|T(x)\|_{L'} : x \in L, \|x\|_L \leq 1\}$

Proof. To prove $\| \|$ is a norm on $B(L, L')$

(1) since $\|T(x)\|_{L'} \geq 0 \ \ \forall x \in L, \|x\|_L \leq 1$, then $\|T\| \geq 0$.

(2) $\|T\| = 0 \iff \sup\{\|T(x)\|_{L'} : x \in L, \|x\|_L \leq 1\} = 0$

$$\iff \|T(x)\|_{L'} = 0 \ \ \forall x \in L, \|x\|_L \leq 1$$

$$\iff T(x) = 0 \ \ \forall x \in L, \|x\|_L \leq 1$$

$$\iff T = \hat{0}$$

(3) Let $T_1, T_2 \in B(L, L')$

$$\|T_1 + T_2\| = \sup\{\|(T_1 + T_2)(x)\|_{L'} : x \in L, \|x\|_L \leq 1\}$$

$$\leq \sup\{\|(T_1(x)\|_{L'} + \|T_2(x)\|_{L'} : x \in L, \|x\|_L \leq 1\}$$

$$\leq \sup_{x\in L}\{\|(T_1(x)\|_{L'} : \|x\|_L \leq 1\} + \sup_{x\in L}\{\|(T_2(x)\|_{L'} : \|x\|_L \leq 1\}$$

$$= \|T_1\| + \|T_2\|$$

(4) $\|\alpha T\| = \sup\{\|(\alpha . T(x)\|_{L'} : x \in L, \|x\|_L \leq 1\}$

$$= |\alpha| \sup\{\|T(x)\|_{L'} : x \in L, \|x\|_L \leq 1\}$$

$$= |\alpha| \, \|T\|$$

□