Functional Analysis

Fourth Grade,

Department of Mathematics,

College of Education For Pure Sciences: Ibn AL-Haitham
University of Baghdad

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Chapter One

Fundamental Concepts

In this chapter, we introduce the following concepts:

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1.3. The Convexity, Examples of Convex Sets, Some Properties About Convex Sets.

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Linear (Vector) Space

A linear space (also called vector space), denoted by L or V, is a collection of objects called **vectors**, which may be added together and multiplied by numbers, called **scalars** which are taken from a field F. Before defining linear space, we first define an arbitrary field.

Definition (1.1):- Let F be a non-empty set and (+) and (.) be two binary operations on F. The ordered triple $(F, +, .)$ is called **field** if

1. $(F, +)$ is a commutative group
2. $(F - \{e\}, .)$ is a commutative group, where e is the identity with respect to (+).
3. (.) is distributed over (+) (from left and right)

Example (1.2):- Let (+) and (.) are **ordinary addition and multiplications**. Then

1. Each of $(\mathbb{R},+,.)$, $(\mathbb{C},+,.)$, and $(Q,+,.)$ are examples of fields
2. $(Z, -, .)$ is not field (Definition 1.1(1) does not hold)

   and $(Z, +, .)$ is not field (Definition 1.1(2) does not hold)

Definition (1.3):- A ***vector space*** (or ***linear space***) over a field F is a nonempty set L of elements $x, y, \ldots$ (called vectors) with two algebraic operations, these operations are called vector addition (+) and multiplication of vectors by scalars (.) , then we say that $(L,+,.)$ is a vector (Linear) space over F.

1) Vectors addition satisfy :-

$(L,+)$ be a commutative group

2) Multiplication by scalars satisfy :-

a) $\alpha.x \in L \ ,\forall\alpha \in F \,, x \in L$

b) $(\alpha\beta).x = \alpha\,(\beta.x) \quad ,\forall x \in L\ and\ \alpha\,,\beta \in F$

c) $\alpha.(x+y) = \alpha.x + \alpha.y$

d) $(\alpha+\beta).x = \alpha.x + \beta.x$ } Distributive laws

e) $1.x = x \ ,\forall x \in L,\ 1\ is\ the\ unity\ F$

Remarks(1.4):-

(1) The filed $F = \mathbb{C}$ or $\mathbb{R}$

(2) If $F = \mathbb{C}$ then we say that the $(L,+,.)$ is complex vector (complex linear) space

(3) If $F = \mathbb{R} \Rightarrow (L,+,.)$ is called real vector (linear)space

(4) The vectors addition is a mapping such that $+ : L \times L \to L$

(5) The multiplication by scalars (scalar multiplication) is a mapping such that $. : F \times L \to L$

(6) We can denote the zero vector by 0_L and the scalar by 0.

Examples of Linear (Vector) Space

Example (1.5):- The Euclidean space : $\mathbb{R}^n = \{ x = (x_1, x_2, \dots, x_n); x_i \in \mathbb{R}, i = 1, \dots, n\}$, with ordinary addition and multiplication. i.e ,

$$\left.\begin{aligned} x + y &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \\ \alpha . x &= (\alpha . x_1, \alpha . x_2, \dots, \alpha . x_n) \end{aligned}\right\} \forall x, y \in \mathbb{R}^n, \ \forall \alpha \in \mathbb{R},$$

Then $(\mathbb{R}^n, +, .)$ is real Linear space over the field $F = \mathbb{R}$.

Solution: It is clear that the following conditions are satisfied

(1) To prove $(\mathbb{R}^n, +)$ is commutative group

a) **The closure**: $\forall\, x, y \in \mathbb{R}^n$ to prove $x + y \in \mathbb{R}^n$

$x \in \mathbb{R}^n \rightarrow x = (x_1, x_2, \dots, x_n)$

$y \in \mathbb{R}^n \rightarrow y = (y_1, y_2, \dots, y_n)$

$x + y = (x_1 + y_1, \dots, x_n + y_n) \in \mathbb{R}^n$

b) **Associative** : to prove $(x + y) + z = x + (y + z), \forall x, y, z \in \mathbb{R}^n$

$$(x + y) + z = ((x_1, x_2, \dots, x_n) + (y_1, \dots, y_n)) + (z_1, \dots, z_n)$$

$$= (x_1 + y_1, \dots, x_n + y_n) + (z_1, \dots, z_n)$$

$$= \big((x_1 + y_1) + z_1, \quad \dots \quad, (x_n + y_n) + z_n\big) \ (+ \text{ asso. on } \mathbb{R})$$

$$= (x_1 + (y_1 + z_1), \quad \dots \quad, (x_n + (y_n + z_n))$$

$$= (x_1, \dots, x_n) + (y_1 + z_1, \dots, y_n + z_n)$$

$$= (x_1, \dots, x_n) + ((y_1, \dots, y_n) + (z_1, \dots \dots, z_n))$$

$$= x + (y + z)$$

c) **Identity**: It is clear that, there exist a unique identity $e = (0,0, \dots ,0) = 0_L$

$$\forall x \in \mathbb{R}^n \ \ s.t.\ \ x + 0_L \ = 0_L \ + x = x$$

d) **Inverse**: $\forall\ x \in \mathbb{R}^n, \exists(-x)$the invers of x s.t. $x + (-x) = (-x) + x = 0_L$

e) **Commutative**: $\forall x, y \in \mathbb{R}^n$, we have $\quad x + y = y + x$

$\Longrightarrow$ $(\mathbb{R}^n, +)$ is commutative group

(2) **Scalar Multiplication :**

a) To prove $\alpha . x \in \mathbb{R}^n$, $\forall x \in \mathbb{R}^n$and $\alpha \in \mathbb{R}$

$$\alpha.(x_1, x_2, \dots, x_n) = (\alpha . x_1, \dots, \alpha . x_n) \in \mathbb{R}^n (\text{since } \alpha . x_i \in \mathbb{R},$$

$$\forall\ i = 1,2,3, \dots \dots, n)$$

b) $(\alpha\beta).x = (\alpha\beta).(x_1, x_2, \dots, x_n)$

$$= ((\alpha\beta).x, \dots, (\alpha\beta).x_n)$$

$$= (\alpha.(\beta.x_1), \dots\ \alpha.(\beta.x_n))$$

$$= \ \alpha.(\beta.x_1, \dots, \beta.x_n)$$

$$= \alpha.(\beta.(x_1, \dots \dots, x_n)$$

$$= \alpha.(\beta.x)$$

c) $\alpha.(x + y) = \ \alpha.((x_1, \dots, x_n) + (y_1, \dots, y_n))$

$$= \ \alpha.((x_1 + y_1\ , \dots, x_n + y_n))$$

$$= (\alpha.(x_1 + y_1), \dots, \alpha.(x_n + y_n)$$

$$= ((\alpha.x_1 + \alpha.y_1), \dots \dots, (\alpha.x_n + \alpha.y_n))$$

$$= (\alpha.x_1, \dots \dots, \alpha.x_n) + (\alpha.y_1, \dots \dots, \alpha.y_n)$$

$= \alpha.(x_1, \ldots., x_n) + \alpha.(y_1, \ldots \ldots.., y_n)$

$= \alpha.x + \alpha.y$

d) $(\alpha + \beta).x = (\alpha + \beta).(x_1, \ldots \ldots, x_n)$

$= ((\alpha + \beta).x_1, \ldots \ldots (\alpha + \beta).x_n))$

$= (\alpha.x_1 + \beta.x_1, \ldots..., \alpha.x_n + \beta.x_n)$

$= \alpha.(x_1, \ldots \ldots, x_n) + \beta.(x_1, \ldots.., x_n)$

$= \alpha.x + \beta.x$

e) $1.x = 1.(x_1, \ldots., x_n) = (1.x, \ldots \ldots, 1.x)$

$= (x_1, \ldots., x_n) = x.$

Thus, $(\mathbb{R}^n, +, .)$ is linear space over $\mathbb{R}$.

Example (1.6):- Consider the space $\mathbb{C}^n$ with two operctions defined as in the previous example, then $(\mathbb{C}^n, +, .)$ is complex linear space over $\mathbb{C}$. **(H.W.)**

Example (1.7):- Show that the space $(l^2, +, .)$ is linear space over F where $l^2 = \{x = (x_1, x_2, x_3, \ldots \ldots), x_i \in F, \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$, with ordinary addition and multiplication. i.e ,

$x + y = (x_1, x_2, x_3, \ldots.) + (y_1, y_2, y_3, \ldots.) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 \ldots \ldots)$

$\alpha . x = \alpha.(x_1, x_2, x_3, \ldots) = (\alpha.x_1, \alpha.x_2, \alpha.x_3 \ldots.), \forall\, x, y \in l^2, \alpha \in F$?

Solution:

1. **To prove that $(l^2, +)$ is commutative group?**

❖ **The closure** : Let $x, y \in l^2$ to prove $x + y \in l^2$

$x \in l^2 \Longrightarrow x = (x_1, x_2, \dots);\ x_i \in F$ and $\sum_{i=1}^{\infty}|x_i|^2 < \infty$

$y \in l^2 \Longrightarrow y = (y_1, y, \dots);\ y_i \in F$ and $\sum_{i=1}^{\infty}|y_i|^2 < \infty$

$x + y = (x_1 + y_1, x_2 + y_2, \dots\ );\ x_i + y_i \in F,$ for all $i = 1,2,3, \dots$

To prove $\sum_{i=1}^{\infty}|x_i + y_i|^2 < \infty$.

$|x_i + y_i|^2 = |(x_i + y_i)^2|$

$= |x_i^2 + 2x_iy_i + y_i^2|$

$\leq |x_i|^2 + 2|x_iy_i| + |y_i|^2$

$\leq |x_i|^2 + |x_i|^2 + |y_i|^2 + |y_i|^2,$ because $(\ 2|x_iy_i| \leq |x_i|^2 + |y_i|^2\ )$

$= 2|x_i|^2 + 2|y_i|^2$

Taking the sum to the both sides of the above inequality

$\sum_{i=1}^{\infty}|x_i + y_i|^2 \leq 2\sum_{i=1}^{\infty}|x_i|^2 + 2\sum_{i=1}^{\infty}|y_i|^2 < \infty + \infty = \infty$

- The associative: **(H.W.)**
- **The identity** : $\exists\ e = (0,0, \dots) \in l^2\ \ \forall x = (x_1, x_2, \dots\ ) \in l^2$

such that $e + x = x + e = x$

It is clear that $\sum_{i=1}^{\infty}|0|^2 < \infty$ **(H.W.)**

- **The inverse** : $\forall\ x = (x_1, x_2, \dots\ ) \in l^2,\ \exists -x = (-x_1, -x_2, \dots\ ) \in l^2$

such that $-x + x = x + -x = 0$. It is clear that $\sum_{i=1}^{\infty}|-x_i|^2 < \infty$ **(H.W.)**

2. **Multiplication by scalars :**

❖ To prove that $\alpha . x \in l^2$, $\forall x \in l^2, \alpha \in F$

$\alpha . x = (\alpha . x_1 , \alpha . x_2 , \dots)$ to prove $\sum_{i=1}^{\infty} |\alpha x_i|^2 < \infty$

$\sum_{i=1}^{\infty} |\alpha . x_i|^2 = \sum_{i=1}^{\infty} |\alpha|^2 \, |x_i|^2 = |\alpha|^2 \sum_{i=1}^{\infty} |x_i|^2 < \infty.$

The rest of the conditions are homework.

Remark (1.8):- In general the inequality holds , for any $1 \leq p < \infty$

$$|x_i + y_i|^p \leq 2^{\mathrm{p}-1} \, [|x_i|^p + |y_i|^p].$$

Example (1.9):- Show that the space $(l^p , + , .)$ is linear space over F where $1 \leq p < \infty$ and $l^p = \{x = (x_1, x_2, x_3, \dots \dots), x_i \in F , \sum_{i=1}^{\infty} |x_i|^p < \infty\}$ with ordinary addition and multiplication ? **(H.W.)**

Example (1.10):- Consider the space $l^\infty = \{ x = (x_1, x_2, x_3 \dots \dots), x_i \in F, |x_i| < c_x , \ i = 1,2, \dots . \}$

where c_x is a real number which may depend on x but does not depend on i . Then $(l^\infty, + , .)$ is linear space over F when **+** and **.** are defined as in Example (1.7).

Solution:

To prove that $(l^\infty , +)$ **is commutative group?**

❖ **The closure** : Let $x, y \in l^\infty$ to prove $x + y \in l^\infty$

$x \in l^\infty \Rightarrow x = (x_1, x_2, \dots);\ x_i \in F$ and $|x_i| < c_x \quad \forall i \in N$

$y \in l^{\infty} \Longrightarrow y = (y_1, y, \dots);\ y_i \in F$ and $|y_i| < c_y \quad \forall i \in N$

$x + y = (x_1 + y_1, x_2 + y_2, \dots \quad);\ x_i + y_i \in F,$ for all $i = 1,2,3, \dots$

To prove $\quad |x_i + y_i| < c_q; c_q > 0.$

Now, $|x_i + y_i| \leq |x_i| + |y_i| \leq c_x + c_y = c_q.$

Thus, $x + y \in l^{\infty}.$

- ❖ The rest of the conditions are homework.

Example (1.11):- The space $(C^b[a,b], +, .)$ is linear space over $\mathbb{R}$ where + and defined as $(f+g)(x) = f(x) + g(x), \quad \forall f, g \in C^b[a,b], \quad x \in \mathbb{R}$

$$(\alpha f)(x) = \alpha f(x),\ \alpha \in \mathbb{R}$$

Solution:

The space $C^b[a,b] = \{ f: [a,b] \to \mathbb{R}$ continuous and bounded function $\}$

(1) To prove that prove $(C^b[a,b], +)$ is commutative group

(b) $\forall f, g \in C^b[a,b]$ T.p. $f + g \in C^b[a,b]$ —— f, g are continuous
\\ f, g are bounded

$\therefore f$ and g are continuous functions $\Longrightarrow f + g$ also continuous function …..(i)

$\therefore f$ and g are bounded functions

$$\left.\frac{\exists k_1 \geq 0 \quad s.t \quad |f(x)| \leq k_1}{\text{and } \exists k_2 \geq 0 \;\; s.t \; |g(x)| \leq k_2}\right\} \forall x \in \mathbb{R}$$

Now , $|(f+g)(x)| = |f(x) + g(x)|$

$$\leq |f(x)| + |g(x)| \leq k_1 + k_2 = k \,, \qquad k \geq 0$$

$\Longrightarrow f + g$ is bounded function ...... (ii)

From (i) & (ii) we have $f + g \in C^b[a,b]$

(b) $\forall f, g, h \in C^b[a,b]$ to prove that $(f+g)+h = f+(g+h)$

$$((f+g)+h)(x) = (f+g)(x) + h(x)$$
$$= \big(f(x) + g(x)\big) + h(x)$$
$$= f(x) + (g(x) + h(x))$$
$$= f(x) + (g+h)(x)$$
$$= (f + (g+h))(x)$$

(c) To prove that, there exist a unique function, for all $f \in C^b[a,b]$

Define $\acute{0}: [a,b] \to R \quad s.t \;\; \acute{0}(x) = 0 \,, \forall x \in [a,b]$

s.t $f + \acute{0} = \acute{0} + f = 0 \,, \forall f \in C^b[a,b]$

($\check{0}$ is continuous and bounded since its constant) $\Longrightarrow \acute{0} \in C^b[a,b]$

(d) To prove that , $\forall f \in C^b[a,b], \; \exists - f \in C^b[a,b]$

Such that $\quad f + (-f) = (-f) + f = \acute{0},$

since $f \in C^b[a,b] \Longrightarrow f$ is continuous and bounded

$\Longrightarrow -f$ is also continuous (by previous proposition) and $-f$ is also bounded

since $(|-f(x)| = |f(x)| \leq k, \quad k \geq 0)$

$= -f \in C^b[a,b]$ and $\big(f + (-f)\big)(x) = f(x) + \big(-f(x)\big) = 0 = 0(x)$

(e) $\forall f, g \in C^b[a,b]$, to prove that $f + g = g + f$

$$(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$$

$\therefore C^b[a,b]$ is comm group

(2) Scalar Multiplication

(a) $\forall f \in C^b[a,b], \alpha \in \mathbb{R}$, to prove that $\alpha f \in C^b[a,b]$

$\therefore f \in C^b[a,b] \Longrightarrow f$ is also continuous $\Longrightarrow \alpha f$ also continuous .... (i)

$f \in C^b[a,b] \Longrightarrow f$ is bounded $\Longrightarrow |f(x)| \leq k, k \geq 0$

But $|(\alpha f)(x)| = |\alpha f(x)| \leq |\alpha| \; f(x) \leq |\alpha| k$

$\Longrightarrow \alpha f$ is bounded .... (ii)

From (i) and (ii) we get, $\alpha f \in C^b[a,b]$

(b) $\forall f \in C^b[a,b]$ *and* $\alpha, \beta \in \mathbb{R}$, to prove $(\alpha\beta)f = \alpha(\beta f)$

$$((\alpha\beta)f)(x) = (\alpha\beta)f(x) = \alpha\big(\beta f(x)\big) = \alpha(\beta f)(x)$$

$$\Longrightarrow (\alpha\beta)f = \alpha(\beta f)$$

(c) $\forall f, g \in C^b[a,b], \alpha \in \mathbb{R}$, to prove $\alpha(f+g) = \alpha f + \alpha g$

$$\big(\alpha(f+g)\big)(x)=\alpha(f+g)(x)=\alpha(f(x)+g(x))$$

$= \alpha f(x) + \alpha g(x) = (\alpha f + \alpha g)(x)$

(d) $\forall f \in C^b[a,b], \alpha,\beta \in \mathbb{R}$, to prove $\quad (\alpha+\beta)f = \alpha f + \beta f$

$$\big((\alpha+\beta)f\big)(x)=(\alpha+\beta)f(x)=\alpha f(x)+\beta\, f(x)$$

$$=(\alpha f)(x)+(\beta f)(x)=(\alpha f+\beta f)(x)$$

(e) Let $f \in C^b[a,b]$ *and* 1 is the unity of $\mathbb{R}$, then $(1f)(x) = 1\, f(x) = f(x)$.

General Properties of Linear Space (without prove)

Theorem(1.12):- Let $(L,+,.)$ be a linear space over F .Then

(1) $0.x = 0_L$, $\forall x \in L$

(2) $\lambda.0_L = 0_L$, $\lambda \in F$

(3) $(-\alpha.x) = (-\alpha).x = \alpha.(-x)$ $,\forall x \in L,\ \alpha \in F$

(4) If $x,y \in L \Longrightarrow \exists!\, z \in L$ such that $x+z=y$

(5) $\alpha.(x-y) = \alpha.x - \alpha.y$, $\forall x,y \in L$, $\alpha \in F$

(6) If $\alpha.x = 0_L \Longrightarrow \alpha = 0$ or $x = 0_L$

(7) If $x \neq 0_L$ *and* $\alpha_1 x = \alpha_2 x \Longrightarrow \alpha_1 = \alpha_2$

(8) If $x \neq 0_L$, $\alpha \neq 0$, $y \neq 0_L$ *and* $\alpha.x = \alpha.y \Longrightarrow x = y$

Linear subspace

Definition(1.13):- Let L be a linear space over F and $\emptyset \neq S \subseteq L$, then we say that S is **linear subspace** of L if S itself is a linear spase over F .

Theorem (1.14):- If L be a linear space over F and $\emptyset \neq S \subseteq L$, then S is linear subspace if satisfy the following conditions

(1) $x + y \in S$, $\forall x, y \in S$

(2) $\alpha . x \in S$, $\forall x \in S$ and $\alpha \in F$

Or satisfy the equivalent condition of two conditions above ,

$\alpha . x + \beta . y \in S, \forall x, y \in S$ and $\alpha, \beta \in F$

Remark(1.15):-

(1) A special subspace of L is improper subspace $S = L$

(2) Every other subspace of $L(\neq \{0\})$is called proper

(3) Another special subspace of any linear space L is S={0}

Example (1.16):- show that $S = \{(x, x_2) \in \mathbb{R}^2 \,;\, x_2 = 3x_1\}$ is subspace of $\mathbb{R}^2$?

Solution :- It is clear that $S \subseteq \mathbb{R}^2$, and $S \neq \emptyset$ because $(0,0) = 0 \in S$

To prove that $\alpha . x + \beta . y \in S$, $\forall \alpha, \beta \in F = \mathbb{R}$, x , $y \in \mathbb{R}^2$

$x = (x_1, x_2), y = (y_1, y_2)$

$\alpha . x + \beta . y = (\alpha x_1, \alpha x_2) + (\beta y_1, \beta y_2)$

$= (\alpha x_1 + \beta y_1 \,, \alpha x_2 + \beta y_2)$

Now, $\alpha x_2 + \beta y_2 = \alpha(3x_1) + \beta(3y_1) = 3(\alpha x_1 + \beta y_1)$

$\Longrightarrow \alpha . x + \beta . y \in S$.

Example (1.17):- show that $S = \{\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} ; a, b \in \mathbb{R}\}$ is subspace of $M_{2\times2}(\mathbb{R})$?

Solution:- It is clear that $S \subseteq M_{2\times2}(\mathbb{R})$, and $S \neq \emptyset$ because $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in S$

$\forall x, y \in S \; and \;\; \alpha , \beta \in F.$

To prove that $\alpha.x + \beta.y = \alpha.\begin{pmatrix} 0 & a_1 \\ b_1 & 0 \end{pmatrix} + \beta.\begin{pmatrix} 0 & a_2 \\ b_2 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & \alpha a_1 \\ \alpha b_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \beta a_2 \\ \beta b_2 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & \alpha a_1 + \beta a_2 \\ \alpha b_1 + \beta b_2 & 0 \end{pmatrix} \in S$

Example (1.18):- show that $S = \{(x_1, x_2) \in \mathbb{R}^2; ax_1 + bx_2 = 0\}$ is subspace of $\mathbb{R}^2$? **(H.W.)**

Example (1.19):-The set $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 ; \; x_1 = 1 + x_2\}$ is not subspace of $\mathbb{R}^3$?

Solution:

Consider $\alpha = 2 \; and \;\; x = (2,1,0) \in S,$ because $(2 = 1 + 1)$

$\alpha.x = 2.(2,1,0) = (4,2,0) \notin S,$ because$(4 \neq 1 + 2)$

Hence, S is not subspace of $\mathbb{R}^3$.

Theorem(1.20):-Let S_1 and S_2 be two subspaces of linear space L . then

(1) $S_1 \cap S_2$ is subspace of linear space L .

(2) $S_1 + S_2$ is subspace of linear space L

(3) $S_1 \subseteq S_1 + S_2$, $S_2 \subseteq S_1 + S_2$.   **(H.W.)**

Exercise :

(1) Which of the following subsets of $\mathbb{R}^3$ be a subspace of $\mathbb{R}^3$

a) $S_1 = \{x = (x_1, x_2, x_3);\ x_1 = x_2 \text{ and } x_3 = 0\}$

b) $S_2 = \{(x_1, x_2, x_3); x_3 = x_2 + 1\}$

c) $S_3 = \{(x_1, x_2, x_3);\ \ x_1, x_2, x_3 \ \geq 0\}$

d) $S_4 = \{(x_1, x_2, x_3);\ x_1 - x_2 + x_3 = k\}$

(2) If S_1 and S_2 are subspaces of linear space L , then $S_1 \cup S_2$ not necessary subspace of L (Give example)

(3) If $S \neq \emptyset$ is any subset of L show that span S is subspace of L .

(4) Show that the Cartesian product $L = L_1 \times L_2$ of two linear spaces over the same field becomes a vector space , we define the two algebraic operations by

$$\left.\frac{(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)}{\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)}\right\} \quad \frac{\left.\begin{array}{c}\forall\ x = (x_1, x_2) \\ \hline y = (y_1, y_2)\end{array}\right\} \in L}{\alpha \in F}$$

(5) Let M be a subspace of a linear space L . The coset of an element $x \in L$ with respect to M is denoted by $x + M$ where

$x + M = \{z, z = x + m, m \in M\}$. Show that $(\frac{L}{M}, +, .)$ is linear space over under algebraic operations defined as

$(x+m)+(y+m)=(x+y)+m, \forall\, x+m, y+m \in \frac{L}{M}$

$\alpha.(x+m)=\alpha.x+m, \forall\, x+m \in \frac{L}{M}\ ,\ \alpha \in F$

Note : The space $(\frac{L}{M},+,.)$ is called quotient space (or factor space) .

Dimensional Linear Spaces

Definition (1.21):- A **linear combination** of vectors $x_1, x_2, \ldots, x_n$ of a linear space L is an expression of the from $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$, where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are any scalars

i.e., x is linear combination of $x_1, x_2, \ldots, x_n$ if $\exists \alpha_1, \alpha_2, \ldots, \alpha_n$ s.t.

$x=\alpha_1 x_1+\alpha_2 x_2+\ \ldots + \alpha_n x_n.$

Example (1. 22):- Let $S=\{(1,2,3),(1,0,2)\}$, Express $x=(-1,2,-1)$, as a linear combination of x_1 and x_2 .

Solution: We must find scalars $\alpha_1, \alpha_2 \in F$ such that $x=\alpha_1.x_1+\alpha_2.x_2$

$(-1,2,-1)=\ \alpha_1.(1,2,3)+\alpha_2.(1,0,2)$

$=(\alpha_1, 2\alpha_1, 3\alpha_1)+(\alpha_2, 0, 2\alpha_2)$

So, $\alpha_1+\alpha_2=-1 \implies \alpha_2=-\alpha_1-1$

$$2\alpha_1 + 0 = 2 \Longrightarrow 2\alpha_1 = 2 \Longrightarrow \alpha_1 = 1$$

and, $3\alpha_1 + 2\alpha_2 = -1$

$\therefore\ \alpha_2 = -1 - 1 = -2$.

Example (1. 23):-If $S = \{(1,2,3), (1,0,2)\}$. Show that $x = (-1,2,0)$, is not linear combination of x_1, x_2 .

Solution:

Let $\alpha_1, \alpha_2 \in F$ and $x_1, x_2 \in S$ such that $x = \alpha_1 . x_1 + \alpha_2 . x_2$, we have

$$\left.\frac{\alpha_1 \quad + \alpha_2 = -1\ 2\alpha_1 + \ \ 0 = \ \ 2}{3\alpha_1 + 2\alpha_2 = 0}\right\}$$

$$\left(\frac{1 \qquad\qquad 1\ :-1}{2 \qquad 0\ :\ 2\ 3 \qquad\qquad 2\ :\ 0}\right) \Longrightarrow \left(\frac{1 \quad 1\ :-1\ 0 \quad -2\ :4}{0 \quad -1\ :3}\right)$$

The system has no solution

$\therefore x$ not linear combination of x_1, x_2.

Example (1.24):- Let $S = \{x_1,\ x_2,\ x_3\}$ where $x_1 = (1,2),\ x_2 = (0,1)\ and\ x_3 = (1,1)$. Express (1,0) as a linear combination of x_1, x_2 and x_3.

Solution:

We must find scalars $\alpha_1, \alpha_2, \alpha_3 \in F$ such that $x = \alpha_1 . x_1 + \alpha_2 . x_2 + \alpha_3 . x_3$

$$(1,0) = \alpha_1 . (1,2) + \alpha_2 . (0,1) + \alpha_3 . (1,1)$$

$$(1,0) = (\alpha_1, 2\alpha_1) + (0, \alpha_2) + (\alpha_3, \alpha_3)$$

$$\alpha_1 + \alpha_3 = 1 \Longrightarrow \alpha_1 = 1 - \alpha_3$$

$$2\alpha_1 + \alpha_2 + \alpha_3 = 0 \Longrightarrow 2(1-\alpha_3) + \alpha_2 + \alpha_3 = 0 \Longrightarrow -\alpha_3 + \alpha_2 = -2$$

$$\alpha_2 = -2 - \alpha_3$$

This system has multiple solutions in this case there are multiple possibilities for the α_i.

Definition (1.25):- Let $\emptyset \neq M \subseteq L$ the smallest subspace of L contains M is called **subspace generated** by M and denoted by [M] or span M.

Remark(1.26):-

1. Let $\emptyset \neq M \subseteq L$, the set of all linear combinations of vectors of M is called span of M .

2. $M \subset$span (M).

3. Span (M) = the intersection of all subspace of L containing M.

Example (1.27):- Find span $\{x_1, x_2\}$ where $x_1 = (1,2,3)$ and $x_2 = (1,0,2)$?

Solution :- The span $\{x_1, x_2\}$ is the set of all vectors $(x, y, z) \in \mathbb{R}^3$ such that

$$(x, y, z) = \alpha_1.(1,2,3) + \alpha_2.(1,0,2)$$

We wish to know for what values of (x, y, z) does this system of equations have solutions for α_1, α_2

$\alpha_1.(1,2,3) + \alpha_2.(1,0,2) = (x, y, z)$

$(\alpha_1, 2\alpha_1, 3\alpha_1) + (\alpha_2, 0, 2\alpha_2) = (x, y, z)$

$\alpha_1 + \alpha_2 = x \Longrightarrow \alpha_2 = x - \alpha_1$

$2\alpha_1 = y \Longrightarrow \alpha_1 = \frac{1}{2}y$

$3\alpha_1 + 2\alpha_2 = z \Longrightarrow 6\alpha_1 + 4\alpha_2 - 2z = 0$

$6\left(\frac{1}{2}y\right) + 4\left(x - \frac{1}{2}y\right) - 2z = 0$

$3y + 4x - 2y - 2z = 0$

$4x + y - 2z = 0$

So, solutions when $4x + y - 2z = 0$

Thus span $\{x_1, x_2\}$ is the plane $4x + y - 2z = 0$

Example (1.28):-Show that $\{x_1, x_2\}$ span $\mathbb{R}^2$, when $x_1 = (1,1), x_2 = (2,1)$.

Solution : we being asked to show that any vectors in $\mathbb{R}^2$ can written as a linear combination of x_1, x_2. Let $(a, b) \in \mathbb{R}^2 and\ (a, b) = \alpha_1.(1,1) + \alpha_2.(2,1)$

$(\alpha_1, \alpha_1) + (2\alpha_2, \alpha_2) = (a, b)$

$\alpha_1 + 2\alpha_2 = a \Longrightarrow \alpha_1 = a - 2\alpha_2$

$\alpha_1 + \alpha_2 = b \Longrightarrow \alpha_2 = b - (a - 2\alpha_2)$

$-\alpha_2 = b - a \Longrightarrow \alpha_2 = a - b$

$\alpha_1 = a - 2(a - b) = 2b - a$ .Note that these two vectors span $\mathbb{R}^2$, that is every vector $\mathbb{R}^2$ can be expressed as a linear combination of them .

Example (1.29):-Show that $S = \{x_1, x_2, x_3\}\ span\ \mathbb{R}^2$, where $x_1 = (1,1), x_2 = (2,1), x_3 = (3,2)$. **(H.W.)**

Definition (1.30):- Let S = { $x_1, \dots, x_n$} be a subset of L , then S is called **linearly independent** if there exist $\alpha_1, \alpha_2, \ \dots, \alpha_n$ such that

if $\alpha_1.x_1 + \alpha_2.x_2 + \cdots + \alpha_n.x_n = 0$ then $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.

Definition (1.31):- Let S = { $x_1, x_2, \dots, x_n$} be a subset of L , then S is said to be **linearly dependent** if it is not linearly independent that is if

$\alpha_1.x_1 + \alpha_2.x_2 + \cdots + \alpha_n.x_n = 0$ but the $\alpha_1, \alpha_2, \ \dots, \alpha_n$ not all zero .

Example (1.32):- Determine $S = \{x_1, x_2\}$ is linearly dependent or independent where $x_1 = (1,2,3)$, $x_2 = (1,0,2)$.

Solution : Let $\alpha_1, \alpha_2 \in F$

$\alpha_1(1,2,3) + \alpha_2(1,0,2) = (0,0,0)$, only solution is trivial solution $\alpha_1 = \alpha_2 = 0$. Thus, S is linearly independent.

Example (1.33):-Determine $S = \{x_1, x_2\}$ is linearly dependent or independent where $x_1 = (1,1,1), x_2 = (2,2,2)$?

Solution: Let $\alpha_1, \alpha_2 \in F$

$\alpha_1(1,1,1) + \alpha_2(2,2,2) = (0,0,0)$

$\alpha_1 + 2\alpha_2 = 0 \Longrightarrow \alpha_1 = 2\alpha_2$

So, S is linearly dependent

Theorem (1.34):- **(without prove)**

(1) Every m vectors set in $\mathbb{R}^n$, if $m > n$ then , the set is linearly dependent

(2) A linearly independent set in $\mathbb{R}^n$ has at most n vectors .

Remark (1.35):- Let L linear space over F , $S \subseteq L$ and $x_0 \in L$, then

(1) If $0_L \in S \Longrightarrow S$ is linear dependent . i.e., every subspace is linear dependent set

(2) If $x_0 \neq 0_2 \Longrightarrow \{x_0\}$ is linearly independent

Definition (1.36):- Let L be a linear space over F .A subset B of L is a **basis** if it is linearly independent and spans L i.e,

(1) B is linearly independent

(2) Span $(B) = L$ L تولد الـ B

The number of elements in a basis for L is called the **dimension** of L and is denoted by dim (L)

Example(1.37):- Consider the linear space $(\mathbb{R}^3, +, ..)$

The dimension of L is 3. i.e., $dim\,(\mathbb{R}^3) = 3$

Since $B = \{(1,0,0), (0,1,0), (0,0,1)\}$ is basis for $\mathbb{R}^3$

Remark (1.38):-

(1) Every linear space $L \neq \{0\}$ has a basis

(2) If $L = \{0\} \Longrightarrow dim\,(L) = 0$

(3) If L finite dimension linear space and S be a subspace of $L,$ then $dim\, S \leq dim\, L.$

(4) If $dim\, S = \dim L \Longrightarrow S = L$

(5) If S $= \{\, x_1, \dots, x_n\}$ be a linearly independent in L then there exists $c > 0$ such that $\|x\| = \| \sum_{i=1}^{n} \alpha_i x_1 \| \geq c \, \sum_{i=1}^{n} |\alpha_i|$ **.**

(6) The dimension of quotient space (or factor space) is called codimension of M and denoted by $codm(M) = dim\,(\frac{L}{M}).$

The Convexity

Definition (1.39):- Let A be a subset of linear space L then , we say that A is **convex set** if satisfy the following condition :

$\forall x, y \in A\,, \lambda \in [0,1]$ then $\lambda x + (1-\lambda) y \in A$.

Examples About Convex Sets

Example(1.40):- If $A = (a, b) \subset \Longrightarrow A$ is convex set

Solution :

Let $x, y \in A\,, \lambda \in [0,1]$

$$x \in (a,b) \Rightarrow a < x < b \Rightarrow \lambda a < \lambda x < \lambda b \quad \ldots (1)$$

$y \in (a,b) \Rightarrow a < y < b \Rightarrow (1-\lambda)a < (1-\lambda)y < (1-\lambda)b \quad (2)$

By summing up (1) and (2)

$$\lambda a + (1-\lambda)a < \lambda x + (1-\lambda)y < \lambda b + (1-\lambda)b$$

$a < \lambda x + (1-\lambda)y < b.$

Example(1.41):- Every linear subspace is convex, but the converse is not true in general

Solution :

Let M be a subspace of linear space L $\Longrightarrow \alpha x + \beta y \in M, \forall x, y \in M, \alpha, \beta \in F$. Put $\alpha = \lambda, \beta = 1 - \lambda, 0 \leq \lambda \leq 1$

$\Longrightarrow \lambda x + (1-\lambda)y \in M, \ \forall x, y \in M, 0 \leq \lambda \leq 1 \ \Longrightarrow M$ is convex set

For the converse , consider the following example

Let $L = \mathbb{R}^2, M = \{(x,y) \in \mathbb{R}^2, x \geq 0, y \geq 0\}$, then M is convex set but not subspace.

- To prove M is convex set.

Let $(x_1, y_1), (x_2, y_2) \in M$ and $0 \leq \lambda \leq 1$. To prove $\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2) \in M$.

$(x_1, y_1) \in M \Rightarrow x_1 \geq 0, y_1 \geq 0$ and $(x_2, y_2) \in M \Rightarrow x_2 \geq 0, y_2 \geq 0$.

Thus, $\lambda x_1 + (1-\lambda)y_1 \geq 0$ and $\lambda x_2 + (1-\lambda)y_2 \geq 0$.

Then , $\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) = (\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2) \in M$.

- ❖ Show that M is not a subspace **(H.W.)**

Theorem(1.42): (Some Properties About Convex Sets)

1. The singleton set is convex set
2. The intersection of convex set is convex.
3. The empty set and the whole space are convex. **(H.W.)**
4. If A is convex set $\Longrightarrow$ αA also convex $\alpha \in F$. **(H.W.)**
5. If A and B are convex set $\Longrightarrow A + B$ also convex set.

Proof (1) Let $A = \{x\}$ to prove A is convex set.

Take $x \in A$ and $\lambda \in [0,1]$ then $\lambda x + (1-\lambda)x = x \in M$.

Proof (2) Let A and B are convex sets To prove $A \cap B$ is convex set.

Let $x, y \in A \cap B$ and $0 \leq \lambda \leq 1$ to prove $\lambda x + (1-\lambda)y \in A \cap B$.

$x, y \in A$ and A is convex $\Rightarrow$ $x + (1-\lambda)y \in A$ …. (1)

$x, y \in B$ and B is convex $\Rightarrow$ $x + (1-\lambda)y \in B$ …. (2)

From (1)&(2) we get $x + (1-\lambda)y \in A \cap B$. Then, $A \cap B$ is convex set.

Proof (5) Let $a_1 + b_1, a_2 + b_2 \in A + B$, then $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

To prove $\lambda\,(a_1 + b_1) + (1-\lambda)(a_2 + b_2) \in A + B$, $\forall \lambda \in [0,1]$

Since A is convex set and $a_1, a_2 \in A \Rightarrow \lambda\, a_1 + (1-\lambda)a_2 \in A \;\; \forall \lambda \in [0,1]$…(1)

Since B is convex set and $b_1, b_2 \in B \Rightarrow \lambda\, b_1 + (1-\lambda)b_2 \in B \;\; \forall \lambda \in [0,1]$…(2)

By summing up (1) and (2) we get

$\lambda\, a_1 + (1-\lambda)a_2 + \lambda\, b_1 + (1-\lambda)b_2 \in A + B$

i.e., $\lambda\,(a_1 + b_1) + (1-\lambda)(a_2 + b_2) \in A + B$. Thus, $A + B$ is a convex set.

Linear operator and linear functional

Definition(1.43):- Let L and L' are linear spaces over the same field F . A mapping $T: L \to L'$ is called **Linear operator** or (**Linear transformation**) if

$$T(\alpha.x + \beta.y) = \alpha T(x) + \beta T(y), \forall x, y \in L \text{ and } \alpha, \beta \in F$$

Note : The linear operator $T: L \to F$ is said to be **linear functional**.

Examples of Linear Functional

Example (1.44):- Let $T: \mathbb{R} \to \mathbb{R}$ such that $T(x) = mx, m \in \mathbb{R}$ the T is linear functional

Solution : Let $x, y \in \mathbb{R}, \alpha, \beta \in \mathbb{R}$

$$T(\alpha x+\beta y)=m(\alpha x+\beta y)=m(\alpha x)+m(\beta y)$$
$$=\alpha(mx)+\beta(my)$$
$$=\ \alpha T(x)+\beta T(y)$$

Example(1.45):- Let $T:\mathbb{R}\to\mathbb{R}$ s.t. $T(x)=mx+b$, $m,b\neq 0\in\mathbb{R}$ then T is not linear functional ?

Solution: Let x , $y\in\mathbb{R}$

$$T(\alpha x+\beta y)=m(\alpha x+\beta y)+b=(m\alpha)x+(m\beta)y+b$$
$$=\alpha(mx)+\beta(my)+b\ \ldots..\qquad (1)$$

Now, $\alpha T(x)+\beta T(y)=\ \alpha(mx+b)+\beta(my+b)$

$$=\alpha mx+\beta my+(\alpha+\beta)b\ \ldots\ldots\quad (2)$$

Let $\alpha=1,\beta=2$, $we\ get\ 1\neq 2$

Example(1.46):- Let $T:C[a,b]\to\mathbb{R}\ such\ that\ T\ (f)=\int_a^b f(x)dx$

Show that T is linear functional.

Solution : Let $f,g\in C[a,b]$, $\alpha,\beta\in\mathbb{R}$. To prove that

$$T(\alpha f+\beta g)=\alpha T(f)+\beta T(g)$$

$$T(\alpha f+\beta g)=\int_a^b(\alpha f+\beta g)(x)dx$$
$$=\ \int_a^b\alpha f(x)\,dx+\ \int_a^b\beta g(x)dx$$
$$=\ \alpha\int_a^b f(x)dx+\ \beta\int_a^b g(x)dx=\ \alpha T(f)+\beta T(g).$$

Example(1.47):- Show that $T:\mathbb{R}^2\to\mathbb{R}^2$ such that

$T(x_1, x_2) = (x_1 + x_2, x_1 - x_2 + 1)$ is not linear operator

Solution : $T(0) = T((0,0)) = (0,1) \neq (0,0) \Longrightarrow T$ not linear functional.

Exercise

(1) If $T: \mathbb{R}^2 \to \mathbb{R}$ s.t $T(x_1, x_2) = x^2{}_1 + x^2{}_2$. Show that T is not linear functional.

(2) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t $T(x_1, x_2) = (x_1 + x_2, 0)$. Is T linear transformation ?

(3) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t $T(x_1, x_2) = (\alpha x_1, x_2), \alpha \in \mathbb{R}$. Is T linear transformation ?

(4) If $T: \mathbb{R}^3 \to \mathbb{R}^3$ s.t $T(x_1, x_2, x_3) = (x_2, x_1 - x_2, 0)$. Is T linear transformation ?

Some Theorems About Linear operators

Theorem(1.48):- Let L, L', L'' are linear spaces over the same field F such that $T: L \to L'$ and g: $L' \to L''$ linear operators then

(1) $T(0_L) = 0_{L'}$ and $g(0_{L'}) = 0_{L''}$

(2) $T(x - y) = T(x) - T(y)$

(3) goT is linear operator .

Theorem(1.49):- Let L, L' are linear spaces over the same field F and $T_1, T_2 : L \to L'$ are two linear operators. Define $+$ and $.$ as follows:

$$(T_1 + T_2)(x) = T_1(x) + T_2(x), \forall x \in L$$

$$(\lambda T_1)(x) = \lambda . T_1(x), \quad \forall x \in L, \ \lambda \in F$$

Then, $T_1 + T_2$ and αT_1 are linear operators

Proof :- Let $x, y \in L \quad and \ \alpha, \beta \in F$ then

(1) $(T_1 + T_2)(\alpha . x + \beta . y) = T_1(\alpha . x + \beta . y) + T_2(\alpha . x + \beta . y)$

$= \alpha . T_1(x) + \beta . T_1(y) + \alpha . T_2(x) + \beta . T_2(y)$

$= \alpha . (T_1(x) + T_2(x)) + \beta . (T_1(y) + T_2(y))$

$= \alpha . (T_1 + T_2)(x) + \beta . (T_2 + T_2)(y)$

(2) $\lambda T_1(\alpha . x + \beta . y) = \lambda . [\alpha . T_1(x) + \beta . T_1(y)]$

$= (\lambda \alpha) . T_1(x) + (\lambda \beta) . T_1(y)$

$= \alpha . \left(\lambda . T_1(x)\right) + \beta \left(\lambda . T_1(y)\right)$

$= \alpha . (\lambda T_1)(x) + \beta . (\lambda T_1)(y).$

Chapter 2

Normed Linear Space

Definition 2.1.
Let $L(F)$ be a linear space over a field F. A mapping $\|.\| : L \to \mathbb{R}$ is called **norm** if the following conditions hold

(1) $\|x\| \geq 0 \quad \forall x \in L$. (Positivity)

(2) $\|x\| = 0$ if and only if $x = \mathbf{0_L}$.

(3) $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in L$. (Triangle Inequality)

(4) $\|\alpha x\| = |\alpha| \, \|x\| \quad \forall x \in L, \forall \alpha \in F$.

$(L, \| \, \|)$ is called **normed linear space**.

Remark 2.2.

From now on, the field F is either $\mathbb{R}$ or C.

Theorem 2.3.

Let $(L, \| \, \|)$ be a normed linear space. Then, for each $x, y \in L$

(1) $\|\mathbf{0_L}\| = 0$.

(2) $\|x\| = \|-x\|$.

(3) $\|x - y\| = \|y - x\|$.

(4) $|\ \|x\| - \|y\|\ | \leq \|x - y\|$. (Reverse Triangle Inequality)

(5) $|\ \|x\| - \|y\|\ | \leq \|x + y\|$. (Reverse Triangle Inequality)

(6) Every subspace of a normed space is itself normed space with respect to the same norm.

Proof. (1) $\|\mathbf{0_L}\| = \|0\mathbf{0_L}\|$ (see Theorem (1.3)(1))

$= 0\,\|\mathbf{0_L}\| = 0$.

(2) $\|-x\| = |-1|\,\|x\| = \|x\| \quad \forall x \in L$.

(3) $\|x - y\| = \|-(y - x)\| = \|y - x\|$ (by part (2)).

(4) We must prove $-\|x - y\| \leq \|x\| - \|y\| \leq \|x - y\|$

$\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$ (by Definition 2.1(3)).

Hence, $\|x\| - \|y\| \leq \|x - y\|$ (I)

Similarly, $\|y\| = \|y - x + x\| \leq \|y - x\| + \|x\|$ (by Definition 2.1(3)).

Hence, $\|y\| - \|x\| \leq \|x - y\|$ (II)

Hence, by (I) and (II), we get $\|x - y\| \geq |\|x\| - \|y\|| \quad \forall x, y \in L$.

(5) We must prove $-\|x + y\| \leq \|x\| - \|y\| \leq \|x + y\|$

$\|x\| = \|x + y - y\| \leq \|x + y\| + \|-y\|$ (by Definition 2.1(3)).

Hence, $\|x\| - \|y\| \leq \|x + y\|$ (III)

Similarly, $\|y\| = \|y + x - x\| \leq \|y + x\| + \|-x\|$ (by Definition 2.1(3)).

Hence, $\|y\| - \|x\| \leq \|x + y\|$

$\|x\| - \|y\| \geq -\|x + y\|$ (IV)

Hence, by (III) and (IV), we get $-\|x + y\| \leq \|x\| - \|y\| \leq \|x + y\|$ $\forall x, y \in L$.

□

2.1 Examples of Normed Linear Space

Example 2.4.

Let $L = \mathbb{R}$ be a linear space over $\mathbb{R}$ with $\| \ \| : L \to \mathbb{R}$ such that $\|x\| = |x|$. Show that $(\mathbb{R}, \| \ \|)$ is a normed space.

Solution: We show that

(1) $\|x\| = |x| \geq 0 \quad \forall x \in \mathbb{R}$; hence $\|x\| \geq 0$.

(2) Let $x \in \mathbb{R}$, $\|x\| = 0 \iff |x| = 0 \iff x = 0$.

(3) $\forall x \in \mathbb{R}, \forall \alpha \in \mathbb{R}$,

$\|\alpha x\| = |\alpha x| = |\alpha| \, |x| = |\alpha| \, \|x\|$.

(4) $\|x + y\| = |x + y| \leq |x| + |y| = \|x\| + \|y\| \quad \forall x, y \in \mathbb{R}$.

Example 2.5.

Let $L = C$ be a complex linear space over C with $\| \ \| : C \to \mathbb{R}$ such that $\|z\| = |z| = \sqrt{a^2 + b^2} \quad \forall z = a + ib$. Show that $(C, \| \ \|)$ is a normed space.

Solution: We show that

(1) $\|z\| = |z| = \sqrt{a^2 + b^2} \geq 0 \quad \forall z = a + ib \in C$; hence $\|z\| \geq 0$.

(2) Let $z = a + ib \in C$

$\|z\| = |z| = \sqrt{a^2 + b^2} = 0 \iff a^2 + b^2 = 0 \iff a^2 = b^2 = 0 \iff a = b = 0 \iff z = 0 + 0i = 0$.

(3) Let $z, w \in C$

$$\begin{aligned} \|z + w\|^2 = |z + w|^2 &= (z + w)(\overline{z + w}) \qquad (|z|^2 = z\overline{z}) \\ &= (z + w)(\overline{z} + \overline{w}) \\ &= z\overline{z} + w\overline{w} + w\overline{z} + \overline{w}z \\ &= z\overline{z} + w\overline{w} + \underbrace{w\overline{z} + \overline{w\overline{z}}} \qquad (\overline{w}z = \overline{\overline{\overline{w}z}} = \overline{w\overline{z}}\ ) \end{aligned}$$

$$= z\overline{z} + w\overline{w} + 2Re\ w\overline{z} \qquad (z + \overline{z} = 2Rez)$$

$$\leq z\overline{z} + w\overline{w} + 2\,|w|\,|z| \qquad (Re\ w\overline{z} \leq |w|\,|z|)$$

$$= |z|^2 + |w|^2 + 2\,|w|\,|z| = \|z\|^2 + \|w\|^2 + 2\,\|w\|\,\|z\|$$

$$= (\|z\| + \|w\|)^2.$$

Thus, $\|z+w\|^2 \leq (\|z\| + \|w\|)^2$, hence, $\|z+w\| \leq \|z\| + \|w\|$.

(4) Let $z \in C, \alpha \in C$, $\|\alpha z\| = |\alpha z| = |\alpha(a+ib)|$

$$= \sqrt{(\alpha a)^2 + (\alpha b)^2} = \sqrt{\alpha^2(a^2+b^2)}$$

$$= \sqrt{\alpha^2}\sqrt{a^2+b^2} = |\alpha|\,|z| = |\alpha|\,\|z\|.$$

As an application to Example 2.5: Let $z = 2+3i, w = 1-i$, then

$\|z+w\| = \|(2+1) + (3i - i)\| = \|3+2i\| = \sqrt{3^2+2^2} = \sqrt{13}.$

$\|5z\| = \|10+15i\| = \sqrt{10^2+15^2} = \sqrt{325} = 5\sqrt{13}.$

$5\,\|z\| = 5\sqrt{2^2+3^2} = 5\sqrt{13}.$

Example 2.6.

Show that the linear space $C^b(\mathbb{R})$ is a normed space under the norm

$\|f\| = \sup\{|f(x)| : x \in \mathbb{R}\}, \quad \forall f \in C^b(\mathbb{R}).$

(1) Since $|f(x)| \geq 0 \;\; \forall x \in \mathbb{R}$. Then, $\|f\| = \sup |f(x)| \geq 0$. Hence, $\|f\| \geq 0$.

(2) $\|f\| = 0 \iff \sup\{|f(x)| : x \in \mathbb{R}\} = 0$

$$\iff |f(x)| = 0 \;\; \forall x \in \mathbb{R}$$

$$\iff f(x) = 0 \;\; \forall x \in \mathbb{R} \iff f = \hat{0} \text{ (zero mapping)}$$

(3) Let $f, g \in C^b(\mathbb{R})$. Then

$\|f+g\| = \sup\{|f(x)+g(x)| : x \in \mathbb{R}\}$

$$\leq \sup\{|f(x)| + |g(x)| : x \in \mathbb{R}\}$$

$$\leq \sup\{|f(x)| : x \in \mathbb{R}\} + \sup\{|g(x)| : x \in \mathbb{R}\} = \|f\| + \|g\|.$$

Hence, $\|f + g\| \leq \|f\| + \|g\|$.

(4) Let $f \in C^b(\mathbb{R}), \alpha \in \mathbb{R}$. Then

$\|\alpha f\| = \sup\{|(\alpha f)(x)| : x \in \mathbb{R}\}$

$= \sup\{|\alpha|\,|f(x)| : x \in \mathbb{R}\}$

$= |\alpha| \sup\{|f(x)| : x \in \mathbb{R}\}$ (By Theorem 2.7 below where $A = |f(x)|$ and $\beta = |\alpha|$)

$= |\alpha|\,\|f\|$.

Theorem 2.7.

If A is a bounded above set and $\beta > 0$, then βA is bounded above and $\sup(\beta A) = \beta \sup(A)$.

As an application to Example 2.6: Let $f, g \in C^b(\mathbb{R})$ such that $f(x) = sin(x)$ and $g(x) = 2cos(x) + 1$. Hence,

$\|f\| = \sup\{|sin(x)| : x \in \mathbb{R}\} = 1$ (since $|sin(x)| \leq 1, \ \forall x \in \mathbb{R}$).

$\|g\| = \sup\{|2cos(x) + 1| : x \in \mathbb{R}\}$.

But $|2cos(x) + 1| \leq 2\,|cos(x)| + 1$

$\leq 2(1) + 1 = 3.$ (since $|cos(x)| \leq 1, \ \forall x \in \mathbb{R}$).

So, $\|g\| = 3$.

Example 2.8.

The linear space $C^b[a, b]$ of all real valued continuous functions on $[a, b]$ is a normed space under the norm defined in Example 2.6. (**H.W.**)

Example 2.9.

The linear space $C[0,1]$ of all real valued continuous functions on $[0,1]$ is a normed space with the norm defined as

$\|f\| = \int_0^1 |f(x)| \ dx \ \ \forall f \in C[0,1]$.

solution: (1) Since $|f(x)| \geq 0, \ \ \forall x \in [0,1]$, then $\int_0^1 |f(x)| \ dx \geq 0$. Thus, $\|f\| \geq 0$.

(2) $\|f\| = 0 \iff \int_0^1 |f(x)| \ dx = 0$

$$\iff |f(x)| = 0 \ \ \forall x \in [0,1]$$

$$\iff f(x) = 0 \ \ \forall x \in [0,1]$$

$$\iff f = \hat{0} \text{ (zero mapping)}.$$

(3) Let $f, g \in C[0,1]$. Then

$\|f + g\| = \int_0^1 |f(x) + g(x)| \, dx$

$\leq \int_0^1 (|f(x)| + |g(x)|) \ dx$

$= \int_0^1 |f(x)| \ dx + \int_0^1 |g(x)| \ dx = \|f\| + \|g\|$

(4) Let $f \in C[0,1], \alpha \in \mathbb{R}$. Then

$\|\alpha f\| = \int_0^1 |\alpha f(x)| \ dx = \int_0^1 |\alpha| \, |f(x)| \ dx = |\alpha| \int_0^1 |f(x)| \ dx = |\alpha| \, \|f\|$.

As an application to Example 2.9: Let $f \in C[0,1]$ such that $f(x) = x^3$ and $g(x) = -x^2$. Find $\|f\|$, $\|g\|$ and $\|f + g\|$.

$\|f\| = \int_0^1 |f(x)| \ dx = \int_0^1 \left|x^3\right| \ dx = \int_0^1 x^3 \ dx = \frac{1}{4}$.

$\|g\| = \int_0^1 |g(x)| \ dx = \int_0^1 \left|-x^2\right| \ dx = \int_0^1 x^2 \ dx = \frac{1}{3}$.

$$\|f + g\| = \int_0^1 |(f + g)(x)| \ dx = \int_0^1 \left| \underbrace{x^3 - x^2}_{\leq 0} \right| \ dx$$

$$= \int_0^1 (x^2 - x^3) \ dx = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

Example 2.10.

Consider the linear space F^n over F ($F = \mathbb{R}$ or C). Define $\| \| : F^n \to \mathbb{R}$ by $\|x\| = \max\{|x_1|, ..., |x_n|\}$ $\forall x = (x_1, ..., x_n) \in F^n$. Then $(F^n, \| \|)$ is a normed space.

solution: (1) For any $x = (x_1, ..., x_n) \in F^n$, $|x_i| \geq 0$, $\forall i = 1, ..., n$.

Then $\max\{|x_1|, ..., |x_n|\} \geq 0$, then $\|x\| \geq 0$.

(2) $\|x\| = 0$, where $x = (x_1, ..., x_n) \in F^n$

$$\iff \max\{|x_1|, ..., |x_n|\} = 0$$

$$\iff |x_1| = ... = |x_n| = 0 \iff x_1 = ... = x_n = 0$$

$$\iff x = (x_1, ..., x_n) = (0, ..., 0) = \mathbf{0}_{F^n}$$

(3) Let $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in F^n$

$x + y = (x_+ y_1, x_2 + y_2, ..., x_n + y_n)$

$$\|x + y\| = \max\{|x_1 + y_1|, ..., |x_n + y_n|\}$$

$$\leq \max\{|x_1| + |y_1|, ..., |x_n| + |y_n|\}$$

$$\leq \max\{|x_1|, ..., |x_n|\} + \max\{|y_1|, ..., |y_n|\} = \|x\| + \|y\|$$

(4) Let $x = (x_1, ..., x_n) \in F^n$ and $\alpha \in F$

$$\|\alpha x\| = \max\{|\alpha x_1|, ..., |\alpha x_n|\}$$

$$= \max\{|\alpha| |x_1|, ..., |\alpha| |x_n|\} = |\alpha| \max\{|x_1|, ..., |x_n|\} = |\alpha| \|x\|$$

<u>**As an application to Example 2.10:**</u> Consider the linear space $\mathbb{R}^3$ over $\mathbb{R}$. Let $x = (x_1, x_2, x_3) = (1, 2, -5), y = (y_1, y_2, y_3) = (0, -7, 3)$. Then ,

(1) $\|x\| = \max\{|1|, |2|, |-5|\} = 5$ and

$\|y\| = \max\{|0|, |-7|, |3|\} = 7$.

$\|x+2y\| = \max\{|1|, |-12|, |11|\} = 12$

(2) Find $\|2x-y\|$, $\|2x+3y\|$, $\|3x\|$

(3) Show that

$$\max\{|x_1|+|y_1|, |x_2|+|y_2|, |x_3|+|y_3|\} \leq \max\{|x_1|, |x_2|, |x_3|\}+\max\{|y_1|, |y_2|, |y_3|\}.$$

Exercise 2.11.

(1) Let $L = C^2$ be a linear space over $F = C$. Define $\| \| : C^2 \to \mathbb{R}$ such that $\|x\| = a|x_1| + b|x_2|$, $\forall x = (x_1, x_2) \in C^2$ and $a, b > 0$. Show that $\| \|$ is a norm on C^2. (**H.W.**)

(2) Consider the linear space $\mathbb{R}^2$. Let $\|x\| = \min\{|x_1|, |x_2|\}$, $\forall x = (x_1, x_2) \in \mathbb{R}^2$. Show that $\| \|$ is not a norm on $\mathbb{R}^2$.

solution: Let $x = (0, -3) \in \mathbb{R}^2$

$\|x\| = \min\{|0|, |-3|\} = \min\{0, 3\} = 0$

Since $X \neq \mathbf{0}_{\mathbb{R}^2}$, but $\|x\| = 0$. Condition (2) of the definition of the norm is not valid. Hence, $\| \|$ is not a norm on $\mathbb{R}^2$.

(3) Consider the linear space $\mathbb{R}^2$. Let $\|x\| = |x_1|^2 + |x_2|^2$, $\forall x = (x_1, x_2) \in \mathbb{R}^2$. Show that $\| \|$ does not satisfies condition (4).

solution: Let $x = (1, 3), \alpha = 2$

$|\alpha| \|x\| = 2(|x_1|^2 + |x_2|^2) = 2(1^2 + 3^2) = 20$

$\|\alpha x\| = \|2(1, 3)\| = \|(2, 6)\| = 2^2 + 6^2 = 40$

Thus, $|\alpha| \|x\| = 20 \neq \|\alpha x\| = 40$.

Some Important Inequalities

To give more examples about normed space, it is important to present some inequalities.

If $l^p = \{x = (x_1, x_2, ...) : x_i \in \mathbb{R}(or\ C)$ and $\sum_{i=1}^{\infty} |x_i|^p < \infty\}$ be a set of sequence space (see Example 1.6). Let $x = (x_1, x_2, ...) \in l^p$, $y = (y_1, y_2, ...) \in l^q$. Then

(1) Holder's Inequality

$$\sum_{i=1}^{\infty}|x_iy_i| \leq \big[\sum_{i=1}^{\infty}|x_i|^p\big]^{\frac{1}{p}}\big[\sum_{i=1}^{\infty}|y_i|^q\big]^{\frac{1}{q}},$$

where $p>1, q>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

(2) Cauchy Schwarz's Inequality

$$\sum_{i=1}^{\infty}|x_iy_i| \leq \big[\sum_{i=1}^{\infty}|x_i|^2\big]^{\frac{1}{2}}\big[\sum_{i=1}^{\infty}|y_i|^2\big]^{\frac{1}{2}},$$

Note that Cauchy Schwarz's inequality is a special case of Holder's inequality where $p=q=2$.

(3) Minkowski's Inequality

If $p \geq 1$

$$\big[\sum_{i=1}^{\infty}|x_i+y_i|^p\big]^{\frac{1}{p}} \leq \big[\sum_{i=1}^{\infty}|x_i|^p\big]^{\frac{1}{p}} + \big[\sum_{i=1}^{\infty}|y_i|^p\big]^{\frac{1}{p}},$$

Remark 2.12.

The three inequalities above hold for the linear spaces $L=\mathbb{R}^n$ and $L=C^n$.

Example 2.13.

Let $L=\mathbb{R}^2$ be a linear space over $\mathbb{R}$. If $x=(-1,2), y=(0,5)\in\mathbb{R}^2$.

(1) Verify Cauchy Shwarz inequality ($p=q=2$).

(2) Verify Minkowski's inequality ($p=3$).

Now we can give the following examples

Example 2.14.

(1) Show that the linear space $\mathbb{R}^n$ over $\mathbb{R}$ (or C^n over C) is a normed space with $\|x\|_2 = \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} \ \forall x \in \mathbb{R}^n$ or $C^n, x = (x_1, ..., x_n)$.

(2) Show that the linear space $\mathbb{R}^n$ over $\mathbb{R}$ (or C^n over C) is a normed space with $\|x\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{\frac{1}{p}} \ \forall x \in \mathbb{R}^n$ or $C^n, x = (x_1, ..., x_n)$ and $1 \leq p < +\infty$. (**H.W.**)

(3) Show that $(l^p, \| \ \|_p)$ is a normed space where $\|x\|_p = \left[\sum_{i=1}^{+\infty} |x_i|^p\right]^{\frac{1}{p}} \quad \forall x = (x_1, x_2, ...) \in l^p$ and $1 \leq p < +\infty$.

Solution (1): Let $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{R}^n$ (or C^n) and $\alpha \in \mathbb{R}$ (or C).

(1) Since $|x_i| \geq 0, \quad \forall i = 1, ..., n$. Then, $\left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} \geq 0$; that is $\|x\|_2 \geq 0$.

(2) $\|x\|_2 = 0 \iff \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} = 0 \iff \sum_{i=1}^{n} |x_i|^2 = 0$

$$\iff |x_i|^2 = 0, \ \forall i = 1, ..., n$$

$$\iff x_i = 0, \ \forall i = 1, ..., n$$

$$\iff x = (x_1, ..., x_n) = \mathbf{0}_{\mathbb{R}^n}$$

(3) $\|x + y\|_2 = \|(x_1 + y_1, ..., x_n + y_n)\|_2$

$= \left[\sum_{i=1}^{n} |x_i + y_i|^2\right]^{\frac{1}{2}} \leq \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} + \left[\sum_{i=1}^{n} |y_i|^2\right]^{\frac{1}{2}}$ (Minkowski's Inequality)

$$= \|x\|_2 + \|y\|_2$$

(4) $\|\alpha x\|_2 = \|(\alpha x_1, ..., \alpha x_n)\|_2 = \left[\sum_{i=1}^{n} |\alpha x_i|^2\right]^{\frac{1}{2}}$

$$= \left[\sum_{i=1}^{n} |\alpha|^2 |x_i|^2 \right]^{\frac{1}{2}}$$

$$= |\alpha| \left[\sum_{i=1}^{n} |x_i|^2 \right]^{\frac{1}{2}} = |\alpha| \, \|x\|_2.$$

Solution (3): Let $x = (x_1, x_2, ...), y = (y_1, y_2, ...) \in l^p$ and $\alpha \in \mathbb{R}$ (or C).

(1) Since $|x_i| \geq 0, \ \forall i \in N$. Then, $\left[\sum_{i=1}^{\infty} |x_i|^p \right]^{\frac{1}{p}} \geq 0$; that is $\|x\|_p \geq 0$.

(2) $\|x\|_p = 0 \iff \left[\sum_{i=1}^{\infty} |x_i|^p \right]^{\frac{1}{p}} = 0 \iff \sum_{i=1}^{\infty} |x_i|^p = 0$

$$\iff |x_i|^2 = 0, \ \forall i \in N$$

$$\iff x_i = 0, \ \forall i \in N$$

$$\iff x = (0, 0, ...)$$

(3) $\|x + y\|_p = \|(x_1 + y_1, ..., x_n + y_n)\|_p$

$= \left[\sum_{i=1}^{\infty} |x_i + y_i|^p \right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^{\infty} |x_i|^p \right]^{\frac{1}{p}} + \left[\sum_{i=1}^{\infty} |y_i|^p \right]^{\frac{1}{p}}$ (Minkowski's Inequality)

$$= \|x\|_p + \|y\|_p$$

(4) $\|\alpha x\|_p = \|(\alpha x_1, ...)\|_p = \left[\sum_{i=1}^{\infty} |\alpha x_i|^p \right]^{\frac{1}{p}}$

$$= \left[\sum_{i=1}^{\infty} |\alpha|^p |x_i|^p \right]^{\frac{1}{p}}$$

$$= |\alpha| \left[\sum_{i=1}^{\infty} |x_i|^p \right]^{\frac{1}{p}} = |\alpha| \, \|x\|_p$$

As an application to Example 2.14(1):

(1) Let $(\mathbb{R}^3, \| \ \|_2)$ be a normed space and $x = (x_1, x_2, x_3) = (1, -2, 4)$. Then, find $\|x\|_2$.

(2) Let $(C^2, \| \ \|_2)$ be a normed space and $x = (x_1, x_2) = (1 + i, -2i)$. Then, find $\|x\|_2$.

2.2 Product of Normed Spaces

Definition 2.15.
Let $(L, \| \|_L), (L', \| \|_L{}')$ be normed linear spaces over a field F. Let

$L \times L' = \{(x, y) : x \in L, y \in L'\}$ be the Cartesian product of L and L'.

Define $+$ on $L \times L'$ by

$$(x_1, y_1) + (x_2, y_2) = (\underbrace{x_1 + x_2}_{\text{sum on L}}, \underbrace{y_1 + y_2}_{\text{sum on } L'}), \quad \forall (x_1, y_1) + (x_2, y_2) \in L \times L'.$$

Define a scalar multiplication

$\alpha.(x, y) = (\alpha x, \alpha y), \quad \forall (x, y) \in L \times L', \forall \alpha \in F.$

Proposition 2.16.

Show that $(L \times L', +, .)$ is a linear space over F. (**H. W.**)

Remark 2.17.

The product linear space defined above can be made a normed space by different ways as we show in the following example.

Example 2.18.

Define $\| \| : L \times L' \to \mathbb{R}$ such that

(1) $\|(x, y)\|_1 = \|x\|_L + \|y\|_{L'}$

(2) $\|(x, y)\|_2 = \max\{\|x\|_L , \|y\|_{L'}\}$

(3) $\|(x, y)\|_3 = \min\{\|x\|_L , \|y\|_{L'}\}$ (**H. W.**)

Show that $(L \times L', \| \|_1), (L \times L', \| \|_2)$ are normed spaces.

Is $(L \times L', \| \|_3)$ is normed space?

Solution (1): To show $(L \times L', \| \|_1)$ is a normed space,

(i) Since $\|x\|_L \geq 0$ and $\|y\|_{L'} \geq 0 \ \ \forall x \in L, \forall y \in L'$, then

$$\|x\|_L + \|y\|_{L'} = \|(x,y)\|_1 \geq 0.$$

(ii) $\|(x,y)\|_1 = 0 \iff \|x\|_L + \|y\|_{L'} = 0$

$$\iff \|x\|_L = \|y\|_{L'} = 0$$

$$\iff x = y = 0 \quad ((L, \| \ \|_L), (L', \| \ \|_{L'}) \text{ are normed spaces})$$

$$\iff (x,y) = (0,0)$$

(iii) For each $(x_1,y_1),(x_2,y_2) \in L \times L'$

$$\begin{aligned}
\|(x_1,y_1) + (x_2,y_2)\|_1 &= \|(x_1+x_2, y_1+y_2)\|_1 \\
&= \|x_1+x_2\|_L + \|y_1+y_2\|_{L'} \\
&\leq \|x_1\|_L + \|x_2\|_L + \|y_1\|_{L'} + \|y_2\|_{L'} \\
&= (\|x_1\|_L + \|y_1\|_{L'}) + (\|x_2\|_L + \|y_2\|_{L'}) \\
&= \|(x_1,y_1)\|_1 + \|(x_2,y_2)\|_1
\end{aligned}$$

(iv) For each $(x,y) \in L \times L'$ and for each $\alpha \in F$

$$\|\alpha(x,y)\|_1 = \|(\alpha x, \alpha y)\|_1 = \|\alpha x\|_L + \|\alpha y\|_{L'}$$

$$= |\alpha| \, \|x\|_L + |\alpha| \, \|y\|_{L'} = |\alpha| \, (\|x\|_L + \|y\|_{L'}) = |\alpha| \, \|(x,y)\|_1$$

Solution (2): Now, we show that $\|(x,y)\|_2 = \max\{\|x\|_L, \|y\|_{L'}\}$ is a norm on $L \times L'$

(i) Since $\|x\|_L \geq 0$ and $\|y\|_{L'} \geq 0 \ \ \forall x \in L, \forall y \in L'$, then

$\max\{\|x\|_L, \|y\|_{L'}\} = \|(x,y)\|_2 \geq 0.$

(ii) $\|(x,y)\|_2 = 0 \iff \max\{\|x\|_L, \|y\|_{L'}\} = 0$

$$\iff \|x\|_L = \|y\|_{L'} = 0$$

$$\iff x = y = 0 \quad ((L, \|\ \|_L), (L', \|\ \|_{L'}) \text{ are normed spaces})$$

$$\iff (x, y) = (0, 0)$$

(iii) For each $(x_1, y_1), (x_2, y_2) \in L \times L'$

$$\|(x_1, y_1) + (x_2, y_2)\|_2 = \|(x_1 + x_2, y_1 + y_2)\|_2$$

$$= \max\{\|x_1 + x_2\|_L, \|y_1 + y_2\|_{L'}\}$$

$$\leq \max\{\|x_1\|_L + \|x_2\|_L, \|y_1\|_{L'} + \|y_2\|_{L'}\}$$

$$\leq \max\{\|x_1\|_L, \|y_1\|_{L'}\} + \max\{\|x_2\|_L, \|y_2\|_{L'}\}$$

$$= \|(x_1, y_1)\|_2 + \|(x_2, y_2)\|_2$$

(iv) For each $(x, y) \in L \times L'$ and for each $\alpha \in F$

$$\|\alpha(x, y)\|_2 = \|(\alpha x, \alpha y)\|_2 = \max\{\|\alpha x\|_L, \|\alpha y\|_{L'}\}$$

$$= \max\{|\alpha|\,\|x\|_L, |\alpha|\,\|y\|_{L'}\}$$

$$= |\alpha| \max\{\|x\|_L, \|y\|_{L'}\} = |\alpha|\,\|(x, y)\|_2$$

As an application to Example 2.18: Let $L = (\mathbb{R}, |\ |)$ and $L' = (\mathbb{R}^2, \|\ \|_2)$ where $\|x\|_2 = \left[\sum_{i=1}^2 |x_i|^2\right]^{\frac{1}{2}}$. If $x = 3 \in L = \mathbb{R}$ and $y = (1, -2) \in L' = \mathbb{R}^2$. Find $\|(x, y)\|_1$ and $\|(x, y)\|_2$

Solution: $\|(x, y)\|_1 = \|(3, (1, -2))\|_1 = \|3\|_{\mathbb{R}} + \|(1, -2)\|_{\mathbb{R}^2}$

$$= |3| + \left[\sum_{i=1}^2 |y_i|^2\right]^{\frac{1}{2}}$$

$$= 3 + \left[|1|^2 + |-2|^2\right]^{\frac{1}{2}} = 3 + \sqrt{5}.$$

Find $\|(x, y)\|_2$ (**H.W.**)

2.3 Normed space and Metric space

Definition 2.19.
Let X be a non empty set and $d : X \times X \to \mathbb{R}$ be a mapping. Then d is called metric if

(1) $d(x, y) \geq 0 \quad \forall x, y \in X$

(2) $d(x, y) = 0 \iff x = y \quad \forall x, y \in X$

(3) $d(x, y) = d(y, x) \quad \forall x, y \in X$

(4) $d(x, y) \leq d(x, z) + d(z, y) \quad \forall x, y, z \in X.$

(X, d) is called **metric space**

Theorem 2.20.
Let $(L, \| \|)$ be a normed linear space. Let $d : L \times L \to \mathbb{R}$ defined by $d(x, y) = \|x - y\| \quad \forall x, y \in L$. Prove that (L, d) is a metric space. (i.e., every normed space is a metric space). The metric d is called metric **induced** by the norm.

Proof. To prove (L, d) is a metric space.

(i) By definition of norm, $\|x - y\| \geq 0 \quad \forall x, y \in L$. Hence, $d(x, y) = \|x - y\| \geq 0$

(ii) $d(x, y) = \|x - y\| = \|y - x\| = d(y, x)$

(iii) $d(x, y) = 0 \iff \|x - y\| = 0 \iff x - y = 0 \iff x = y$

(iv) $d(x, y) = \|x - y\| = \|x - z + z - y\| \leq \|x - z\| + \|z - y\| = d(x, z) + d(y, z)$ □

Lemma 2.21.

Let d be a metric induced by a normed space $(L, \| \|)$ (i.e., $d(x, y) = \|x - y\|$). Then d satisfies the following:

(i) $d(x + a, y + a) = d(x, y) \quad \forall x, y, a \in L.$

(ii) $d(\alpha x, \alpha y) = |\alpha|\, d(x, y) \quad \forall x, y \in L,\ \forall a \in F.$

Proof. (1) $d(x + a, y + a) = \|x + a - (y + a)\| = d(x, y) \quad \forall x, y, a \in L$

(2) $d(\alpha x, \alpha y) = \|\alpha x - \alpha y\| = \|\alpha(x - y)\| = |\alpha|\, \|x - y\| = |\alpha|\, d(x, y).$ □

Remark 2.22.

Not every metric space is a normed space as we show in the next example

Example 2.23.

Let d be the discrete metric on a space X. Then d can't be obtained from a norm on L (i.e., $(L, \| \|)$, where

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & x \neq y\ . \end{cases}$$

Solution: Suppose d induced by a norm on L. Then, by previous Lemma,

$d(\alpha x, \alpha y) = |\alpha|\, d(x, y)\ \ \forall x, y \in X$ and $\forall \alpha \in F.$

Let $x, y \in L$ such that $x \neq y$. Then $\alpha x \neq \alpha y$ such that $d(x, y) = 1, d(\alpha x, \alpha y) = 1$ (1)

But $|\alpha|\, d(x, y) = |\alpha|$ (2)

Hence, $d(\alpha x, \alpha y) = 1 \neq |\alpha| = |\alpha|\, d(x, y)$ for any $\alpha \neq \pm 1$. Thus, d can not be induced by a normed space.

Example 2.24.

Let $d(x, y) = |x| + |y| \quad \forall x, y \in \mathbb{R}$. Then, d is a metric on $\mathbb{R}$ (check!). However, d is not induced by a normed space. To show this, let $x = 1, y = 3, a = 2 \in \mathbb{R}$.

$d(x, y) = d(1, 3) = |1| + |3| = 4$

On the other hand, $d(x + a, y + a) = d(3, 5) = |3| + |5| = 8$

Thus, $d(x, y) \neq d(x + a, y + a)$. By Lemma 2.21, d is not induced by a norm.

2.4 Generalizations of Some Concepts from Metric Space

In what follow, we give generalizations of some known concepts from metric space such as open (closed) ball, open (closed) set, interior set, closure of a set, convergent sequence, Cauchy sequence, and bounded sequence.

Definition 2.25.
Let $(L, \| \|)$ be a normed linear space. Let $x_0 \in L, r \in \mathbb{R}, r > 0$. Then the set

$$B_r(x_0) = \{x \in L : \|x - x_0\| < r\}$$

is called an **open ball** with center x_0 and radius r. Similarly,

$$\bar{B}_r(x_0) = \{x \in L : \|x - x_0\| \leq r\}$$

is called an **closed ball** with center x_0 and radius r.

Definition 2.26.

Let $(L, \| \|)$ be a normed space and $A \subseteq L$. Then A is said to be

- **open set** if $\forall x \in A, \exists r > 0$ such that $B_r(x) \subseteq A$.
- **closed set** if $A^c = L \setminus A$ is open set

Remark 2.27.

Let $(L, \| \|)$ be a normed space. Then

(1) L, ϕ are closed and open.

(2) The union of any family of open sets is open

(3) The union of finite family of closed sets is closed

(4) The intersection of finite family of open sets is open

(5) The intersection of any family of closed sets is closed.

Theorem 2.28.

Any finite subset of a normed space is closed.

Proof. Let L be a normed space and $A \subseteq L$.

If $A = \phi$, then A is closed (by Remark 2.27(1))

If $A = \{x\}$ to prove A is closed (i.e., to prove $L \setminus A$ is open)

Let $y \in L \setminus A = L \setminus \{x\}$ so that $y \neq x$. Put $\|x - y\| = r > 0$. Thus, $x \notin B_r(y)$ and hence $B_r(y) \subseteq A^c = L \setminus \{x\}$. Thus, A^c is open and thus A is closed.

If $A = \{x_1, .., x_n\}, n \in Z_+, n > 1$ then $A = \cup_{i=1}^{n} \{x_i\}$. By Remark 2.27(3), A is closed $\square$

Exercise 2.29.

Let $(L, \| \|)$ be a normed space. Prove that

(i) The set $A_1 = \{x \in L : \|x\| \leq 1\}$ is closed

(ii) The set $A_2 = \{x \in L : \|x\| < 1\}$ is open

(iii) The set $C = \{x \in L : \|x\| = 1\}$ is closed

Solution:

(i) $A_1 = \{x \in L : \|x\| \leq 1\} = \overline{B}_1(0)$.

So, A_1 is a closed set (by Definition 2.25)

(ii) $A_2 = \{x \in L : \|x\| < 1\} = B_1(0)$.

So, A_1 is an open set (by Definition 2.25)

(iii) $C = \{x \in L : \|x\| = 1\}$

$L \setminus C = \{x \in L : \|x\| < 1\} \cup \{x \in L : \|x\| > 1\}$

Let $C_1 = \{x \in L : \|x\| < 1\}$ is open set

Let $C_2 = \{x \in L : \|x\| > 1\}$

So, $L \setminus C_2 = \{x \in L : \|x\| \leq 1\}$ which is closed set. Hence, C_2 is an open set.

Thus, $L \setminus C = C_1 \cup C_2$ is an open set (by Remark 2.27(2)).

Definition 2.30.

Let L be a normed space and $A \subseteq L$. A point $x \in L$ is called **limit point** of A if for each open set G containing x, we have $(G \cap A) \setminus \{x\} \neq \phi$.

The set of all limit points of A is denoted by A' and is called **derived set**.

The closure of A is denoted by $\overline{A}$ and is defined as $\overline{A} = A \cup A'$.

Proposition 2.31.

Let L be a normed linear space and $A \subseteq L$. Then $x \in \overline{A}$ if and only if $\forall r > 0, \exists y \in A, \|x - y\| < r$.

Proof. ($\Rightarrow$) Let $x \in \overline{A} = A \cup A'$

If $x \in A'$ then for each open set G, $x \in G, (G \cap A) \setminus \{x\} \neq \phi$.

Since $B_r(x)$ is an open set then $\forall r > 0$, we have $B_r(x) \cap A \setminus \{x\} \neq \phi$. Thus,

$\exists y \in B_r(x) \cap A, y \neq x \implies \|y - x\| < r$ **(I)**

If $x \in A$ then $\exists y = x$ such that $\|y - x\| = 0 < r$ **(II)**

From **(I)** and **(II)**, we get the required result.

($\Leftarrow$) If for each $r > 0, \exists y \in A$ such that $\|y - x\| < r$; that is $\forall r > 0, \exists y \in A, y \in B_r(x)$

$\implies \forall r > 0, (B_r(x) \cap A) \setminus \{x\} \neq \phi \implies x \in A'$. Thus, $x \in \overline{A}$. □

2.5 Convergence in Normed Space

Definition 2.32.
Let $\langle x_n \rangle$ be a sequence in a normed space $(L, \| \, \|)$. Then $\langle x_n \rangle$ is said to be **convergent** in L if $\exists x \in L$ such that $\forall \epsilon > 0, \exists k \in Z_+$ such that $\|x_n - x\| < \epsilon, \ \forall n > k$

We write $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty}(x_n) = x$; that is

$\|x_n - x\| \to 0 \iff x_n \to x$.

$\langle x_n \rangle$ is **divergent** if it is not convergent.

Theorem 2.33.

If $\langle x_n \rangle$ is a convergent sequence in $(L, \| \, \|)$, then its limit is unique. i.e., If $\langle x_n \rangle \to x$ and $\langle x_n \rangle \to y$ then $x = y$.

Proof. Let $\epsilon > 0$. Since $\langle x_n \rangle \to x$ and $\langle x_n \rangle \to y$, then $\exists k_1, k_2 \in Z_+$ such that

$\|x_n - x\| < \frac{\epsilon}{2}, \ \ \forall n > k_1$ and $\|x_n - y\| < \frac{\epsilon}{2}, \ \ \forall n > k_2$

Let $k = \max\{k_1, k_2\}$, so $\forall n > k$

$$\|x - y\| = \|x_n - y - x_n + x\| = \|(x_n - y) - (x_n - x)\|$$

$$\leq \|x_n - y\| + \|x_n - x\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\implies \|x - y\| < \epsilon, \ \ \forall \epsilon > 0$. Thus, $\|x - y\| = 0$, so $x = y$. □

Theorem 2.34.

Let $A \subseteq L$ where L is a normed space, let $x \in L$. Then

$x \in \overline{A} \iff \exists \langle x_n \rangle$ a sequence in A such that $\langle x_n \rangle \to x$.

Proof. ($\Rightarrow$) Let $x \in \overline{A} = A \cup A'$

If $x \in A$ then the sequence $\langle x, x, x, ... \rangle \to x$ **(I)**

If $x \notin A, i.e., x \in A'$ then for each open set G, $x \in G, (G \cap A) \setminus \{x\} \neq \phi$.

Since $B_r(x)$ is an open set then $\forall r > 0$, we have $B_r(x) \cap A \setminus \{x\} \neq \phi$. Set $0 < r = \dfrac{1}{n} \in Z_+$. Then $\forall n \in Z_+, (B_{\frac{1}{n}}(x) \cap A) \setminus \{x\} \neq \phi$

Let $x_n \in B_{\frac{1}{n}}(x) \cap A$, s.t $x_n \neq x$, hence, $\|x_n - x\| < \dfrac{1}{n}, \ \ \forall n \in Z_+$ **(*)**

Thus, $\exists \langle x_n \rangle \in A$ such that $\|x_n - x\| < \frac{1}{n}, \ \ \forall n \in Z_+$.

To show $\langle x_n \rangle \to x$; that is $\|x_n - x\| < \epsilon, \ \ \forall \epsilon > 0$

Let $\epsilon > 0$ so by Archmedian theorem $\exists k \in Z_+$ such that $\dfrac{1}{k} < \epsilon$

Hence, $\forall n > k, \dfrac{1}{n} < \dfrac{1}{k} < \epsilon$

From **(*)**, $\forall n > k, \|x_n - x\| < \dfrac{1}{n} < \dfrac{1}{k} < \epsilon$. Thus, $x_n \to x$ **(II)**

From **(I)** and **(II)**, we get the required result.

($\Leftarrow$) If $\exists \langle x_n \rangle$ a sequence in A such that $\langle x_n \rangle \to x$. To prove $x \in \overline{A} = A \cup A'$

If $x \in A$ then $x \in \overline{A}$

If $x \notin A$. Let G be an open set in L such that $x \in G$. Then $\exists r > 0$ such that $B_r(x) \subseteq G$. Since $r > 0$ and $x_n \to x, \exists k \in Z_+$ such that $\|x_n - x\| < r, \quad \forall n > k$.

This implies, $x_n \in B_r(x) \quad \forall n > k$ and since $x_n \in A \ \ \forall n \in Z_+$. Then $(B_r(x) \cap A) \setminus \{x\} \neq \phi$. Since $B_r(x) \subseteq G$, then $(G \cap A) \setminus \{x\} \neq \phi$. So, $x \in A'$, and therefore $x \in \overline{A}$. □

Theorem 2.35.

Let $\langle x_n \rangle, \langle y_n \rangle$ be two sequences in normed space $(L, \| \ \|)$ such that $x_n \to x$ and $y_n \to y$. Then

(1) $\langle x_n \rangle \pm \langle y_n \rangle \to x \pm y$

(2) $\lambda \langle x_n \rangle \to \lambda x \quad$ for any scalar λ

(3) $\|\langle x_n \rangle\| \to \|x\|$

Proof. **(1)** Since $x_n \to x$, then

for each $\epsilon > 0, \exists k_1 \in Z_+$ such that $\|x_n - x\| < \dfrac{\epsilon}{2}, \ \ \forall n > k_1$

Also since $y_n \to y$, then

for each $\epsilon > 0, \exists k_2 \in Z_+$ such that $\|y_n - y\| < \dfrac{\epsilon}{2}, \ \ \forall n > k_2$

Let $k = \max\{k_1, k_2\}$. Then, for each $n > k$

$\|x_n - x\| < \dfrac{\epsilon}{2}$ and $\|y_n - y\| < \dfrac{\epsilon}{2}$ **(I)**

Now, for each $n > k$,

$$\|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\| \leq \|x_n - x\| + \|y_n - y\|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad (\text{from } \mathbf{(I)})$$

Thus, $x_n + y_n \to x + y$ as required.

(2) Let $\epsilon > 0$. Since $x_n \to x, \exists k \in Z_+$ s.t $\|x_n - x\| < \frac{\epsilon}{|\lambda|}, \quad \forall n > k$ **(II)**

But $\|\lambda x_n - \lambda x\| = |\lambda| \underbrace{\|x_n - x\| < \frac{\epsilon}{|\lambda|}}_{\text{using (II)}} |\lambda| = \epsilon$

Thus, $\lambda\langle x_n\rangle \to \lambda x$

(3) Let $\epsilon > 0$. Since $x_n \to x, \exists k \in Z_+$ s.t $\|x_n - x\| < \epsilon, \quad \forall n > k$ **(III)**

But $|\ \|x_n\| - \|x\|\ | \leq \underbrace{\|x_n - x\| < \epsilon}_{\text{using (III)}} \quad \forall n > k$. Hence, $\|x_n\| \to \|x\|$. □

Definition 2.36.
Let $\langle x_n\rangle$ be a sequence in a normed space $(L, \|\ \|)$. Then $\langle x_n\rangle$ is said to be **Cauchy sequence** if $\forall \epsilon > 0, \exists k \in Z_+$ s.t $\|x_n - x_m\| < \epsilon$, $\forall n, m > k$.

Theorem 2.37.

Every convergent sequence in a normed space $(L, \|\ \|)$ is a Cauchy sequence.

Proof. Let $\langle x_n\rangle$ be a convergent sequence in L. Then $\exists x \in L$ such that $x_n \to x$ and so $\forall \epsilon > 0, \exists k \in Z_+$ such that $\|x_n - x\| < \frac{\epsilon}{2} \quad \forall n > k$ **(I)**

Now, for $n, m > k$,

$$\|x_n - x_m\| = \|(x_n - x) + (x - x_m)\| \leq \underbrace{\|x_n - x\| + \|x_m - x\| < \frac{\epsilon}{2} + \frac{\epsilon}{2}}_{\text{using (I)}} = \epsilon.$$

Thus, $\langle x_n\rangle$ is a Cauchy sequence. □

Definition 2.38.
Let $\langle x_n\rangle$ be a sequence in a normed space $(L, \|\ \|)$. Then $\langle x_n\rangle$ is said to be **bounded sequence** if $\exists k \in \mathbb{R}, k > 0$ such that $\|x_n\| \leq k, \ \forall n \in Z_+$.

Theorem 2.39.

Every Cauchy sequence $\langle x_n\rangle$ in a normed space $(L, \|\ \|)$ is bounded.

Proof. Let $\epsilon = 1$. Since $\langle x_n\rangle$ is a Cauchy sequence, $\exists k \in Z_+$ such that $\|x_n - x_m\| < 1, \quad \forall n, m > k$. Hence, $\|x_n - x_{k+1}\| < 1, \quad \forall n > k$ (by considering $m = k + 1$) **(I)**

By Theorem 2.3(4), we have $| \, \|x_n\| - \|x_{k+1}\| \, | \leq \underbrace{\|x_n - x_{k+1}\| < 1}_{\text{using } \textbf{(I)}} \quad \forall n > k$

Thus, $\|x_n\| - \|x_{k+1}\| < 1 \quad \forall n > k$

Then, $\|x_n\| < 1 + \|x_{k+1}\| \quad \forall n > k$

Let $M = \max\{\|x_1\|, \|x_2\|, ..., \|x_k\|, 1 + \|x_{k+1}\|\}$

Hence, $\|x_n\| \leq M \quad \forall n \in Z_+$. So, $\langle x_n \rangle$ is bounded. □

Corollary 2.40.

Every convergent sequence in a normed space $(L, \| \, \|)$ is bounded.

Proof. From Theorem 2.37, Every convergent sequence in a normed space $(L, \| \, \|)$ is Cauchy, and from Theorem 2.39, every Cauchy sequence in a normed space $(L, \| \, \|)$ is bounded. □

2.6 Convexity in Normed Linear Space

Definition 2.41. ***(revisit)***
A subset A of a linear space L is said to be **convex** if $\forall x, y \in A, \lambda \in [0, 1]$ then $\lambda x + (1 - \lambda)y \in A$.

Example 2.42.

Let $A = (1, 3) \subset \mathbb{R}$. Is A convex set?

Solution: Let $x, y \in A, \lambda \in [0, 1]$

Since $1 < x < 3 \implies 1\lambda < \lambda x < 3\lambda$ (I)

Since $1 < y < 3 \implies 1(1 - \lambda) < (1 - \lambda)y < 3(1 - \lambda)$ (II)

By summing up (I) and (II)

$$\lambda + (1 - \lambda) < \lambda x + (1 - \lambda)y < 3\lambda + 3(1 - \lambda)$$

$$1 < \lambda x + (1-\lambda)y << 3$$

Thus, $\lambda x + (1-\lambda)y \in A$. Hence, A is convex set.

Proposition 2.43.

Let L linear space. Then

(1) Every subspace of L is convex

(2) If $A, B \subset L$ are convex sets then $A \cap B$ is convex (**H.W.**)

(3) If $A, B \subset L$ are convex sets then $A + B$ is convex

Proof. **(1)** Let L be a linear space over a field $F = \mathbb{R}$ or C, let A be a subspace of L. Hence, by Theorem 1.13, $\forall x, y \in A$ and $\forall \alpha, \beta \in F$ we have $\alpha x + \beta y \in A$.

Take $\alpha = \lambda \in [0,1]$ and $\beta = 1-\lambda$. Hence, $\alpha x + \beta y = \lambda x + (1-\lambda)y \in A$.

Thus, A is a convex set.

(3) Let $a_1 + b_1, a_2 + b_2 \in A + B$, then $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

To prove $\lambda(a_1 + b_1) + (1-\lambda)(a_2 + b_2) \in A + B, \ \ \forall \lambda \in [0,1]$.

Since A convex and $a_1, a_2 \in A \implies \lambda a_1 + (1-\lambda)a_2 \in A \quad \forall \lambda \in [0,1]$ (I)

Since B convex and $b_1, b_2 \in B \implies \lambda b_1 + (1-\lambda)b_2 \in B \quad \forall \lambda \in [0,1]$ (II)

By summing up (I) and (II) we get

$\lambda a_1 + (1-\lambda)a_2 + \lambda b_1 + (1-\lambda)b_2 \in A + B$

i.e., $\lambda(a_1 + b_1) + (1-\lambda)(a_2 + b_2) \in A + B$. Thus, $A + B$ is a convex set. □

Remark 2.44.

The union of two convex sets is not necessary convex. For example, let $A = (3,7) \cup (7,12)$. Then A is not convex. To show this, take $x = 6, y = 8, \lambda = \dfrac{1}{2}$ then $\lambda x + (1-\lambda)y = \dfrac{1}{2}(6) + \dfrac{1}{2}(8) = 7 \notin A \cup B$.

Proposition 2.45.

Let $(L, \| \|)$ be a normed linear space, let $x_0 \in L$. Then $B_r(x_0)$ and $\overline{B}_r(x_0)$ are convex sets.

Proof. To prove $B_r(x_0)$ is a convex set. Let $x, y \in B_r(x_0)$, and let $\lambda \in [0, 1]$. Then,

$\|x - x_0\| < r$ and $\|y - x_0\| < r$ (I)

We must prove $\lambda x + (1 - \lambda)y \in B_r(x_0)$; that is we must prove

$\|\lambda x + (1 - \lambda)y - x_0\| < r$

$\|\lambda x + (1 - \lambda)y - x_0\| = \|\lambda x + \lambda \mathbf{x_0} - \lambda \mathbf{x_0} + (1 - \lambda)y - x_0\|$ (adding and subtracting λx_0)

$$= \|\lambda(x - x_0) + (1 - \lambda)(y - x_0)\|$$

$$\leq |\lambda| \, \|(x - x_0)\| + |1 - \lambda| \, \|y - x_0\| < \lambda r + (1 - \lambda)r = r$$

(by (I) and since $\lambda > 0$ then $|\lambda| = \lambda, |1 - \lambda| = 1 - \lambda$)

Thus, $\lambda x + (1 - \lambda)y \in B_r(x_0)$ and hence $B_r(x_0)$ is convex. Similarly, $\overline{B}_r(x_0)$ is a convex set. □

Proposition 2.46.

Let $(L, \| \|)$ be a normed linear space and $A \subseteq L$ and convex then $\overline{A}$ is a convex set.

Proof. Let $x, y \in \overline{A}$ and $\lambda \in [0, 1]$. To prove $\lambda x + (1 - \lambda)y \in \overline{A}$

Let $r > 0$. Since $x, y \in \overline{A}$ then by Proposition 2.31, $\exists a, b \in A$ such that

$\|x - a\| < r$ and $\|y - b\| < r$ (I)

Since A is convex then $\lambda a + (1 - \lambda)b \in A$

Now, $\big\|\lambda x + (1 - \lambda)y - \big(\lambda a + (1 - \lambda)b\big)\big\| = \|\lambda(x - a) + (1 - \lambda)(y - b)\|$

$$\leq \lambda \|x - a\| + (1-\lambda)\|y - b\|$$

$$< \lambda r + (1-\lambda) r \quad \text{(from (I))}$$

$$= r$$

Thus, $\left\| \left(\lambda x + (1-\lambda)y\right) - \big(\underbrace{\lambda a + (1-\lambda)b}_{\in A}\big) \right\| < r$

Thus, from Proposition 2.31, $\lambda x + (1-\lambda)y \in \overline{A}$. □

Remark 2.47.

The converse of the above proposition is not true. For example, let $A = [1,2) \cup (2,5] \subset (\mathbb{R}, |\ |)$ then $\overline{A} = [1,5]$ is a convex set. But A is not convex, since if $x = 1, y = 3, \lambda = \dfrac{1}{2}$ then $\lambda x + (1-\lambda)y = \frac{1}{2} + \frac{1}{2}(3) = 2 \notin A$.

2.7 Continuity in Normed Linear Space

Definition 2.48.

Let $(L, \|\ \|_L), (L', \|\ \|_L{}')$ be normed linear spaces. A mapping $f : L \to L'$ is called **continuous** at $x_0 \in L$ if for each $\epsilon > 0, \exists \delta > 0$ (depend on x_0) such that

$$\forall x \in L, \ \text{if} \ \ \|x - x_0\|_L < \delta \ \text{then} \ \ \|f(x) - f(x_0)\|_L{}' < \epsilon.$$

i.e., $\forall x \in L$, if $x \in B_\delta(x_0)$ then $f(x) \in B_\epsilon(f(x_0))$

Theorem 2.49.

Let $(L, \|\ \|_L), (L', \|\ \|_L{}')$ be normed linear spaces. A mapping $f : L \to L'$ is continuous at $x_0 \in L$ if and only if $\forall \langle x_n \rangle \in L$ with $x_n \to x_0$ implies that $f(x_n) \to f(x_0)$.

Proof. ($\Rightarrow$) Let f be a contiunuous mapping at x_0 and let $\langle x_n \rangle$ be a sequence in L such that $x_n \to x_0$. To prove $f(x_n) \to f(x_0)$.

Let $\epsilon > 0$, then $\exists \delta > 0$ such that $\forall x \in L$

if $\|x - x_0\|_L < \delta$ then $\|f(x) - f(x_0)\|_L{}' < \epsilon$ (From continuity of f at x_0).

Since $x_n \to x_0$ and $\delta > 0, \exists k \in Z_+$ such that $\|x_n - x_0\|_L < \delta, \quad \forall n > k$.

Hence, $\|f(x_n) - f(x_0)\|_L{}' < \epsilon, \quad \forall n > k$; that is $f(x_n) \to f(x_0)$.

($\Leftarrow$) Suppose that $x_n \to x_0$ implies that $f(x_n) \to f(x_0)$. To prove f is continuous at x_0.

Assume that f is not continuous at x_0, so $\exists \epsilon > 0$ such that $\forall \delta > 0, \exists x \in L$ and

$\|x - x_0\|_L < \delta$ but $\|f(x) - f(x_0)\|_L{}' \geq \epsilon$.

Now, $\forall n \in Z_+, \dfrac{1}{n} > 0$, then $\exists x_n \in L$ such that

$\|x_n - x_0\|_L < \dfrac{1}{n}$ but $\|f(x_n) - f(x_0)\|_L{}' \geq \epsilon$. This means $x_n \to x_0$ but $f(x_n) \nrightarrow f(x_0)$ in L' which is a contradiction. Thus, f is continuous at x_0. □

Theorem 2.50.

Let $(L, \| \ \|)$ be a normed space and let $f : (L, \| \ \|) \to (\mathbb{R}, | \ |)$ such that $f(x) = \|x\| \ \ \forall x \in L$. Then f is continuous at x_0.

Proof. Let $x_n \to x_0$ in L. Then $\forall \epsilon > 0, \exists k \in Z_+$ such that

$\|x_n - x_0\| < \epsilon \ \ \forall n > k$ **(I)**

But $| \ \|x_n\| - \|x_0\| \ | \leq \|x_n - x_0\| \quad \forall n > k$

$\Longrightarrow | \ \|x_n\| - \|x_0\| \ | < \epsilon \ \ \forall n > k$ (Using **(I)**)

$\Longrightarrow |f(x_n) - f(x_0)| < \epsilon \ \ \forall n > k$ (Using (since $f(x) = \|x\|$))

$f(x_n) \to f(x_0)$; that is f is continuous at x_0. □

Remark 2.51.

Let $(L_1, \| \|_1), (L_2, \| \|_2)$ and $(L_3, \| \|_3)$ be normed spaces and let $f : L_1 \times L_2 \to L_3$ be a mapping. Then f is continuous at $(x_0, y_0) \in L_1 \times L_2$ if and only if $\forall \langle (x_n, y_n) \rangle \in L_1 \times L_2$ and $\langle (x_n, y_n) \rangle \to (x_0, y_0)$ then $f(x_n, y_n) \to f(x_0, y_0)$.

Theorem 2.52.

Let $(L, \| \|$ be a normed space over a field F. Then

(1) The mapping $f : L \times L \to L$ such that $f(x, y) = x + y \quad \forall x, y \in L$ is continuous at any point in $L \times L$.

(2) The mapping $g : F \times L \to L$ such that $g(\lambda, x) = \lambda x \quad \forall x \in L, \forall \lambda \in F$ is continuous at any point in $F \times L$.

Proof. **(1)** Let (x_0, y_0) be an arbitrary point in $L \times L$ and $(x_n, y_n) \to (x_0, y_0)$. Then, $x_n \to x_0$ and $y_n \to y_0$ such that

$\|x_n - x_0\| \to 0$ and $\|y_n - y_0\| \to 0$ as $n \to +\infty$.

We must prove $f(x_n, y_n) \to f(x_0, y_0)$. i.e., $\|f(x_n, y_n) - f(x_0, y_0)\| \to 0$

Now,
$$\begin{aligned}\|f(x_n, y_n) - f(x_0, y_0)\| &= \|(x_n + y_n) - (x_0 + y_0)\| \\ &= \|(x_n - x_0) + (y_n - y_0)\| \\ &\leq \|x_n - x_0\| + \|y_n - y_0\|\end{aligned}$$

Thus, $\|f(x_n, y_n) - f(x_0, y_0)\| \to 0$ as $n \to +\infty$; that is f is continuous at (x_0, y_0). Since (x_0, y_0) is arbitrary, f is continuous at $L \times L$.

(2) Let (λ_0, x_0) be an arbitrary point in $F \times L$ and $(\lambda_n, x_n) \to (\lambda_0, x_0)$. Then, $\lambda_n \to \lambda_0$ and $x_n \to x_0$.

Hence, $|\lambda_n - \lambda_0| \to 0$, $\|x_n - x_0\| \to 0$ as $n \to +\infty$.

We must prove $g(\lambda_n, x_n) \to g(\lambda_0, x_0)$. i.e., $\|g(\lambda_n, x_n) - g(\lambda_0, x_0)\| \to 0$

$$\begin{aligned}
\|g(\lambda_n, x_n) - g(\lambda_0, x_0)\| &= \|\lambda_n x_n - \lambda_0 x_0\| \\
&= \|\lambda_n x_n - \boldsymbol{\lambda_n x_0} + \boldsymbol{\lambda_n x_0} - \lambda_0 x_0\| \\
&= \|\lambda_n (x_n - x_0) + (\lambda_n - \lambda_0) x_0\| \\
&\leq |\lambda_n| \, \|x_n - x_0\| + |\lambda_n - \lambda_0| \, \|x_0\|
\end{aligned}$$

But $\|x_n - x_0\| \to 0$ and $|\lambda_n - \lambda_0| \to 0$ so that

$\|g(\lambda_n, x_n) - g(\lambda_0, x_0)\| \to 0$ as $n \to \infty$; that is $g(\lambda_n, x_n) \to g(\lambda_0, x_0)$. Thus, g is continuous at (λ_0, x_0). □

Theorem 2.53.

Let $(L, \| \|_L), (L', \| \|_L{}')$ be normed spaces and let $f : L \to L'$ be a linear transformation. If f is continuous at 0 then f is continuous at any point.

Proof. Let $x_0 \in L$ be an arbitrary point and let $x_n \to x_0$.

To prove $f(x_n) \to f(x_0)$ (using Theorem 2.49).

Since $x_n \to x_0$, then $x_n - x_0 \to 0$

But f is continuous at 0, thus $f(x_n - x_0) \to f(0)$

Since f is a linear transformation, then $f(x_n) - f(x_0) \to f(0) = 0$

It follows that $f(x_n) \to f(x_0)$. □

Remark 2.54.

The condition f is a linear transformation in the above theorem is necessary condition. For example: consider the normed space $(\mathbb{R}, | |)$. Let f is defined as

$$f(x) = \begin{cases} x & \text{if } x \leq 1 \\ x+1 & \text{if } x > 1 . \end{cases}$$

It is clear that f is continuous at 0 and discontinuous at 1.

Also f is not linear transformation because if $x = 5, y = 6$ and $\alpha = \beta = 1$

$f(\alpha x + \beta y) = f(5 + 6) = f(11) = 11 + 1 = 12$

and $\alpha f(x) + \beta f(y) = f(5) + f(6) = (5 + 1) + (6 + 1) = 13$

Hence $f(\alpha x + \beta y) \neq \alpha f(x) + \beta f(y)$.

Theorem 2.55.

Let $(L, \| \|_L), (L', \| \|_L{}')$ be normed spaces and let $f : L \to L'$ be a linear transformation. If f is continuous at a point $x_1 \in L$ then f is continuous at each point.

Proof. Let $x_1 \in L$ and assume that f is continuous at x_1. Let $x_2 \in L$ be any point. To prove that f is continuous at x_2. Let $x_n \to x_2$ in L. Then, $x_n - x_2 \to 0$ and hence $x_n - x_2 + x_1 \to x_1$. Since f is continuous at x_1 then $f(x_n - x_2 + x_1) \to f(x_1)$.

Since f is a linear transformation, then $f(x_n) - f(x_2) + f(x_1) \to f(x_1)$.

Hence, $f(x_n) - f(x_2) \to 0$, and thus, $f(x_n) \to f(x_2)$.

Therefore, f is continuous at x_2. Thus, f can not be continuous at some points and discontinuos at some points. □

Example 2.56.

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) . \end{cases}$$

Show that f is not continuous at $(0,0)$.

Solution: Let $x_n = \dfrac{1}{n}$ and $y_n = \dfrac{-1}{n}$ $\forall n \in N$.

Then, $x_n \to 0$ and $y_n \to 0$. Thus, $(x_n, y_n) \to (0,0)$. But

$$f(x_n, y_n) = \frac{\frac{1}{n}(\frac{-1}{n})}{\frac{1}{n^2} + \frac{1}{n^2}} = \frac{\frac{-1}{n^2}}{\frac{2}{n^2}} = \frac{-1}{2}$$

Hence, $f(x_n, y_n) \to \frac{-1}{2}$ but $f(0,0) = (0,0)$. Thus, $f(x_n, y_n) \nrightarrow f(0,0)$.

Thus, f is not continuous at $(0,0)$.

2.8 Boundedness in Normed Linear Space

Definition 2.57. ***Bounded Set***

Let $(L, \| \|_L)$ be a normed space and let $A \subset L$. A is called a **bounded set** if there exists $k > 0$ such that $\|x\| \leq k \quad \forall x \in A$.

Example 2.58.

Consider $(\mathbb{R}, | |)$ and let $A = [-1, 1)$. Since $|x| \leq 1$, then A is bounded.

Example 2.59.

Consider $(\mathbb{R}^2, \| \|)$ be a normed space such that

$\|x\| = \left[\sum_{i=1}^{2} |x_i|^2\right]^{\frac{1}{2}}$ be the Eucledian norm, for each $x = (x_1, x_2) \in \mathbb{R}^2$.

Let $A = \{(x_1, x_2) \in \mathbb{R}^2 : -1 \leq x_1 \leq 1, \ \ x_2 \geq 0\}$. Then, A is unbounded.

Theorem 2.60.

Let $(L, \| \|_L)$ be a normed space and let $A \subseteq L$. Then the following statements are equivalent.

(1) A is bounded.

(2) If $\langle x_n \rangle$ is a sequence in A and $\langle \alpha_n \rangle$ is a sequence in F such that $\alpha_n \to 0$ then $\alpha_n x_n \to 0$.

Proof. (1)⇒(2) Since A is bounded, $\exists k > 0$ such that $\|x_n\| \leq k \quad \forall x_n \in A$. Since $\alpha_n \to 0$ as $n \to \infty$, then $|\alpha_n| \to 0$. Hence,

$\|\alpha_n x_n - 0\| = \|\alpha_n x_n\| = |\alpha_n| \|x_n\| \leq |\alpha_n| k \quad$ (since $\|x_n\| \leq k$)

But $|\alpha_n| \to 0$, thus $|\alpha_n| k \to 0$. Therefore, $\|\alpha_n x_n - 0\| \to 0$ and hence $\alpha_n x_n \to \mathbf{0}_X$.

(2)⇒(1) Suppose A is not bounded. Then, $\forall k \in Z_+, \exists x_k \in A$ such that $\|x_k\| > k$.

Put $\alpha_k = \dfrac{1}{k}$. Hence, $\alpha_k \to 0$. But

$\|\alpha_k x_k\| = \left\|\frac{1}{k} x_k\right\| = \dfrac{1}{k} \|x_k\| > \dfrac{1}{k}.k = 1$

Then, $\|\alpha_k x_k\| > 1$, thus $\alpha_k x_k \nrightarrow 0$ which contradicts (2). □

Definition 2.61. ***Bounded Mapping***
Let $(L, \| \|_L), (L', \| \|_L{}')$ be two normed space and $f : L \to L'$ be a linear transformation. f is called **bounded mapping** if for each $A \subseteq L$ bounded then $f(A) = \{f(a) : a \in A\}$ is bounded set in L'.

Example 2.62.

Let Consider $(\mathbb{R}, | |)$ and $(\mathbb{R}^2, \| \|)$ be a normed space such that

$\|x\| = \left[\sum_{i=1}^{2} |x_i|^2\right]^{\frac{1}{2}} = [|x_1|^2 + |x_2|^2]^{\frac{1}{2}} = [x_1^2 + x_2^2]^{\frac{1}{2}} \quad \forall (x_1, x_2) \in \mathbb{R}^2.$

Define $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x_1, x_2) = x_1 + x_2 \quad \forall (x_1, x_2) \in \mathbb{R}^2$. Show that f is a linear transformation (**H.W.**). Let $A \subseteq \mathbb{R}^2$ and A is bounded. Show that $f(A)$ is bounded.

Solution: Let $A \subseteq \mathbb{R}^2$ and A is bounded to prove $f(A) = \{f(x_1, x_2) : (x_1, x_2) \in A\}$ is bounded.

Note that $\forall (x_1, x_2) \in A \implies f(x_1, x_2) = x_1 + x_2 \in f(A)$

$|fx_1, x_2| = |x_1 + x_2| \leq |x_1| + |x_2| \qquad \textbf{(I)}$

Since A is bounded then $\exists k > 0$ such that $\|(x_1, x_2)\| \leq k \quad \forall (x_1, x_2) \in A$

$\implies (x_1^2 + x_2^2)^{\frac{1}{2}} \leq k \implies x_1^2 + x_2^2 \leq k^2$

Since $x_1^2 \leq x_1^2 + x_2^2 \leq k^2$, then $x_1^2 \leq k^2 \implies |x_1| \leq k \qquad \textbf{(II)}$

Similarly, $x_2^2 \leq x_1^2 + x_2^2 \leq k^2$, then $x_2^2 \leq k^2 \implies |x_2| \leq k \qquad \textbf{(III)}$

Substitute **(II)** and **(III)** in **(I)**

$$|f(x_1, x_2)| = |x_1 + x_2| \leq \underbrace{|x_1| + |x_2| \leq k + k}_{\text{by } \textbf{(II) and (III)}} = 2k$$

i.e., $|f(x_1, x_2)| \leq 2k$. Thus, $f(A)$ is bounded, and hence, f is bounded.

Theorem 2.63.

Let $(L, \| \|_L), (L', \| \|_L{}')$ be normed spaces and $f : L \to L'$ be a linear transformation. Then f is bounded if and only if $\exists k > 0$ such that $\|f(x)\|_L{}' \leq k \|x\|_L \qquad \forall x \in L$.

Proof. ($\Rightarrow$) If f is bounded and let $A = \{x \in L : \|x\|_L \leq 1\}$.

It is clear A is bounded, and hence, $f(A)$ is bounded in L' (by definition of bnd function).

Thus, $\exists k > 0$ such that $\|f(x)\|_L{}' \leq k \qquad \forall x \in A \qquad \textbf{(I)}$

(1) If $x = \mathbf{0_L}$ then $f(\mathbf{0_L}) = \mathbf{0'_L}$, and thus, $\|f(\mathbf{0_L})\| = 0 \leq k \|\mathbf{0_L}\| = 0$.

(2) If $x \neq \mathbf{0_L}$, put $y = \dfrac{x}{\|x\|}$ such that $\|y\| = \left\|\dfrac{x}{\|x\|}\right\| = \dfrac{1}{\|x\|}.\|x\| = 1$.

Hence, $y \in A$. Thus, $\|f(y)\| \leq k$ **(II)**

$\|f(y)\| = \left\|f(\frac{x}{\|x\|})\right\| = \left\|\frac{1}{\|x\|}f(x)\right\| = \frac{1}{\|x\|}\|f(x)\|$

By **(II)**, $\|f(y)\| \leq k$, thus $\frac{1}{\|x\|}\|f(x)\| \leq k$. i.e., $\|f(x)\| \leq k.\|x\|$ as required.

($\Leftarrow$) Let A be a bounded set. Then, $\exists k_1 > 0$ such that $\|x\| \leq k_1 \quad \forall x \in A$

Since $\|f(x)\| \leq k\|x\| \quad \forall x \in X$, hence $\|f(x)\| \leq k\|x\| \quad \forall x \in A$. Then we get $\|f(x)\| \leq kk_1 \quad \forall x \in A$. Thus, $\|f(x)\| \leq k_2 \quad \forall x \in A$ where $k_2 = kk_1$; that is, $f(A)$ is a bounded set. □

Theorem 2.64.

Let $(L, \|\ \|_L), (L', \|\ \|_L{}')$ be normed spaces and $f : L \to L'$ be a linear transformation. Then f is bounded if and only if f is continuous.

Proof. ($\Leftarrow$) Suppose that f is continuous and not bounded,

hence $\forall n \in Z_+, \exists x_n \in L$ such that $\|f(x_n)\|_L{}' > n\|x_n\|_L$.

Let $y_n = \frac{x_n}{n\|x_n\|}$. Then, $\|f(y_n)\| = \left\|f(\frac{x_n}{n\|x_n\|})\right\| = \frac{\|f(x_n)\|}{n\|x_n\|} > \frac{n\|x_n\|}{n\|x_n\|} = 1$

Thus, $\|f(y_n) - f(0)\| = \|f(y_n)\| > 1$, i.e., $f(y_n) \nrightarrow f(0)$ **(I)**

but $\|y_n\| = \left\|\frac{x_n}{n\|x_n\|}\right\| = \frac{\|x_n\|}{n\|x_n\|} = \frac{1}{n}$

as $n \to \infty$, we get $\|y_n\| \to 0$, and hence, $y_n \to \mathbf{0}_L$.

It follows that $f(y_n) \to \underbrace{f(\mathbf{0}_L) = \mathbf{0}'_L}_{\text{By Theorem 1.19(i)}}$ (Since f is a linear transformation)

This contradicts **(I)**, thus, f is bounded.

($\Rightarrow$) Assume that f is bounded to prove f is continuous for all $x \in L$. Let $x_0 \in L$ and $\epsilon > 0$, to find $\delta > 0$ such that

$\forall x \in L, \|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$.

$\|f(x) - f(x_0)\| = \|f(x - x_0)\|$ (f is linear transformation)

Since f is bounded, then $\exists k > 0$ s.t. $\|f(x)\| \leq k\,\|x\| \quad \forall x \in L$ **(I)**

$$\text{Hence, } \|f(x) - f(x_0)\| = \underbrace{\|f(x - x_0)\| \leq k\,\|x - x_0\|}_{\text{By } \textbf{(I)}}$$

$$< k\delta \qquad (\text{Since } \|x - x_0\| < \delta)$$

$$= k.\frac{\epsilon}{k} \qquad (\text{By choosing } \delta = \frac{\epsilon}{k} = \epsilon)$$

Thus, $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$.

Hence, f is continuous at $x_0 \in L$. Since x_0 is an arbitrary, then f is cont. $\forall x \in L$. □

Theorem 2.65.

Let $(L, \|\ \|_L), (L', \|\ \|_L{}')$ be normed spaces and $f : L \to L'$ be a linear transformation. If L is a finite dimensional space then f is bounded (hence, continuous).

Example 2.66.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x, y) = x + y \quad \forall (x, y) \in \mathbb{R}^2$.

f is a linear transformation function (check!)

and $dim(\mathbb{R}^2) = 2$. Hence, f is bounded (hence, continuous).

2.9 Bounded Linear Transformation

Definition 2.67.
Let $(L, \|\ \|_L), (L', \|\ \|_L{}')$ be normed spaces over a field F. The set of all bounded linear transformation mappings from L to L' is defined as

$B(L, L') = \{T : T : L \to L' \text{ is a linear bounded (hence, cont.) trans.}\}$

Theorem 2.68.

Prove that $B(L, L')$ is a linear subspace (over a field F) of the space of linear transformation mappings with respect to usual addition and usual scalar multiplication.

Proof. Let $\alpha, \beta \in F$ and $T_1, T_2 \in B(L, L')$. To prove $\alpha T_1 + \beta T_2 \in B(L, L')$

Since T_1, T_2 are linear transformations, then by Theorem 1.30, $\alpha T_1, \beta T_2$ are linear trans.

Now, $\alpha T_1, \beta T_2$ are linear trans., by Theorem 1.30,$\alpha T_1 + \beta T_2$ is linear transformation.

Next, we show $\alpha T_1 + \beta T_2$ is bounded.

Since T_1, T_2 are bounded, then $\exists k_1, k_2 > 0$ such that $\forall x \in L$ we have

$\|T_1(x)\|'_L \leq k_1 \|x\|_L$ and $\|T_2(x)\|'_L \leq k_2 \|x\|_L$ **(I)**

Then, $\|(\alpha T_1 + \beta T_2)(x)\|'_L = \|(\alpha T_1)(x) + (\beta T_2)(x)\|'_L$

$$= \|\alpha.T_1(x) + \beta.T_2(x)\|'_L \qquad \text{(Definition of scalar multiplication)}$$

$$\leq \|\alpha.T_1(x)\|'_L + \|\beta.T_2(x)\|'_L$$

$$= |\alpha| \, \|T_1(x)\|'_L + |\beta| \, \|T_2(x)\|'_L$$

$$\leq |\alpha| \, k_1 \|x\|_L + |\beta| \, k_2 \|x\|_L$$

$$= (|\alpha| \, k_1 + |\beta| \, k_2) \|x\|_L = k \|x\|_L \qquad (k = |\alpha| \, k_1 + |\beta| \, k_2)$$

Hence, $\alpha T_1 + \beta T_2$ is bounded.

Since $\alpha T_1 + \beta T_2$ is bounded and linear transformation, then $\alpha T_1 + \beta T_2 \in B(L, L')$. □

Theorem 2.69.

Let $(L, \| \|_L), (L', \| \|_L{}')$ be normed space. Prove that $B(L, L')$ is a normed space such that $\forall T \in B(L, L')$ we have

$\|T\| = \sup\{\|T(x)\|_{L'} : x \in L, \|x\|_L \leq 1\}$

Proof. To prove $\| \|$ is a norm on $B(L, L')$

(1) since $\|T(x)\|_{L'} \geq 0 \ \ \forall x \in L, \|x\|_L \leq 1$, then $\|T\| \geq 0$.

(2) $\|T\| = 0 \iff \sup\{\|T(x)\|_{L'} : x \in L, \|x\|_L \leq 1\} = 0$

$$\iff \|T(x)\|_{L'} = 0 \ \ \forall x \in L, \|x\|_L \leq 1$$

$$\iff T(x) = 0 \ \ \forall x \in L, \|x\|_L \leq 1$$

$$\iff T = \hat{0}$$

(3) Let $T_1, T_2 \in B(L, L')$

$$\|T_1 + T_2\| = \sup\{\|(T_1 + T_2)(x)\|_{L'} : x \in L, \|x\|_L \leq 1\}$$

$$\leq \sup\{\|(T_1(x)\|_{L'} + \|T_2(x)\|_{L'} : x \in L, \|x\|_L \leq 1\}$$

$$\leq \sup_{x \in L}\{\|(T_1(x)\|_{L'} : \|x\|_L \leq 1\} + \sup_{x \in L}\{\|(T_2(x)\|_{L'} : \|x\|_L \leq 1\}$$

$$= \|T_1\| + \|T_2\|$$

(4) $\|\alpha T\| = \sup\{\|(\alpha.T(x)\|_{L'} : x \in L, \|x\|_L \leq 1\}$

$$= |\alpha| \sup\{\|T(x)\|_{L'} : x \in L, \|x\|_L \leq 1\}$$

$$= |\alpha| \, \|T\|$$

□

Chapter Three

Banach Space

In this chapter, we introduce the following

1.1. Banach Space.

1.2. Examples of Banach Space.

1.3. General Properties of Banach Space.

Banach Space

Definition (3.1):-

Let L be a normed space, we say that L is a complete space if every Cauchy sequence is convergent.

The complete normed space is called Banach space.

$i.e,$ $(L, \|.\|)$ is Banach space if

(1) $(L, \|.\|)$ is normed space

(2) $(L, \|.\|)$ is complete space.

Examples of Banach Spaces

Example (3.2) :-

The space F^n with the norm $\|x\| = (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}}$, $x \in F^n$, is Banach space ?

Solution :- To prove that :

(1) $(F^n. \|.\|)$ is normed space (H.W)

(2) $(F^n. \|.\|)$ is complete space ?

Let $< x_m >$ be a Cauchy sequence in $F^n \Longrightarrow x_m \in F^n$, for each m =1,2,3,...

$\langle x_m \rangle = \langle x_1, x_2, \dots, x_m, \dots \rangle$

$= \langle (x_{11}, x_{12}, \dots, x_{1n}), (x_{21}, x_{22}, \dots, x_{2n}), \dots, (x_{m1}, x_{m2}, \dots, x_{mn}), \dots \rangle$

Then $\forall \epsilon > 0, \exists k \in Z_+$ such that

$\|x_m - x_j\| < \epsilon, \ \forall m, j > k \implies \|x_m - x_j\|^2 < \epsilon^2 \quad \dots (1)$

Since,

$x_m = (x_{m1}, x_{m2}, \dots, x_{mn}), x_{mi} \in F, i = 1, \dots, n$

$x_j = (x_{j1}, x_{j2}, \dots, x_{jn}), \quad x_{ji} \in F, i = 1, \dots, n$

$x_m - x_j = (x_{m1} - x_{j1}, x_{m2} - x_{j2}, \dots, x_{mn} - x_{jn})$

So, $\|x_m - x_j\|^2 = \sum_{i=1}^{n} |x_{mi} - x_{ji}|^2 \quad \dots (2)$

From (1) & (2) $\implies \sum_{i=1}^{n} |x_{mi} - x_{ji}|^2 < \epsilon^2$

$$\implies |x_{mi} - x_{ji}|^2 < \epsilon^2, \forall m, j > k$$

$\implies |x_{mi} - x_{ji}| < \epsilon \implies < x_{mi} > \forall i = 1, 2, \dots \dots, n$ is Cauchy in F

But F is complete space $\implies \forall i = 1, 2, \dots \dots, n, \ \exists x_i \in F \ s.t$

$x_{mi} \to x_i \implies \forall \epsilon > 0, \exists k_i \in N$ such that

$$|x_{mi} - x_i| < \frac{\epsilon}{\sqrt{n}}, \forall m > k_i$$

Put, $x = (x_1, x_2, \dots, x_n)$. Let $k = \{k_1, k_2, \dots, k_n\}$

$\Longrightarrow \forall m > k, \quad |x_{mi} - x_i| < \frac{\epsilon}{\sqrt{n}} \Longrightarrow |x_{mi} - x_i|^2 < \frac{\epsilon^2}{n}$

$\Longrightarrow \sum_{i=1}^{n} |x_{mi} - x_i|^2 < n.\frac{\epsilon^2}{n} = \epsilon^2$

But $\|x_m - x\|^2 = \sum_{i=1}^{n} |x_{mi} - x_i|^2 < \epsilon^2$

$$\Longrightarrow \|x_m - x\|^2 < \epsilon^2 \ \Longrightarrow \ \|x_m - x\| < \epsilon$$

$\Longrightarrow < x_m >$ is convergent sequence $\Longrightarrow (F^n, \|.\|)$ is Banach space .

Example (3.3) :-

The space F^n with the norm $\|x\| = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$, $p \geq 1$, $x \in F^n$, is Banach space ?

(H.W.)

Example(3.4) :-

a .The space $\mathbb{R}^n (or\ \mathbb{C}^n)$ with the norm

$\|x\| = \{|x_1|, \ldots, |x_n|\}$, $\forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ (or $\mathbb{C}^n$) is a Banach space .

Solution : Let $\langle x_m \rangle$ be a Cauchy sequence in F^n

$\langle x_m \rangle = \langle x_1, x_2, \ldots, x_m, \ldots \rangle$

$= \langle (x_{11}, x_{12}, \ldots, x_{1n}), (x_{21}, x_{22}, \ldots, x_{2n}), \ldots, (x_{m1}, x_{m2}, \ldots, x_{mn}), \ldots \rangle$

Then $\forall \epsilon > 0, \exists k \in Z_+$ such that $\|x_m - x_j\| < \epsilon \ \forall m, j > k$ (1)

Since $x_m, x_j \in F^n$, then

$x_m = (x_{m1}, x_{m2}, \dots, x_{mn}), x_{mi} \in F, i = 1, \dots, n$

$x_j = (x_{j1}, x_{j2}, \dots, x_{jn}), x_{ji}, \in F, i = 1 , \dots, n$

$x_m - x_j = (x_{m1} - x_{j1}, x_{m2} - x_{j2} , \dots, x_{mn} - x_{jn})$

Then,

$\|x_m - x_j\| = \{|x_{m1} - x_{j1}| , |x_{m2} - x_{j2}| , \dots, |x_{mn} - x_{jn}|\} < \epsilon \quad \forall m, j > k$

It follows that $|x_{mi} - x_{ji}| < \epsilon, \ \forall i = 1, \dots, n \ and \ \forall m, j > k$

Hence $\langle x_{mi} \rangle$ is a Cauchy sequence in $\mathbb{R}$(or $\mathbb{C}$)

So, it is convergent to $x_i \ in \ F$

Hence , for any $\in > 0, \exists k_i \in Z_+$ such that $|x_{mi} - x_i| < \in, \forall \ m_i > k_i$

$put \ l = \{ k_1, \dots, k_n\}$. then for each $\in > 0$

$|x_{mi} - x_i| < \in, \forall \ m > l, \forall i = 1, \dots, n$

For each $\epsilon > 0$,

$\|x_m - x\| = \{|x_{m1} - x_1|, |x_{m2} - x_2|, \dots, |x_{mn} - x_n|\} < \in, \forall \ \ m > l$

Thus $\langle x_m \rangle$ be a Cauchy sequence in $\mathbb{R}^n$(or $\mathbb{C}^n$)and $x_m \to x.$ Thus, $\mathbb{R}^n$(or $\mathbb{C}^n$) is a Banach space

Example (3.4):-

b . Show that $(l^{\infty}, \|.\|)$ is Banach space where $\|x\| = sup\ |x_i|$, $\forall x = (x_1, x_2, \dots) \in l^{\infty}$?

Solution :- (1) To prove that $(l^{\infty}, \|.\|)$ is normed space **(H.W.)**

(2) To prove that $(l^{\infty}, \|.\|)$ is complete space

Let $< x_m >$ be a Cauchy sequence in $l^{\infty} \Longrightarrow x_m \in l^{\infty}$

$$x_m = (x_{m1}, x_{m2}, \dots, x_{mn}, \dots)$$

$$x_m - x_j = (x_{m1} - x_{j1}, \dots, x_{mn} - x_{jn}, \dots)$$

$$\|x_m - x_j\| = sup\ |x_{mi} - x_{ji}| < \epsilon\ ,\ \ \forall m, j > k$$

$$\Longrightarrow \left|x_{mi} - x_{ji}\right| < \epsilon\ , \forall m\ , j > k$$

$\Longrightarrow \left|x_{mi} - x_{ji}\right| < \epsilon\ , \forall m\ , j > k \Longrightarrow < x_m >$ is Cauchy in F , but F is complete $\Longrightarrow$ $\forall i,\ \exists x_i \in F$ such that

$$x_{mi} \to x_i \Longrightarrow \forall \epsilon > 0, \exists k_i \in Z_+ \text{ such that } |x_{mi} - x_i| < \epsilon$$

Let $k = \{k_1, k_2, \dots\} \Longrightarrow |x_{mi} - x_i| < \epsilon, \qquad \forall\ m > k \dots (1)$

Put $x = (x_1, x_2, \dots)$ to prove that $x \in l^{\infty}$

And $x_m \to x$. Now , since $x_m \in l^{\infty} \Longrightarrow \exists k_m \in \mathbb{R}^+$ such that

$$|x_{mi}| < k_m\ , \forall i\ , \text{but } \ x_i = (x_i - x_{mi}) + \ x_{mi}$$

$$|x_i| \le |x_i - x_{mi}| + |x_{mi}| < \epsilon + k_m \Longrightarrow x \in l^{\infty}$$

By (1) we get, $\quad sup\ |x_{mi} - x_i| < \epsilon\ ,\ \forall m > k$

$\Longrightarrow \|x_m - x\| < \epsilon\ \Longrightarrow\ < x_m >$ is Cauchy sequence

$\Longrightarrow (l^{\infty}, \|.\|)$ is complete. So, $(l^{\infty}, \|.\|)$ is Banach space.

Example (3.5) :-

The space $C[a,b]$ with the norm $\|f\| = \{|f(x)|: x \in [a,b]\}$ $\forall f \in C[a,b]$ is a Banach space

Solution : Let $\langle f_n \rangle$ be a Cauchy sequence in $C[a,b]$

Then $\forall \in > 0, \exists k \in Z_+$ such that $\|f_m - f_n\| < \in, \forall m,n > k$

Hence , $\forall \in > 0, \exists k \in Z_+$ such that:

$\{|f_m(x) - f_n(x)|: x \in C[a,b]\} < \in \ \forall m,n > k$

It follows that $|f_m(x) - f_n(x)| < \in \ \forall x \in C[a,b], \ \ \forall m,n \ > k$

Hence, $\langle f_n(x) \rangle$ is a Cauchy sequence in $\mathbb{R}$.

Since $\mathbb{R}$ is a Banach space , then $\langle f_n(x) \rangle$ is convergent to $f(x)$ in $\mathbb{R}$ thus ,

$\forall \in > 0, \exists k \in N$ such that $|f_m(x) - f(x)| < \in \ \ \forall m \geq k$

Thus , $\|f_m - f\| = \{|f_m(x) - f(x)|: x \in [a,b]\} < \in \ \ \ \forall m \geq k$

Hence , $f_m \to f \ as \ m \ \to \infty$ thus , $C[a,b]$ is a Banach space .

Example(3.6) :-

The space $(C[0,1], \|.\|)$ is not Banach space where $\|f\| = \int_0^1 |f(x)|\, dx$

Solution :- The space $(C[0,1], \|.\|)$ is normed space but not complete space , since there exist Cauchy sequence but not convergent , for example consider

$$f_n(x) = \begin{cases} 1 & if \quad 0 \le x \le \frac{1}{2} \\ -nx + \frac{1}{2}n + 1 & if \quad \frac{1}{2} < x \le \frac{1}{2} + \frac{1}{n} \\ 0 & if \quad \frac{1}{2} + \frac{1}{n} < x \le 1 \end{cases}$$

To prove $f_n(x)$ is Cauchy ? Let $m > n > 3$

$$\|f_m - f_n\| = \int_0^1 |(f_m - f_n)(x)dx|$$

$$= \int_0^1 |f_m(x) - f_n(x)|\, dx$$

$$= \int_0^{\frac{1}{2}} |f_m(x) - f_n(x)|\, dx + \int_{\frac{1}{2}}^1 |f_m(x) - f_n(x)|\, dx$$

$$= \int_0^{\frac{1}{2}} |1 - 1|\, dx + \int_{\frac{1}{2}}^1 |f_m(x) - f_n(x)|\, dx$$

$$\le \int_{\frac{1}{2}}^1 |f_m(x)|\, dx + \int_{\frac{1}{2}}^1 |f_n(x)|\, dx$$

$$= \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{m}} \left|-mx + \frac{1}{2}m + 1\right| \, dx + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} \left|-nx + \frac{1}{2}n + 1\right| dx$$

$$= \frac{1}{2m} + \frac{1}{2n} \ .$$

So , $\|f_m - f_n\| \leq \frac{1}{2m} + \frac{1}{2n}$, as $n, m \to \infty \Longrightarrow \|f_m - f_n\| \to 0$

$\Longrightarrow < f_m >$ is Cauchy sequence

But $f_m \to f$, where $f(x) = \begin{cases} 1 & if\ 0 \leq x \leq \frac{1}{2} \\ 0 & if\ \frac{1}{2} < x \leq 1 \end{cases}$

and f is not continuous function $\to f \notin C[a,b]$

$\Longrightarrow < f_m >$ not convergent $\Longrightarrow C[0,1]$ is not Banach space.

Some Important Theorems in Banach Space

Theorem (3.7): Let H be a subspace of Banach space L . then H is Banach space iff H is closed in L

Proof :- $\Longrightarrow$)

If H is Banach space $\Longrightarrow H$ is complete . To prove that H is closed ?

Let $x \in \overline{H} \Longrightarrow$ there exist a sequence $< x_n >$ in H s. t $x_n \to x$.

So , $< x_n >$ is Cauchy sequence

Since H is complete $\Longrightarrow \exists y \in H \;\; s.t \;\; x_n \to y$, But the limit point is unique

$So, x = y \implies x \in H \implies H = \bar{H} \implies H$ is closed

$\Longleftarrow$) Suppose that H is closed set in L. To prove that H is a Banach space ?

It is clear that H is normed space (because every subspace of normed space is normed space). Now , let $< x_n >$ be a Cauchy sequence in $H \subseteq L$

$\Longrightarrow$ the sequence $< x_n >$ is Cauchy sequence in L , but L is complete.

$\Longrightarrow < x_n >$ is convergent sequence in L . i.e, $\exists x \in L \;\; s.t \;\; x_n \to x$

Since $x_n \in H \Longrightarrow x \in \bar{H}$ (by theorem) , but H is closed $. i.e, \; H = \bar{H}$

$So, x \in H \implies < x_n >$ convergent in $H \implies H$ is complete

Theorem (3.8):- Every finite dimensional normed space is complete space

Proof :- Let $dim \; L = n > 0$ and $\{x_1, x_2, \dots, x_n\}$ basis for L. T.p L is complete

Let $< x_m >$ be a Cauchy sequence in $L \Longrightarrow \|x_m - x_j\| < \in, \quad \forall m, j > k$,

i.e., $\|x_m - x_j\| \to 0$, $\quad \forall m, j > k \qquad \dots.. \qquad (1)$

Since, $x_m, xj \in L$. By previous lemma $\Longrightarrow x_m = \sum_{i=1}^{n} \alpha_{mi} x_i \;\; , \alpha_{mi} \in F$

$x_j = \sum_{i=1}^{n} \alpha_{ji} x_i \;\; , \alpha_{ji} \in F \; and \;\; x_m - x_j = \sum_{i=1}^{n} (\alpha_{mi} - \alpha_{ji}) x_i$

Since $\{x_1, x_2, \dots, x_n\}$ is linearly independent $\Longrightarrow \exists c > 0 \;\; s.t$

$\|x_m - x_j\| = \| \sum_{i=1}^{n} (\alpha_{mi} - \alpha_{ji}) \| \geq c \sum_{i=1}^{n} |\alpha_{mi} - \alpha_{ji}|$ ……… (2)

From (1) & (2) we get $\sum_{i=1}^{n} |\alpha_{mi} - \alpha_{ji}| \to 0 \ \ as\ m, j \to \infty$.

$\Longrightarrow |\alpha_{mi} - \alpha_{ji}| \to 0$ as $m, j \to \infty, \forall i$.

$\Longrightarrow < \alpha_{mi} >$ is Cauchy in F & F is complete

$\Longrightarrow \alpha_{mi} \to \alpha_i, \ \forall i = 1,2, \dots, n$

$i.e, x_m \to x$, where $x = \sum_{i=1}^{n} \alpha_i x_i$

$\Longrightarrow L$ is complete space

Corollary(3.9):- Every finite dimensional subspace of a Banach space is closed set.

Proof :- Let M is finite dimensional $\Longrightarrow M$ complete $\Longrightarrow M$ is closed (by above theorem)

Dentition (3.10) Quotient Space

Let X be a linear space over F. Let H be a subspace of a linear space L. Let $x + H = \{z;\ z = x + y, x \in L , y \in H\}$.

Define addition and scalar multiplication by

$$(x_1 + y) + (x_2 + y) = (x_1 + x_2) + y, \ \ \forall \ \ x_1 + y , \ \ x_2 + y \in L/H$$

$$\alpha.(x + y) = \alpha.x + y, \forall\ x + y \in \frac{L}{H} , \ \alpha \in F$$

Then the space $(\frac{L}{H},+,.)$ is called quotient space (or factor space)

Proposition (3.11) :-

Prove that $(\frac{L}{H},+,\ .)$ is a linear space over F. **(H.W.)**

Theorem(3.12) :- Let L be a normed space and H be a closed subset of L, then L/H is normed space with $\|.\|_1$ where

$\|x+H\|_1 = inf\ inf\ \{\|x+y\|: y \in H\}$

Proof (1) T.P $\|x+H\|_1 \geq 0$

For any $x+H \in L/H$

$\|x+y\| \geq 0,\ \forall y \in H$

$\{\|x+y\|: y \in H\} \geq 0$

$\|x+H\|_1 = inf\ \{\|x+y\|: y \in H\} \geq 0$

(2) T.P $\|x+H\|_1 = 0 \Leftrightarrow x+H = H = 0_{L/H}$

$(\Longrightarrow) If\ \|x+H\|_1 = 0 \Longrightarrow inf\ \{\|x+y\|: y \in H\} = 0$

Hence , $\exists \langle y_n \rangle \in H$ such that $\|x+y_n\| \to 0$ as $n \to 0$

Hence , $x+y_n \to 0$ as $n \to 0$.Thus , $y_n \to -x$.

Therefore , $\exists\langle y_n\rangle \in H$ such that $y_n \to -x$ thus $-x \in \overline{H}$ (by theorem).

Now, since H is closed , then $-x \in \overline{H} = H$, i.e. , $-x \in H$

Since H is a subspace then $x \in H \; and \; x + H = H$, that is , $x + H = 0_{L/H}$

$(\Longleftarrow) If \; x + H = H = 0_{x/y} \; then \; x \in H. i.e., x + H \in H, \; \forall y \in H$

Hence , $\|x + H\|_1 = inf \; \{\|x + y\|: y \in H\} = inf \; \{\|z\|: z \in H\}$

Since $0 \in Y$ and $\|0\| = 0,$ so $inf \; \{\|z\|: z \in H\} = 0.$ Thus , $\|x + H\|_1 = 0$

(3) T.P $\|\alpha.(x + H)\|_1 = |\alpha|\|x + H\|_1, \; \alpha \in F$

If $\alpha = 0$ then (3) holds

If $\alpha \neq 0$ then

$\|\alpha.(x + H)\|_1 = inf \; \{\|\alpha(x + y)\|: y \in H\}$

$= inf \; \{|\alpha|\|x + y\|: y \in H\}$

$= |\alpha| \; inf \; \{\|x + y\|: y \in H\}$

By using the proposition (If A is bounded below and $\alpha \geq 0$, then $inf \; (\alpha A) = \alpha \; inf \; (A)$)

$= |\alpha|\|x + H\|_1$

(4) Let $x_1 + H , x_2 + H \in L/H$

$\|(x_1 + H) + (x_2 + H)\|_1 = \|(x_1 + x_2) + H\|_1$

$inf\ \{\|x_1 + x_2 + y\|: y \in H\}$

$inf\ \{\|x_1 + x_2 + z_1 + z_2\|: z_1, z_2 \in H\}$

$\leq inf\ \{\|x_1 + z_1\| + \|x_2 + z_2\|: z_1, z_2 \in H\}$

$= inf\ \{\|x_1 + z_1\|: z_1\ \in H\} + inf\ \{\|x_2 + z_2\|: z_2 \in H\}$

$= \|x_1\ + H\|_1 + \|x_2 + H\|_1$

Thus L/H is a normed space

Proposition (3.13) :- If L is a Banach space and H is a closed subspace of L .Then L/H is a Banach space.

Proof : $L/H = \{x + H: x \in L\}. Let$ $\langle x_n \rangle$ be a Cauchy sequence in L / H then , $x_n = x_n + H\ where\ x_n \in L, \forall n \in Z_+$

$\forall \in > 0, \exists k \in Z_+$ such that $\|x_m - x_n\| < \in \forall n, m > k$

So , $\forall \in > 0, \exists k \in Z_+$ such that $\|x_m - x_n + H\| < \in \forall n, m > k$

Then , $\forall \in > 0, \exists k \in Z_+$ such that:

$inf\ \{\|x_m - x_n + H\|: y \in H\} < \in \forall n, m > k$ .This implies $\forall y \in H,$ $\langle x_n + y \rangle$ is a Cauchy in L

Since L is a Banach space , then $\exists z \in L$ such that $x_n + y \to z = (z - y) + y$

$= w + y, \ \forall y \in H$

Thus , $x_n + H \to w + H$. Thus L / H is a Banach space .

Chapter 4

Inner Product Space

Definition 4.1.
Let L is a linear space over F. A mapping $\langle\ ,\rangle : L \times L \to F$ is called an **inner product on** L if the following axioms hold

(1) $\langle x, x\rangle \geq 0 \quad \forall x \in L$.

(2) $\langle x, x\rangle = 0 \iff x = 0_L$.

(3) $\overline{\langle x, y\rangle} = \langle y, x\rangle \quad \forall x, y \in L$, where $\overline{\langle x, y\rangle}$ =conjugate of $\langle x, y\rangle$.

(4) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle \quad \forall x, y, z \in L$ and $\alpha, \beta \in F$.

$(L, \langle\ ,\rangle)$ is called **inner product space** (briefly, I.P.S) or **Pre-Hilbert space**.

Remark 4.2.

(1) If $F = \mathbb{R}$ then axiom (3) becomes $\langle x, y\rangle = \langle y, x\rangle \quad \forall x, y \in L$.

(2) Every subspace of inner product space is an inner product space.

4.1 Examples of Inner Product Space

Example 4.3.

Let $L = \mathbb{R}^2$ and let $\langle \ , \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to F$ is defined as $\langle x, y\rangle = x_1y_1 + x_2y_2 \quad \forall x, y \in \ \mathbb{R}^2$ where $x = (x_1, x_2), y = (y_1, y_2)$. Show that $\langle \ , \rangle$ is an inner product on $\mathbb{R}^2$.

<u>**Solution:**</u> (i) We check the I.P.S axioms

(1) $\langle x, x\rangle = x_1^2 + x_2^2 \geq 0 \quad \forall x = (x_1, x_2) \in \mathbb{R}^2$

(2) $\langle x, x\rangle = 0 \iff x_1^2 + x_2^2 = 0 \iff x_1 = x_2 = 0 \iff x = (0, 0)$

(3) $\langle x, y\rangle = x_1y_1 + x_2y_2 = \overline{\langle x, y\rangle}$ (since $F = \mathbb{R}$)

(4) Let $\alpha, \beta \in \mathbb{R}$ and let $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2)$

$$\begin{aligned}\langle \alpha x + \beta y, z\rangle &= \langle(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2), (z_1, z_2)\rangle \\ &= (\alpha x_1 + \beta y_1)z_1 + (\alpha x_2 + \beta y_2)z_2 \\ &= (\alpha x_1z_1 + \alpha x_2z_2) + (\beta y_1z_1 + \beta y_2z_2) \\ &= \alpha(x_1z_1 + x_2z_2) + \beta(y_1z_1 + y_2z_2) \\ &= \alpha\langle x, z\rangle + \beta\langle y, z\rangle\end{aligned}$$

Thus, $\langle \ , \rangle$ is an inner product on $\mathbb{R}^2$.

<u>**As an application to Example 4.3:**</u>

Let $x = (2, 1), y = (0, -3), z = (3, 4)$. Find $\langle x, z\rangle, \langle x, x\rangle, \langle x + y, z\rangle$.

Remark 4.4.

As a generalization of Example 4.3, let $L = \mathbb{R}^n$ and $\langle\ ,\ \rangle : \mathbb{R}^n \times \mathbb{R}^n \to F$ is defined as $\langle x, y\rangle = x_1y_1 + x_2y_2 + ... + x_ny_n \quad \forall x, y \in \mathbb{R}^n$ where $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$. Then, $(\mathbb{R}^n, \langle\ ,\ \rangle)$ is an inner product space (**check!**). The space $(\mathbb{R}^n, \langle\ ,\ \rangle)$ is called **usual inner space**.

Example 4.5.

Let $L = \mathbb{R}^2$, which of the following is an inner product on L.

(i) $\langle x, y\rangle = 3x_1y_1 + x_2y_2$ (**H.W.**)

(ii) $\langle x, y\rangle = x_1^2y_1^2 + x_2^2y_2^2$

where $x = (x_1, x_2), y = (y_1, y_2)$

Solution: (i) We check the I.P.S axioms

(ii) The first three axioms of the definition of inner product hold but the forth condition does not satisfy.

If $\alpha = \beta = 1$ and let $x = (1, -1), y = (-1, 0), z = (-2, 2)$. Then

$\langle \alpha x + \beta y, z\rangle = \langle (0, -1), (-2, 2)\rangle = 0^2(-2)^2 + (-1)^2 2^2 = 4$

and $\alpha\langle x, z\rangle + \beta\langle y, z\rangle = \langle (1, -1), (-2, 2)\rangle + \beta\langle (-1, 0), (-2, 2)\rangle$

$$= 1^1.(-2)^2 + (-1)^2.2^2 + (-1)^2.2^2 + 0^2.2^{=}12$$

Thus, $\langle \alpha x + \beta y, z\rangle \neq \alpha\langle x, z\rangle + \beta\langle y, z\rangle$.

Example 4.6.

Let $L = F^n$ be a linear space and let $\langle\ ,\rangle : F^n \times F^n \to F$ defined as

$\langle x, y\rangle = \sum_{i=1}^{n} x_i\overline{y}_i = x_1\overline{y}_1 + ... + x_n\overline{y}_n \quad \forall x, y \in\ F^n$ where $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$. Show that $\langle\ ,\rangle$ is an inner product on F^n.

<u>**Solution**</u>:

(1) $\langle x, x\rangle = \sum_{i=1}^{n} x_i\overline{x}_i = \sum_{i=1}^{n} |x_i|^2 \geq 0$

(2) $\langle x, x\rangle = 0 \iff \sum_{i=1}^{n} |x_i|^2 = 0 \iff x_i = 0\ \ \forall i = 1, ..., n$

$$\iff x = (x_1, ..., x_n) = (0, ..., 0) = 0_{F^n}$$

(3) $\langle \overline{x, y}\rangle = \overline{\sum_{i=1}^{n} x_i\overline{y}_i} = \sum_{i=1}^{n} \overline{x}_i y_i = \sum_{i=1}^{n} y_i\overline{x}_i = \langle y, x\rangle$

(4) Let $\alpha, \beta \in F$ and let $x, y, z \in F^n$

$\alpha x + \beta y = (\alpha x_1 + \beta y_1, ..., \alpha x_n + \beta y_n)$

$\langle \alpha x + \beta y, z\rangle = \sum_{i=1}^{n} (\alpha x_i + \beta y_i)\overline{z}_i = \alpha \sum_{i=1}^{n} x_i\overline{z}_i + \beta \sum_{i=1}^{n} y_i\overline{z}_i = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$.

Thus, $\langle\ ,\rangle$ is an inner product on C^n.

<u>**As an application to Example 4.6:**</u>

Let $L = C^2$ and $\langle x, y\rangle = \sum_{i=1}^{2} x_i\overline{y}_i \quad \forall x, y \in\ C^2$ where $x = (x_1, x_2), y = (y_1, y_2)$. If $x = (2 + 3i, 1 + i), y = (1 + i, 1 - i), z = (2, 1 + i)$

Find $\langle x, x\rangle, \langle x + y, z\rangle, \langle x, y + z\rangle$

<u>**Solution**</u>: $\langle x, x\rangle = (2 + 3i)(\overline{2 + 3i}) + (1 + i)(\overline{1 + i})$

$$= (2 + 3i)(2 - 3i) + (1 + i)(1 - i)$$

$$= (4 + 9) + (1 + 1) = 15$$

$x + y = (3 + 4i, 2)$

$\langle x + y, z\rangle = (3 + 4i)2 + 2(\overline{1 + i}) = (6 + 8i) + 2(1 - i) = 8 + 6i$

$\langle x, y + z\rangle =$

Example 4.7.

Let $L = C[0,1]$ be a linear space over $\mathbb{R}$, and let $\langle \ , \ \rangle : L \times L \to \mathbb{R}$ is defined by $\langle f, g\rangle = \displaystyle\int_0^1 f(x)g(x)\,dx$. Prove that $\langle \ , \ \rangle$ is an inner product on L.

Solution: (1) $\langle f, f\rangle = \displaystyle\int_0^1 f(x)f(x)\,dx = \int_0^1 [f(x)]^2\,dx \geq 0$

(2) $\langle f, f\rangle = 0 \iff \displaystyle\int_0^1 [f(x)]^2\,dx = 0 \iff [f(x)]^2 = 0 \ \forall x \in [0,1]$

$$\iff f(x) = 0 \ \forall x \in [0,1] \iff f = \hat{0}$$

(3) Let $\alpha, \beta \in \mathbb{R}$ and $f, g, h \in L$

$$\langle \alpha f + \beta g, h\rangle = \int_0^1 (\alpha f + \beta g)(x)h(x)\,dx$$

$$= \int_0^1 (\alpha f(x) + \beta g(x)\ h(x)\,dx$$

$$= \alpha \int_0^1 f(x)\ h(x)\,dx + \beta \int_0^1 g(x)\ h(x)\,dx$$

$$= \alpha\langle f, h\rangle + \beta\langle g, h\rangle$$

(4) $\langle f, g\rangle = \displaystyle\int_0^1 f(x)g(x)\,dx = \int_0^1 g(x)f(x)\,dx = \langle g, f\rangle$

As an application to Example 4.7:

Let $f(x) = x + 1, \;\; g(x) = x^2, \;\; h(x) = 3x + 2 \quad \forall x \in [0, 1]$

Find $\langle f, f\rangle, \langle f + g, h\rangle, \langle f, h\rangle, \langle 2f + 3g, h\rangle, \langle f - g, h - g\rangle$

Example 4.8.

Let $L = \mathbb{R}$ and $\langle\; ,\; \rangle : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $\langle x, y\rangle = |xy| \quad \forall x, y \in \mathbb{R}$. Is $(L, \langle\; ,\; \rangle)$ I.P.S? (**H.W.**)

4.2 Some Properties of Inner Product Space

Theorem 4.9.

Let $(L, \langle\ ,\ \rangle)$ be an inner product space (I.P.S). Then, $\forall x, y, z \in L$ and $\alpha, \beta \in F$

(1) $\langle x, 0_L\rangle = \langle 0_L, x\rangle = 0$

(2) $\langle x, \alpha y + \beta z\rangle = \overline{\alpha}\langle x, y\rangle + \overline{\beta}\langle x, z\rangle$.

Proof. (1) $\langle 0_L, x\rangle = \langle 0_L + 0_L, x\rangle$

$$= \langle 0_L, x\rangle + \langle 0_L, x\rangle$$

Hence, $\langle 0_L, x\rangle + 0 = \langle 0_L, x\rangle + \langle 0_L, x\rangle$

Thus, $0 = \langle 0_L, x\rangle$ (I)

Now, $\overline{\langle 0_L, x\rangle} = \langle x, 0_L\rangle$

$$\overline{0} = \langle x, 0_L\rangle$$

$$0 = \langle x, 0_L\rangle$$

(2) $\langle x, \alpha y + \beta z\rangle = \overline{\langle \alpha y + \beta z, x\rangle}$

$$= \overline{\alpha\langle y, x\rangle + \beta\langle z, x\rangle}$$

$$= \overline{\alpha}\ \overline{\langle y, x\rangle} + \overline{\beta}\ \overline{\langle z, x\rangle}$$

$$= \overline{\alpha}\ \langle x, y\rangle + \overline{\beta}\ \langle x, z\rangle$$

□

Corollary 4.10.

If $(L, \langle\ ,\ \rangle)$ is an I.P.S. Then

(i) $\left\langle \sum_{i=1}^{n} \alpha_i x_i, y\right\rangle = \sum_{i=1}^{n} \alpha_i\langle x_i, y\rangle$ where $x_1, ..., x_n, y \in L$

(ii) $\big\langle x, \sum_{i=1}^{n} \beta_i y_i \big\rangle = \sum_{i=1}^{n} \overline{\beta}_i \big\langle x, y_i \big\rangle$ where $x, y_1, ..., y_n \in L$

(iii) $\big\langle \sum_{i=1}^{n} \alpha_i x_i, \sum_{j=1}^{m} \beta_j y_j \big\rangle = \sum_{i=1}^{n} \alpha_i \big(\sum_{j=1}^{m} \overline{\beta}_j \langle x_i, y_j \rangle \big)$

where $x_1, ..., x_n, y_1, ..., y_m \in L$

Proof. (i) We proof using induction.

If $n = 1$ then $\langle \alpha_1 x_1, y \rangle = \alpha_1 \langle x_1, y \rangle$ (by definition of norm)

If $n = 2$ then $\langle \alpha_1 x_1 + \alpha_2 x_2, y \rangle = \alpha_1 \langle x_1, y \rangle + \alpha_2 \langle x_2, y \rangle$ (by definition of norm)

Suppose (i) hold when $n = k$

$\big\langle \sum_{i=1}^{k} \alpha_i x_i, y \big\rangle = \sum_{i=1}^{k} \alpha_i \big\langle x_i, y \big\rangle$ (**I**)

To prove (i) hold when $n = k + 1$

T.p. $\big\langle \sum_{i=1}^{k+1} \alpha_i x_i, y \big\rangle = \sum_{i=1}^{k+1} \alpha_i \big\langle x_i, y \big\rangle$

$$\begin{aligned}
\big\langle \sum_{i=1}^{k+1} \alpha_i x_i, y \big\rangle &= \big\langle \sum_{i=1}^{k} \alpha_i x_i + \alpha_{k+1} x_{k+1}, y \big\rangle \\
&= \big\langle \sum_{i=1}^{k} \alpha_i x_i, y \big\rangle + \big\langle \alpha_{k+1} x_{k+1}, y \big\rangle \\
&= \sum_{i=1}^{k} \alpha_i \big\langle x_i, y \big\rangle + \alpha_{k+1} \big\langle x_{k+1}, y \big\rangle \\
&= \sum_{i=1}^{k+1} \alpha_i \big\langle x_i, y \big\rangle
\end{aligned}$$

(ii) The proof is similar to the proof of (i).

(iii) Let $z = \sum_{j=1}^{m} \beta_j y_j$

$$\begin{aligned}
\big\langle \sum_{i=1}^{n} \alpha_i x_i, \sum_{j=1}^{m} \beta_j y_j \big\rangle &= \big\langle \sum_{i=1}^{n} \alpha_i x_i, z \big\rangle \\
&= \sum_{i=1}^{n} \alpha_i \big\langle x_i, z \big\rangle \qquad \text{(by part (i))}
\end{aligned}$$

$$= \sum_{i=1}^{n} \alpha_i \langle x_i, \sum_{j=1}^{m} \beta_j y_j \rangle$$

$$= \sum_{i=1}^{n} \alpha_i \Big(\sum_{j=1}^{m} \overline{\beta}_j \langle x_i, y_j \rangle \Big) \qquad \text{(by part (ii))}$$ □

Theorem 4.11.

Let $(L, \langle\ ,\ \rangle)$ is an I.P.S. such that $\langle v_1, w\rangle = \langle v_2, w\rangle \quad \forall w \in L$. Then $v_1 = v_2$. Also, if $\langle v_1, w\rangle = 0 \quad \forall w \in L$ then $v_1 = 0_L$.

Proof. By assumption, $\langle v_1 - v_2, w\rangle = \langle v_1, w\rangle - \langle v_2, w\rangle = 0, \quad \forall w \in L$.

Put $w = v_1 - v_2$, then $\langle v_1 - v_2, v_1 - v_2\rangle = 0 \implies v_1 - v_2 = 0 \implies v_1 = v_2$.

Now, $\langle v_1, w\rangle = 0, \quad \forall w \in L \implies \langle v_1, v_1\rangle = 0 \implies v_1 = 0_L$. □

Theorem 4.12. ***General Cauchy Schwarz's Inequality***

Let $(L, \langle\ ,\ \rangle)$ is an I.P.S. and let $\|\ \| : L \to \mathbb{R}$ is defined by $\|x\| = \sqrt{\langle x, x\rangle} \quad \forall x \in L$. Then,

$$\big|\langle x, y\rangle\big| \le \|x\|\,\|y\| \quad \forall x, y \in L.$$

Proof. If $x = 0$ or $y = 0$ then $\langle x, y\rangle = 0$, and hence $\langle x, y\rangle = 0 \le \|x\|\,\|y\|$

If $y \ne 0$, put $z = \dfrac{y}{\|y\|} \qquad$ **(I)**

$$\|z\|^2 = \langle z, z\rangle = \langle \frac{y}{\|y\|}, \frac{y}{\|y\|} \rangle$$

$$= \frac{1}{\|y\|^2}\langle y, y\rangle = \frac{1}{\|y\|^2}\|y\|^2 = 1 \qquad \textbf{(II)}$$

Next, it is enough to show that $\big|\langle x, z\rangle\big| \le \|x\|$

because if $\big|\langle x, z\rangle\big| \le \|x\|$ then from **(I)**

$$\big|\langle x, z\rangle\big| = \left|\langle x, \frac{y}{\|y\|}\rangle\right| = \frac{1}{\|y\|}\big|\langle x, y\rangle\big| \le \|x\|$$

$$\big|\langle x, y\rangle\big| \le \|x\|\,\|y\|$$

Let $\alpha \in F$ then $\langle x-\alpha z, x-\alpha z\rangle \geq 0$

$$\langle x,x\rangle - \alpha\langle z,x\rangle - \overline{\alpha}\langle x,z\rangle + \alpha\overline{\alpha}\langle z,z\rangle \geq 0$$

$$\|x\|^2 - \overline{\alpha}\langle x,z\rangle - \alpha\langle z,x\rangle + \alpha\overline{\alpha} \underbrace{\|z\|^2}_{=1 \text{ from (I)}} \geq 0$$

$$\|x\|^2 - \langle x,z\rangle\overline{\langle x,z\rangle} + \langle x,z\rangle\overline{\langle x,z\rangle} - \overline{\alpha}\langle x,z\rangle - \alpha\langle z,x\rangle + \alpha\overline{\alpha} \geq 0$$

$$\|x\|^2 - \left|\langle x,z\rangle\right|^2 + \langle x,z\rangle\left(\overline{\langle x,z\rangle} - \overline{\alpha}\right) - \alpha\left(\langle z,x\rangle - \overline{\alpha}\right)) \geq 0$$

$$\|x\|^2 - \left|\langle x,z\rangle\right|^2 + \langle x,z\rangle\left(\overline{\langle x,z\rangle - \alpha}\right) - \alpha\left(\overline{\langle x,z\rangle - \alpha}\right)) \geq 0$$

$$\|x\|^2 - \left|\langle x,z\rangle\right|^2 + \left(\langle x,z\rangle - \alpha\right)\left(\overline{\langle x,z\rangle - \alpha}\right) \geq 0$$

$$\|x\|^2 - \left|\langle x,z\rangle\right|^2 + \left|\langle x,z\rangle - \alpha\right|^2 \geq 0 \quad \forall \alpha \in F \quad \textbf{(III)}$$

Put $\alpha = \langle x,z\rangle$, then (**III**) becomes

$$\|x\|^2 - \left|\langle x,z\rangle\right|^2 \geq 0 \implies \left|\langle x,z\rangle\right|^2 \leq \|x\|^2$$

$$\left|\langle x,z\rangle\right| \leq \|x\|$$

$$\left|\langle x, \frac{y}{\|y\|}\rangle\right| \leq \|x\| \quad \text{(using } \textbf{(I)}\text{)}$$

$$\left|\langle x,y\rangle\right| \frac{1}{\|y\|} \leq \|x\|$$

$$\left|\langle x,y\rangle\right| \leq \|x\| \, \|y\| .$$

□

As an application to Theorem 4.12:

If $L = \mathbb{R}^n$ and $\langle x,y\rangle = \sum_{i=1}^n x_i y_i$ for any $X = (x_1, ..., x_n), Y = (y_1, ..., y_n)$. Apply Cauchy Schwarz inequality.

Sloution: We have , $\|x\| = \left[\langle x,x\rangle\right]^{\frac{1}{2}} = \left[\sum_{i=1}^n x_i^2\right]^{\frac{1}{2}}$ and $\|y\| = \left[\langle y,y\rangle\right]^{\frac{1}{2}} = \left[\sum_{i=1}^n y_i^2\right]^{\frac{1}{2}}$

From Theorem 4.12, $\left|\langle x,y\rangle\right| \leq \|x\| \, \|y\|$; that is

$$\left|\sum_{i=1}^{n} x_i y_i\right| \leq \left[\sum_{i=1}^{n} x_i^2\right]^{\frac{1}{2}} \left[\sum_{i=1}^{n} y_i^2\right]^{\frac{1}{2}}$$

Theorem 4.13.

Every inner product space is a normed space and hence a metric space.

Proof. Let $(L, \langle \ , \ \rangle)$ is an I.P.S. and let the function $\| \ \| : L \to \mathbb{R}$ is defined by

$\|x\| = \sqrt{\langle x, x\rangle} \quad \forall x \in L$. To prove $\| \ \|$ is a norm on L

(1) Since $\langle x, x\rangle \geq 0 \quad \forall x \in L \implies \|x\| = \sqrt{\langle x, x\rangle} \geq 0 \quad \forall x \in L$

(2) $\|x\| = 0 \iff \sqrt{\langle x, x\rangle} = 0 \iff \langle x, x\rangle = 0 \iff x = \mathbf{0_X}$

(3) Let $\alpha \in F$ and $x \in L$

$\|\alpha x\|^2 = \langle \alpha x, \alpha x\rangle = \alpha\overline{\alpha}\langle x, x\rangle = |\alpha|^2 \|x\|^2$

Thus, $\|\alpha x\| = |\alpha| \, \|x\|$

(4) T.P. $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in L$

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle \\
&= \langle x, x\rangle + \langle y, x\rangle + \langle x, y\rangle + \langle y, y\rangle \\
&= \|x\|^2 + \overline{\langle x, y\rangle} + \langle x, y\rangle + \|y\|^2 \\
&= \|x\|^2 + 2Re\langle x, y\rangle + \|y\|^2 \\
&\leq \|x\|^2 + 2\left|\langle x, y\rangle\right| + \|y\|^2 \\
&\leq \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2 \quad \text{(by Cauchy Schwarz)}
\end{aligned}$$

$$= (\|x\| + \|y\|)^2$$

Thus, $\|x+y\| \le \|x\| + \|y\|$. □

Theorem 4.14.

Let $(L, \langle\ ,\ \rangle)$ is an I.P.S. and $x, y \in L$. Then

(1) $\|x+y\|^2 = \|x\|^2 + 2Re\langle x, y\rangle + \|y\|^2$ (Polarization Identity)

(2) $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ (Law of Parallelogram)

(3) $\langle x, y\rangle = \frac{1}{4}\left[\, \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 \,\right]$

Proof. (1) $\|x+y\|^2 = \langle x+y, x+y\rangle$

$$= \langle x, x\rangle + \langle y, x\rangle + \langle x, y\rangle + \langle y, y\rangle$$

$$= \|x\|^2 + \overline{\langle x, y\rangle} + \langle x, y\rangle + \|y\|^2$$

$$= \|x\|^2 + 2Re\langle x, y\rangle + \|y\|^2$$

(2) T.P. $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

By part (1), $\|x+y\|^2 = \|x\|^2 + 2Re\langle x, y\rangle + \|y\|^2$ (**I**)

$$\|x-y\|^2 = \langle x-y, x-y\rangle$$

$$= \langle x, x\rangle - \langle y, x\rangle - \langle x, y\rangle + \langle y, y\rangle$$

$$= \|x\|^2 - \overline{\langle x, y\rangle} - \langle x, y\rangle + \|y\|^2$$

$$= \|x\|^2 - 2Re\langle x, y\rangle + \|y\|^2 \quad (\mathbf{II})$$

By summing up (**I**) and (**II**) we get $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

(3) By parts (1) and (2), we have

$$\|x+y\|^2 - \|x-y\|^2 = \|x\|^2 + 2Re\langle x,y\rangle + \|y\|^2 - \left(\|x\|^2 - 2Re\langle x,y\rangle + \|y\|^2\right)$$

$$= 2Re\langle x,y\rangle + 2Re\langle x,y\rangle$$

$$= \langle\overline{x,y}\rangle + \langle x,y\rangle + \langle\overline{x,y}\rangle + \langle x,y\rangle$$

$$= 2\langle y,x\rangle + 2\langle x,y\rangle \qquad (\mathbf{I})$$

$$\|x+iy\|^2 = \langle x+iy, x+iy\rangle$$

$$= \langle x,x\rangle + i\langle y,x\rangle + \bar{i}\langle x,y\rangle + \langle y,y\rangle$$

$$= \|x\|^2 + i\langle y,x\rangle - i\langle x,y\rangle + \|y\|^2$$

$$\|x-iy\|^2 = \langle x-iy, x-iy\rangle$$

$$= \langle x,x\rangle - i\langle y,x\rangle - \bar{i}\langle x,y\rangle + \langle y,y\rangle$$

$$= \|x\|^2 - i\langle y,x\rangle + i\langle x,y\rangle + \|y\|^2$$

Hence we get,

$$i\,\|x+iy\|^2 - i\,\|x-iy\|^2 = i\left[\,\|x\|^2 + i\langle y,x\rangle - i\langle x,y\rangle + \|y\|^2\,\right] - i\left[\,\|x\|^2 - i\langle y,x\rangle + i\langle x,y\rangle + \|y\|^2\,\right]$$

$$= i\,\|x\|^2 - \langle y,x\rangle + \langle x,y\rangle + i\,\|y\|^2 - i\,\|x\|^2 - \langle y,x\rangle + \langle x,y\rangle - i\,\|y\|^2$$

$$= 2\langle x,y\rangle - 2\langle y,x\rangle \qquad (\mathbf{II})$$

By (**I**) and (**II**), we have

$\|x+y\|^2 - \|x-y\|^2 + i\,\|x+iy\|^2 - i\,\|x-iy\|^2 = 2\langle y,x\rangle + 2\langle x,y\rangle + 2\langle x,y\rangle - 2\langle y,x\rangle$

$$\|x+y\|^2 - \|x-y\|^2 + i\,\|x+iy\|^2 - i\,\|x-iy\|^2 = 4\langle x,y\rangle$$

$$\frac{1}{4}\,\|x+y\|^2 - \|x-y\|^2 + i\,\|x+iy\|^2 - i\,\|x-iy\|^2 = \langle x,y\rangle \qquad \square$$

Remark 4.15.
Any normed linear space generated from inner product space must satisfies the three laws of Theorem 4.14.

Example 4.16.
Let $L = C[a,b]$ and let $\|f\| = \max\{|f(x)| : x \in [a,b]\}$. Then the converse of Theorem 4.13. i.e.,

(1) Show that $(L, \|\ \|)$ is a normed linear space (**H.W.**)

(2) Show that L is not generated by I.P.S (i.e, L is not I.P.S)

<u>**Solution**</u>: (2) To show that L is not I.P.S, we shall show that parallelogram law does not hold. i.e., $\|f+g\|^2 + \|f-g\|^2 \neq 2\,\|f\|^2 + 2\,\|g\|^2$ for some $f, g \in C[a,b]$.
Let $f(x) = 1$ and $g(x) = \dfrac{x-a}{b-a} \quad \forall x \in [a,b]$
Note that f, g are continuous on $[a,b]$. Thus, $f, g \in C[a,b]$.
$\|f\| = 1$ and $\|g\| = 1$

$$\|f+g\| = \left\|1 + \frac{x-a}{b-a}\right\| = \max\left\{\left|1 + \frac{x-a}{b-a}\right| : x \in [a,b]\right\} = 2$$

$$\|f-g\| = \left\|1 - \frac{x-a}{b-a}\right\| = \max\left\{\left|1 - \frac{x-a}{b-a}\right| : x \in [a,b]\right\} = 1$$

$\|f+g\|^2 + \|f-g\|^2 = 4+1 = 5$ (**I**)

$2\|f\|^2 + 2\|g\|^2 = 2.1^2 + 2.1^2 = 4$ (**II**)

By (**I**) and (**II**), we get $\|f+g\|^2 + \|f-g\|^2 \neq 2\|f\|^2 + 2\|g\|^2$

i.e., $5 \neq 4$

Example 4.17.

Let $L = \mathbb{R}^2$ and let $\|x\| = |x_1| + |x_2| \quad \forall x = (x_1, x_2) \in \mathbb{R}^2$. Then the converse of Theorem 4.13. i.e.,

(1) Show that $(\mathbb{R}^2, \|\ \|)$ is a normed linear space (**H.W.**)

(2) Show that $\mathbb{R}^2$ is not generated by I.P.S (i.e, $\mathbb{R}^2$ is not I.P.S)

Solution: (2) To show that L is not I.P.S, we shall show that parallelogram law does not hold. i.e., $\|x+y\|^2 + \|x-y\|^2 \neq 2\|x\|^2 + 2\|y\|^2$ for some $x, y \in \mathbb{R}^2$

Let $x = (2,3)$ and $y = (-6,1)$

$\|x\| = |2| + |3| = 5 \implies 2\|x\|^2 = 50$

$\|y\| = |-6| + |1| = 7 \implies 2\|y\|^2 = 98$

$\|x+y\| = \|(-4,4)\| = |-4| + |4| = 8$

$\|x+y\|^2 = 64$

$\|x-y\| = \|(8,2)\| = |8| + |2| = 10$

$\|x-y\|^2 = 100$

Thus, $\|x+y\|^2 + \|x-y\|^2 = 64 + 100 = 164$

and $2\|x\|^2 + 2\|y\|^2 = 50 + 98 = 148$

Hence, $\|x+y\|^2 + \|x-y\|^2 \neq 2\|x\|^2 + 2\|y\|^2$

i.e., $\|\ \|$ does not satisfy paralleogram law.

Example 4.18.

Let $L = \mathbb{R}^2$ and let $\|x\| = \max\{|x_1|, |x_2|\} \quad \forall (x_1, x_2) \in \mathbb{R}^2$. Then

(1) Show that $(\mathbb{R}^2, \| \ \|)$ is a normed linear space (**H.W.**)

(2) Is $\mathbb{R}^2$ generated by I.P.S? (**H.W.**)

Theorem 4.19.

Let $(L, \langle\ ,\ \rangle)$ is an I.P.S. Then

(1) If $(x_n) \to x$ and $(y_n) \to y$ then $\langle x_n, y_n \rangle \to \langle x, y \rangle$

(2) If (x_n) and (y_n) are Cauchy sequences in L then $\langle x_n, y_n \rangle$ is a Cauchy sequence in F.

Proof. (1) $\langle x_n, y_n \rangle = \langle x + (x_n - x), y + (y_n - y) \rangle$

$$= \langle x, y \rangle + \langle x, y_n - y \rangle + \langle x_n - x, y \rangle + \langle x_n - x, y_n - y \rangle$$

$$\langle x_n, y_n \rangle - \langle x, y \rangle = \langle x, y_n - y \rangle + \langle x_n - x, y \rangle + \langle x_n - x, y_n - y \rangle$$

$$\left| \langle x_n, y_n \rangle - \langle x, y \rangle \right| = \left| \langle x, y_n - y \rangle + \langle x_n - x, y \rangle + \langle x_n - x, y_n - y \rangle \right|$$

$$\leq \left| \langle x, y_n - y \rangle \right| + \left| \langle x_n - x, y \rangle \right| + \left| \langle x_n - x, y_n - y \rangle \right|$$

$$\leq \|x\| \, \|y_n - y\| + \|x_n - x\| \, \|y\| + \|x_n - x\| \, \|y_n - y\|$$ (By Cauchy Schwarz)

But $(x_n) \to x$ and $(y_n) \to y$ then $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$

Hence, $\left| \langle x_n, y_n \rangle - \langle x, y \rangle \right| \to 0$, and hence, $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

(2) for any $n, m \in Z_+$

$$\langle x_n, y_n \rangle = \langle (x_n - x_m) + x_m, (y_n - y_m) + y_m \rangle$$

$$= \langle x_n - x_m, y_n - y_m \rangle + \langle x_m, y_m \rangle + \langle x_m, y_n - y_m \rangle + \langle x_n - x_m, y_m \rangle$$

$$\langle x_n, y_n \rangle - \langle x_m, y_m \rangle = \langle x_n - x_m, y_n - y_m \rangle + \langle x_m, y_n - y_m \rangle + \langle x_m, y_n - y_m \rangle$$

$$\left| \langle x_n, y_n \rangle - \langle x_m, y_m \rangle \right| = \left| \langle x_n - x_m, y_n - y_m \rangle + \langle x_m, y_n - y_m \rangle + \langle x_m, y_n - y_m \rangle \right|$$

$$\leq \left| \langle x_n - x_m, y_n - y_m \rangle \right| + \left| \langle x_m, y_n - y_m \rangle \right| + \left| \langle x_m, y_n - y_m \rangle \right|$$

$\leq \|x_n - x_m\| \|y_n - y_m\| + \|x_m\| \|y_n - y_m\| + \|x_n - x_m\| \|y_m\|$ (By Cauchy Schwarz)

But (x_n) and (y_n) are Cauchy sequences, then $\|x_n - x_m\| \to 0$ and $\|y_n - y_m\| \to 0$ as $n \to \infty$. Also, (x_n) and (y_n) are bounded sequences, then as $n \to \infty$

$$\left|\langle x_n, y_n\rangle - \langle x_m, y_m\rangle\right| \to 0$$ □

Corollary 4.20.

Let $(L, \langle\ ,\ \rangle)$ is an I.P.S. Then

(1) If $(x_n) \to x$ then $\|x_n\| \to \|x\|$

(2) If (x_n) is a Cauchy sequences in L then $\langle\ \|x_n\|\ \rangle$ is a convergent sequence in $\mathbb{R}$.

Proof. (1) Since $(x_n) \to x$ then $\langle x_n, x_n\rangle \to \langle x, x\rangle$ (By Theorem 4.19)

Hence, $\|x_n\|^2 \to \|x\|^2$. i.e., $\|x_n\| \to \|x\|$

(2) Since (x_n) is a Cauchy sequences in L, then by Theorem 4.19(2), $\langle x_n, x_n\rangle$ is a Cauchy sequence in F. Since $F = \mathbb{R}$ or C then F is complete. Thus, $\langle \|x_n\|^2 \rangle$ is a convergent sequence in F. Thus, $\langle\ \|x_n\|\ \rangle$ is a convergent sequence in F □

4.3 Hilbert Space

Definition 4.21.
Hilbert space is an I.P.S. $(L, \langle\ ,\ \rangle)$ which is a Banach space with respect to $\|x\| = \sqrt{\langle x, x\rangle}$.

Example 4.22.

Consider the I.P.S. $(\mathbb{R}^n, \langle\ ,\ \rangle)$ (or $(C^n, \langle\ ,\ \rangle)$ such that $\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y}_i$ where $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{R}^n$ (or C^n). (see Example 4.6)

Show that $(\mathbb{R}^n, \langle\ ,\ \rangle)$ (or $(C^n, \langle\ ,\ \rangle)$ is Hilbert space.

Solution: Since $\sqrt{\langle x, x\rangle} = \left[\sum_{i=1}^{n} x_i \overline{x}_i\right]^{\frac{1}{2}} = \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} = \|x\|$

From Example 3.2, $\mathbb{R}^n$ (or C^n) is a Banach space w.r.t. $\|x\| = \sqrt{\langle x, x\rangle}$, and thus, $(\mathbb{R}^n, \langle\ ,\ \rangle)$ (or $(C^n, \langle\ ,\ \rangle)$ is a Hilbert space.

Example 4.23.

The space $C[-1, 1]$ with the inner product defined by $\langle f, g\rangle = \displaystyle\int_{-1}^{1} f(x)\ g(x)\ dx$ is not a Hilbert space.

Solution: Let

$$f_n(x) = \begin{cases} 0 & \text{if} \quad -1 \leq x \leq 0 \\ nx & \text{if} \quad 0 < x < \frac{1}{n} \\ 1 & \text{if} \quad \frac{1}{n} \leq x \leq 1 \end{cases}$$

$\|f_n - f_m\|^2 = \langle f_n - f_m, f_n - f_m\rangle$

Suppose $n > m$, then $\dfrac{1}{n} < \dfrac{1}{m}$. We must find $f_n(x) - f_m(x)$

$$f_n(x) = \begin{cases} 0 & \text{if} \quad -1 \leq x \leq 0 \\ nx & \text{if} \quad 0 < x < \frac{1}{n} \\ 1 & \text{if} \quad \frac{1}{n} \leq x \leq 1 \end{cases}$$

and

$$f_m(x) = \begin{cases} 0 & \text{if} \quad -1 \leq x \leq 0 \\ mx & \text{if} \quad 0 < x < \frac{1}{m} \\ 1 & \text{if} \quad \frac{1}{m} \leq x \leq 1 \end{cases}$$

Then

$$f_n(x) - f_m(x) = \begin{cases} 0 & \text{if} \quad -1 \leq x \leq 0 \\ (n-m)x & \text{if} \quad 0 < x < \frac{1}{n} \\ 1 - mx & \text{if} \quad \frac{1}{n} \leq x \leq \frac{1}{m} \\ 0 & \text{if} \quad \frac{1}{m} \leq x \leq 1 \end{cases}$$

$\|f_n - f_m\|^2 = \int_{-1}^{1} (f_n(x) - f_m(x))^2 \, dx = \int_0^{\frac{1}{n}} (n-m)^2 x^2 \, dx + \int_{\frac{1}{n}}^{\frac{1}{m}} (1 - mx)^2 \, dx$

$$= \frac{(n-m)^2 x^3}{3}\Big]_0^{\frac{1}{n}} + (\frac{-1}{m}) \frac{(1-mx)^3}{3}\Big]_{\frac{1}{n}}^{\frac{1}{m}}$$

$$= \frac{(n-m)^2}{3} \frac{1}{n^3} - \frac{1}{m}\Big[0 - \frac{1}{3}(1 - \frac{m}{n})^3\Big]$$

$$= \frac{(n-m)^2}{3n^3} + \frac{1}{3m}(\frac{n-m}{n})^3$$

$$= \frac{(n-m)^2}{3n^2m}$$

Thus, $\|f_n - f_m\|^2 = \dfrac{(n-m)^2}{3n^2m}$

Since $n > m$, then $n = m + t$

$$\|f_n - f_m\|^2 = \frac{t^2}{3(m+t)^2m} \to 0 \quad \text{as } m \to \infty$$

Hence, $\|f_n - f_m\| \to 0$. Thus, $\langle f_n \rangle$ is a Cauchy sequence.

But $f_n \to f$ where

$$f(x) = \begin{cases} 0 & \text{if} \quad -1 \le x \le 0 \\ 1 & \text{if} \quad 0 < x \le 1 \end{cases}$$

Thus, $f \notin C[-1,1]$. Then, $\langle f_n \rangle$ is not convergent in $C[-1,1]$. i.e., The space is not Hilbert space.

Remark 4.24.

Every Hilbert space is a Banach space but the converse is not true. For example, the space $C[a,b]$ with $\|f\| = \max\{|f(x)| : x \in [a,b]\}$ is a Banach space (see Example 3.5). However, $C[a,b]$ is not a Hilbert space since it does not satisfy parallelogram law; that is $\| \ \|$ can not be obtained from inner product (see Example 4.16).

4.4 Orthogonality and Orthonormality in Inner Product Space

Definition 4.25. ***orthogonal Elements***
Let $(L, \langle \ , \rangle)$ be an I.P.S and $x, y \in L$. Then x is said to be **orthogonal** on y (denoted by $x \perp y$) if and only if $\langle x, y \rangle = 0$.

Example 4.26.

Let $L = \mathbb{R}^2$ is I.P.S such that $\langle x, y \rangle = x_1 y_1 + x_2 y_2$ is usual inner product $\forall x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Let $x = (-6, 3), y = (2, -1), z = (1, 2)$. Show that $x \perp z, y \perp z$ and $y \not\perp x$.

Solution: $\langle x, z \rangle = \langle (-6, 3), (1, 2) \rangle = -6 + 6 = 0$. Hence, $x \perp z$.

$\langle y, z \rangle =$

$\langle y, x \rangle =$

Proposition 4.27.

Let $(L, \langle \ , \rangle)$ be an I.P.S and $x, y \in L$. Then

(i) If $x \perp y$ then $y \perp x$.

(ii) $\mathbf{0}_L \perp x \ \ \forall x \in L$. (**H.W.**)

(iii) if $x \perp x$ then $x = \mathbf{0}_L$. (**H.W.**)

Proof. (1) Let $x \perp y$ then $\langle x, y \rangle = 0$. From Definition 4.1(3), we have $\langle y, x \rangle = \overline{\langle x, y \rangle} = \overline{0}$. i.e., $y \perp x$. □

Proposition 4.28.

Let $(L, \langle\ ,\rangle)$ be an I.P.S and $x, x_1, ..., x_n \in L$ such that x is orthogonal on $x_1, ..., x_n$. Prove that x is orthogonal on any linear combination of $x_1, ..., x_n$.

Proof. Let w be a linear combination of $x_1, ..., x_n$. i.e., there exists $\alpha_i \in F$ such that $w = \sum_{i=1}^{n} \alpha_i x_i$. We must show $\langle x, w\rangle = 0$.

$\langle x, w\rangle = \langle x, \sum_{i=1}^{n} \alpha_i x_i\rangle = \sum_{i=1}^{n} \overline{\alpha_i}\langle x, x_i\rangle$ (by Corollary 4.9(ii))

$= \sum_{i=1}^{n} \overline{\alpha_i}.0$ (From the assumption)

$= 0.$ □

Example 4.29.

(1) Find the value of a that makes the vectors $X = (a, 2, -1), Y = (3, -5, 2)$ orthogonal vectors in $\mathbb{R}^3$ with usual inner product. (**H.W.**)

(2) Let $(L, \langle\ ,\rangle)$ be an I.P.S over $\mathbb{R}$ and let $x, y \in L$ such that $\|x\| = \|y\| = 1$ (i.e., x and y are normal elements). Prove that $x + y \perp x - y$.

Answer: $\langle x + y, x - y\rangle = \langle x, x\rangle - \langle x, y\rangle + \langle y, x\rangle - \langle y, y\rangle = \|x\|^2 - \langle x, y\rangle + \langle x, y\rangle - \|y\|^2 = 0$. Hence, $x + y \perp x - y$.

(3) Let $(L, \langle\ ,\rangle)$ be an I.P.S and let $x, y \in L$ such that $x \perp y$. Prove that $\|x + y\|^2 = \|x\|^2 + \|y\|^2 = \|x - y\|^2$.

Answer: $\|x + y\|^2 = \langle x + y, x + y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$

$= \|x\|^2 + 0 + 0 + \|y\|^2 = \|x\|^2 + \|y\|^2$

Similarly, $\|x - y\|^2 = \|x\|^2 + \|y\|^2$.

(4) Let $(L, \langle\ ,\rangle)$ be an I.P.S and let $x, y \in L$ such that $x \perp y$. Prove that $\|x + \lambda y\| = \|x - \lambda y\|$.

Answer: (H.W.)

(5) Let $(L, \langle\ ,\rangle)$ be an I.P.S and let $x_1, x_2, ..., x_n \in X$ such that $x_i \perp x_j\ \ \forall i \neq j$. Prove that $\|\sum_{i=1}^{n} x_i\|^2 = \sum_{i=1}^{n} \|x_i\|^2$.

Answer: We prove using induction. If $n = 1$, the statement is true.

If $n = 2$. Since $x_1 \perp x_2$ then $\|x_1 + x_2\|^2 = \|x_1\|^2 + \|x_2\|^2$ (by part (3)).

Suppose the statement is true for $n = k$. i.e., $\left\|\sum_{i=1}^{k} x_i\right\|^2 = \sum_{i=1}^{k} \|x_i\|^2$

To prove the statement is true when $n = k + 1$. i.e.,

T.P. $\left\|\sum_{i=1}^{k+1} x_i\right\|^2 = \sum_{i=1}^{k+1} \|x_i\|^2$

$$\left\|\sum_{i=1}^{k+1} x_i\right\|^2 = \left\|\sum_{i=1}^{k} x_i + x_{k+1}\right\|^2 = \left\|\sum_{i=1}^{k} x_i\right\|^2 + \|x_{k+1}\|^2$$

$$= \sum_{i=1}^{k} \|x_i\|^2 + \|x_{k+1}\|^2 \text{ (by induction } n = k)$$

$$= \sum_{i=1}^{k+1} \|x_i\|^2.$$

Definition 4.30. ***Orthogonal to Set***

Let $(L, \langle\ ,\rangle)$ be an I.P.S, $x \in L$, and $A \subseteq X$. Then, x is said to be orthogonal on A ($x \perp A$) if $x \perp a\ \ \forall a \in A$.

Example 4.31.

Consider the space $\mathbb{R}^2$ with usual product space and $A = \{(0, a) : a \in \mathbb{R}\}$.

Then $(2, 0) \perp A$ because $\langle(2, 0), (0, a)\rangle = 2.0 + 0.a = 0$.

Definition 4.32. ***Orthogonal Sets***
Let $(L, \langle\ ,\ \rangle)$ be an I.P.S, and $A, B \subseteq L$. Then, A is said to be orthogonal to B $(A \perp B)$ if $a \perp b,\ \ \forall a \in A, \forall b \in B$.

Example 4.33.

Consider the space $\mathbb{R}^2$ with usual inner product and $A = \{(0, a) : a \in \mathbb{R}\}$ and $B = \{(b, 0) : b \in \mathbb{R}\}$. Show that $A \perp B$.

Answer: for each $(0, a) \in A$ and for each $(b, 0) \in B$, then

$\langle (a, 0), (0, b) \rangle = a.0 + 0.b = 0$. Thus, $A \perp B$.

Proposition 4.34.

Let $(L, \langle\ ,\ \rangle)$ be an I.P.S, and $A, B \subseteq L$ such that $A \perp B$ then $A \cap B = \{\mathbf{0}\}$.

Proof. Let $x \in A \cap B \Rightarrow x \in A$ and $x \in B$ $\qquad (\mathbf{I})$

Since $A \perp B \Rightarrow \langle a, b \rangle = 0, \quad \forall a \in A, \forall b \in B$.

From $(\mathbf{I})$, $\langle a, b \rangle = \langle x, x \rangle = 0$.

Using Definition 4.1(2), $x = \mathbf{0}$, then $A \cap B = \{\mathbf{0}\}$. □

Definition 4.35.
Let $(L, \langle\ ,\ \rangle)$ be an I.P.S. and $\phi \neq A \subseteq L$. Then, the set

$$A^{\perp} = \{x \in L : x \perp a, \quad \forall a \in A\}$$

is called the orthogonal complement on A.

Proposition 4.36.

Let $(L, \langle\ ,\ \rangle)$ be an I.P.S. and $\phi \neq A, B \subseteq L$. Then,

(1) $L^{\perp} = \{\mathbf{0}\}$.

(2) $\{\mathbf{0}\}^{\perp} = L$. (**H.W.**)

(3) $A \cap A^{\perp} = \{\mathbf{0}\}$.

(4) $A \subseteq A^{\perp\perp}$.

(5) If $A \subseteq B$ then $B^{\perp} \subseteq A^{\perp}$. (**H.W.**)

(6) If $A \subseteq B^{\perp}$ then $B \subseteq A^{\perp}$.

Proof. (1) $L^{\perp} = \{x \in L : x \perp L\} = \{x \in L : \langle x, l\rangle = 0, \forall l \in L\} = \{\mathbf{0}\}$.

(3) Let $x \in A \cap A^{\perp} \Rightarrow x \in A$ and $x \in A^{\perp}$ **(I)**

Since $x \in A^{\perp}$ then $x \perp A$ **(II)**

From **(I)** and **(II)**, $x \perp x$. i.e., $\langle x, x\rangle = 0$, thus $x = \mathbf{0}$.

Then, $A \cap A^{\perp} = \{\mathbf{0}\}$.

(4) To prove $A \subseteq A^{\perp\perp}$. Let $x \in A$.

For any $y \in A^{\perp} \Rightarrow y \perp A$. In particular, $y \perp x$ $(x \in A)$

From Proposition 4.27(1), $x \perp y$, $\forall y \in A^{\perp}$. Thus, $x \in A^{\perp\perp}$.

(6) Let $A \subseteq B^{\perp}$, then from part (5), $B^{\perp\perp} \subseteq A^{\perp}$.

Now, from part (4), $B \subseteq B^{\perp\perp} \subseteq A^{\perp}$. Then, $B \subseteq A^{\perp}$. □

Theorem 4.37.

Let $(L, \langle\ ,\rangle)$ be an I.P.S. and $\phi \neq A \subseteq L$. Then, $A^{\perp}$ is a closed subspace of L.

Proof. (1) To prove $A^{\perp}$ is a subspace of L.

Let $x, y \in A^{\perp}$ and $\alpha, \beta \in F$. T.P. $\alpha x + \beta y \in A^{\perp}$

$$\text{T.P. } \langle \alpha x + \beta y, a\rangle = 0, \quad \forall a \in A.$$

Since $x, y \in A^{\perp} \Rightarrow \langle x, a\rangle = \langle y, b\rangle = 0 \qquad$ (**I**)

$\langle \alpha x + \beta y, a\rangle = \alpha\langle x, a\rangle + \overline{\beta}\langle y, a\rangle = \alpha.0 + \overline{\beta}.0 = 0 \qquad$ [from (I)]

Thus, $A^{\perp}$ is a subspace of L.

(2) T.P. $A^{\perp}$ is a closed set (i.e., $A^{\perp} \subseteq \overline{A^{\perp}}$ and $\overline{A^{\perp}} \subseteq A^{\perp}$)

It is clear that $A^{\perp} \subseteq \overline{A^{\perp}} \qquad$ (**I**)

T.P. $\overline{A^{\perp}} \subseteq A^{\perp}$. Let $x \in \overline{A^{\perp}}$ then $\exists (x_n) \in A^{\perp}$ such that $(x_n) \to x$.

Since $(x_n) \in A^{\perp}, \quad \forall n \in N \Rightarrow x_n \perp A \Rightarrow x_n \perp a, \ \forall a \in A$

$$\Rightarrow \langle x_n, a\rangle = 0, \quad \forall a \in A.$$

But $(x_n) \to x$ and $a \to a$. Thus, from Theorem 4.19(1), $\underbrace{\langle x_n, a\rangle}_{=0} \to \langle x, a\rangle$.

$\langle x, a\rangle = 0 \ \ \forall a \in A$. Then, $x \in A^{\perp}$. Thus, $\overline{A^{\perp}} \subseteq A^{\perp} \qquad$ (**II**).

From (**I**) and (**II**), $A^{\perp}$ is a closed set.

□

Definition 4.38. ***Orthonormal Set***

Let $(L, \langle\ ,\rangle)$ be an I.P.S. and $A \subseteq L$. Then, A is called **orthonormal set** if

(1) A is said to be **orthogonal** if $x \perp y \ \ \forall x, y \in A, \ \ x \neq y$.

(2) Each element $x \in A$ is a normal element. i.e., $\langle x, x\rangle^{\frac{1}{2}} = \|x\| = 1 \ \ \forall x \in A$.

Remark 4.39.

Orthonarmal set has no zero element ($\mathbf{0} \notin A$) because $\|\mathbf{0}\| \neq 1$ ($\mathbf{0}$ is not normal element).

Example 4.40.

Let $L = \mathbb{R}^3$ with usual inner product and $A = \{(1,2,2),(2,1,-2),(2,-2,1)\} \subset L$. Show that A is orthogonal but not orthonormal.

<u>Solution:</u> T.P. A is orthogonal set **(H.W.)**.

To show not every vector in A is normal. i.e.,

$\|(1,2,2)\|^2 = \langle (1,2,2),(1,2,2)\rangle = 1+4+4 = 9 \neq 1 \Rightarrow \|(1,2,2)\| \neq 1$.

Thus, A is not orthonormal.

Theorem 4.41.

Let L be an I.P.S. and $x_1, ..., x_n$ be orthonormal vectors in L. Then

$$\sum_{i=1}^{n} |\langle x, x_i\rangle|^2 \leq \|x\|^2 \quad \forall x \in L$$

.

Example 4.42.

Let $L = \mathbb{R}^3$ and $x_1 = \frac{1}{3}(1,2,2), x_2 = \frac{1}{3}(2,1,-2), x_3 = \frac{1}{3}(2,-2,1)$.

Let $x = (2,1,3)$. Then

$|\langle x, x_1\rangle|^2 = [\frac{1}{3}(2+2+6)]^2 = \dfrac{100}{9}$

$|\langle x, x_2\rangle|^2 = [\frac{1}{3}(4+1-6)]^2 = \dfrac{1}{9}$

$|\langle x, x_3\rangle|^2 = [\frac{1}{3}(4-2+3)]^2 = \dfrac{25}{9}$

$\sum_{i=1}^{3} |\langle x, x_i\rangle|^2 = \dfrac{100}{9} + \dfrac{1}{9} + \dfrac{25}{9} = 14.$

on the other hand, $\|x\|^2 = \langle x, x\rangle = 4 + 1 + 9 = 14$.

As in Theorem 4.41, $\sum_{i=1}^{3} |\langle x, x_i\rangle|^2 = \|x\|^2$

Take $x = (1, 1, 1)$ and apply Theorem 4.41. **(H.W.)**

Theorem 4.43.

Let $(L, \langle\ ,\rangle)$ be an I.P.S. Let (x_n) be an orthonormal sequence in L and (λ_n) be a sequence in F such that $\sum_{i=1}^{+\infty} |\lambda_i|^2 < +\infty$. Let $y_n = \sum_{i=1}^{n} \lambda_i x_i$. Then, (y_n) is a Cauchy sequence.

Proof. Let $y_n = \sum_{i=1}^{n} \lambda_i x_i$, $\quad y_m = \sum_{i=1}^{m} \lambda_i x_i$. Assume that $n < m$ then $m = n + k$ for some $k \in N$. We must prove $\|y_m - y_n\| \to 0$.

$y_m - y_n = \sum_{i=1}^{m} \lambda_i x_i - \sum_{i=1}^{n} \lambda_i x_i = \sum_{i=1}^{n+k} \lambda_i x_i - \sum_{i=1}^{n} \lambda_i x_i = \sum_{i=n+1}^{n+k} \lambda_i x_i$.

$$\begin{aligned}
\|y_m - y_n\|^2 = \left\|\sum_{i=n+1}^{n+k} \lambda_i x_i\right\|^2 &= \langle \sum_{i=n+1}^{n+k} \lambda_i x_i,\ \sum_{i=n+1}^{n+k} \lambda_i x_i\rangle \\
&= \sum_{i=n+1}^{n+k} \lambda_i \sum_{i=n+1}^{n+k} \overline{\lambda_i}\langle x_i, x_i\rangle \\
&= \sum_{i=n+1}^{n+k} \lambda_i \overline{\lambda_i}\langle x_i, x_i\rangle \\
&= \sum_{i=n+1}^{n+k} |\lambda_i|^2 \|x_i\|^2 \\
&= \sum_{i=n+1}^{n+k} |\lambda_i|^2 \qquad (\|x_i\|^2 = 1\ \ \forall i)
\end{aligned}$$

As $n \to +\infty$, $\sum_{i=n+1}^{n+k} |\lambda_i|^2 \to 0 \qquad (\sum_{i=1}^{+\infty} |\lambda_i|^2$ convergent)

Thus, $\|y_m - y_n\|^2 \to 0$ which means $\|y_m - y_n\| \to 0$. Hence, (y_n) is a Cauchy sequence. □

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