$$f(x) = \begin{cases} x & \text{if } x \leq 1 \\ x+1 & \text{if } x > 1 \, . \end{cases}$$

It is clear that f is continuous at 0 and discontinuous at 1.

Also f is not linear transformation because if $x = 5, y = 6$ and $\alpha = \beta = 1$

$f(\alpha x + \beta y) = f(5+6) = f(11) = 11 + 1 = 12$

and $\alpha f(x) + \beta f(y) = f(5) + f(6) = (5+1) + (6+1) = 13$

Hence $f(\alpha x + \beta y) \neq \alpha f(x) + \beta f(y)$.

Theorem 2.55.

Let $(L, \| \|_L), (L', \| \|_L{}')$ be normed spaces and let $f : L \to L'$ be a linear transformation. If f is continuous at a point $x_1 \in L$ then f is continuous at each point.

Proof. Let $x_1 \in L$ and assume that f is continuous at x_1. Let $x_2 \in L$ be any point. To prove that f is continuous at x_2. Let $x_n \to x_2$ in L. Then, $x_n - x_2 \to 0$ and hence $x_n - x_2 + x_1 \to x_1$. Since f is continuous at x_1 then $f(x_n - x_2 + x_1) \to f(x_1)$.

Since f is a linear transformation, then $f(x_n) - f(x_2) + f(x_1) \to f(x_1)$.

Hence, $f(x_n) - f(x_2) \to 0$, and thus, $f(x_n) \to f(x_2)$.

Therefore, f is continuous at x_2. Thus, f can not be continuous at some points and discontinuos at some points. $\square$

Example 2.56.

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \; . \end{cases}$$

Show that f is not continuous at $(0,0)$.

<u>Solution</u>: Let $x_n = \dfrac{1}{n}$ and $y_n = \dfrac{-1}{n} \quad \forall n \in N$.

Then, $x_n \to 0$ and $y_n \to 0$. Thus, $(x_n, y_n) \to (0,0)$. But

$$f(x_n, y_n) = \frac{\frac{1}{n}(\frac{-1}{n})}{\frac{1}{n^2} + \frac{1}{n^2}} = \frac{\frac{-1}{n^2}}{\frac{2}{n^2}} = \frac{-1}{2}$$

Hence, $f(x_n, y_n) \to \frac{-1}{2}$ but $f(0,0) = (0,0)$. Thus, $f(x_n, y_n) \nrightarrow f(0,0)$.

Thus, f is not continuous at $(0,0)$.

2.8 Boundedness in Normed Linear Space

Definition 2.57. ***Bounded Set***

Let $(L, \| \, \|_L)$ be a normed space and let $A \subset L$. A is called a **bounded set** if there exists $k > 0$ such that $\|x\| \leq k \quad \forall x \in A$.

Example 2.58.

Consider $(\mathbb{R}, | \, |)$ and let $A = [-1, 1)$. Since $|x| \leq 1$, then A is bounded.

Example 2.59.

Consider $(\mathbb{R}^2, \| \, \|)$ be a normed space such that

$\|x\| = \left[\sum_{i=1}^{2} |x_i|^2\right]^{\frac{1}{2}}$ be the Eucledian norm, for each $x = (x_1, x_2) \in \mathbb{R}^2$.

Let $A = \{(x_1, x_2) \in \mathbb{R}^2 : -1 \leq x_1 \leq 1, \;\; x_2 \geq 0\}$. Then, A is unbounded.

Theorem 2.60.

Let $(L, \| \|_L)$ be a normed space and let $A \subseteq L$. Then the following statements are equivalent.

(1) A is bounded.

(2) If $\langle x_n \rangle$ is a sequence in A and $\langle \alpha_n \rangle$ is a sequence in F such that $\alpha_n \to 0$ then $\alpha_n x_n \to 0$.

Proof. (1)⇒(2) Since A is bounded, $\exists k > 0$ such that $\|x_n\| \leq k \quad \forall x_n \in A$. Since $\alpha_n \to 0$ as $n \to \infty$, then $|\alpha_n| \to 0$. Hence,

$\|\alpha_n x_n - 0\| = \|\alpha_n x_n\| = |\alpha_n| \|x_n\| \leq |\alpha_n| k \quad (\text{since } \|x_n\| \leq k)$

But $|\alpha_n| \to 0$, thus $|\alpha_n| k \to 0$. Therefore, $\|\alpha_n x_n - 0\| \to 0$ and hence $\alpha_n x_n \to \mathbf{0}_X$.

(2)⇒(1) Suppose A is not bounded. Then, $\forall k \in Z_+, \exists x_k \in A$ such that $\|x_k\| > k$.

Put $\alpha_k = \dfrac{1}{k}$. Hence, $\alpha_k \to 0$. But

$\|\alpha_k x_k\| = \left\|\frac{1}{k} x_k\right\| = \dfrac{1}{k} \|x_k\| > \dfrac{1}{k}.k = 1$

Then, $\|\alpha_k x_k\| > 1$, thus $\alpha_k x_k \nrightarrow 0$ which contradicts (2). □

Definition 2.61. ***Bounded Mapping***
Let $(L, \| \|_L), (L', \| \|_L{}')$ be two normed space and $f : L \to L'$ be a linear transformation. f is called **bounded mapping** if for each $A \subseteq L$ bounded then $f(A) = \{f(a) : a \in A\}$ is bounded set in L'.

Example 2.62.

Let Consider $(\mathbb{R}, | |)$ and $(\mathbb{R}^2, \| \|)$ be a normed space such that

$\|x\| = \left[\sum_{i=1}^{2} |x_i|^2 \right]^{\frac{1}{2}} = [|x_1|^2 + |x_2|^2]^{\frac{1}{2}} = [x_1^2 + x_2^2]^{\frac{1}{2}} \quad \forall (x_1, x_2) \in \mathbb{R}^2.$

Define $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x_1, x_2) = x_1 + x_2 \quad \forall (x_1, x_2) \in \mathbb{R}^2$. Show that f is a linear transformation (**H.W.**). Let $A \subseteq \mathbb{R}^2$ and A is bounded. Show that $f(A)$ is bounded.

Solution: Let $A \subseteq \mathbb{R}^2$ and A is bounded to prove $f(A) = \{f(x_1, x_2) : (x_1, x_2) \in A\}$ is bounded.

Note that $\forall (x_1, x_2) \in A \implies f(x_1, x_2) = x_1 + x_2 \in f(A)$

$|fx_1, x_2| = |x_1 + x_2| \leq |x_1| + |x_2| \qquad$ **(I)**

Since A is bounded then $\exists k > 0$ such that $\|(x_1, x_2)\| \leq k \quad \forall (x_1, x_2) \in A$

$\implies (x_1^2 + x_2^2)^{\frac{1}{2}} \leq k \implies x_1^2 + x_2^2 \leq k^2$

Since $x_1^2 \leq x_1^2 + x_2^2 \leq k^2$, then $x_1^2 \leq k^2 \implies |x_1| \leq k \qquad$ **(II)**

Similarly, $x_2^2 \leq x_1^2 + x_2^2 \leq k^2$, then $x_2^2 \leq k^2 \implies |x_2| \leq k \qquad$ **(III)**

Substitute **(II)** and **(III)** in **(I)**

$$|f(x_1, x_2)| = |x_1 + x_2| \leq \underbrace{|x_1| + |x_2| \leq k + k}_{\text{by } \mathbf{(II)} \text{ and } \mathbf{(III)}} = 2k$$

i.e., $|f(x_1, x_2)| \leq 2k$. Thus, $f(A)$ is bounded, and hence, f is bounded.

Theorem 2.63.

Let $(L, \| \|_L), (L', \| \|_L{}')$ be normed spaces and $f : L \to L'$ be a linear transformation. Then f is bounded if and only if $\exists k > 0$ such that $\|f(x)\|_L{}' \leq k \|x\|_L \qquad \forall x \in L$.

Proof. ($\Rightarrow$) If f is bounded and let $A = \{x \in L : \|x\|_L \leq 1\}$.

It is clear A is bounded, and hence, $f(A)$ is bounded in L' (by definition of bnd function).

Thus, $\exists k > 0$ such that $\|f(x)\|_L{}' \leq k \qquad \forall x \in A \qquad$ **(I)**

(1) If $x = \mathbf{0_L}$ then $f(\mathbf{0_L}) = \mathbf{0'_L}$, and thus, $\|f(\mathbf{0_L})\| = 0 \leq k \|\mathbf{0_L}\| = 0$.

(2) If $x \neq \mathbf{0_L}$, put $y = \dfrac{x}{\|x\|}$ such that $\|y\| = \left\|\dfrac{x}{\|x\|}\right\| = \dfrac{1}{\|x\|}.\|x\| = 1$.

Hence, $y \in A$. Thus, $\|f(y)\| \leq k$ **(II)**

$\|f(y)\| = \left\|f(\frac{x}{\|x\|})\right\| = \left\|\frac{1}{\|x\|}f(x)\right\| = \frac{1}{\|x\|}\|f(x)\|$

By **(II)**, $\|f(y)\| \leq k$, thus $\frac{1}{\|x\|}\|f(x)\| \leq k$. i.e., $\|f(x)\| \leq k.\|x\|$ as required.

($\Leftarrow$) Let A be a bounded set. Then, $\exists k_1 > 0$ such that $\|x\| \leq k_1 \quad \forall x \in A$

Since $\|f(x)\| \leq k\|x\| \quad \forall x \in X$, hence $\|f(x)\| \leq k\|x\| \quad \forall x \in A$. Then we get $\|f(x)\| \leq kk_1 \quad \forall x \in A$. Thus, $\|f(x)\| \leq k_2 \quad \forall x \in A$ where $k_2 = kk_1$; that is, $f(A)$ is a bounded set. □

Theorem 2.64.

Let $(L, \| \|_L), (L', \| \|_L{}')$ be normed spaces and $f : L \to L'$ be a linear transformation. Then f is bounded if and only if f is continuous.

Proof. ($\Leftarrow$) Suppose that f is continuous and not bounded,

hence $\forall n \in Z_+, \exists x_n \in L$ such that $\|f(x_n)\|_L{}' > n\|x_n\|_L$.

Let $y_n = \frac{x_n}{n\|x_n\|}$. Then, $\|f(y_n)\| = \left\|f(\frac{x_n}{n\|x_n\|})\right\| = \frac{\|f(x_n)\|}{n\|x_n\|} > \frac{n\|x_n\|}{n\|x_n\|} = 1$

Thus, $\|f(y_n) - f(0)\| = \|f(y_n)\| > 1$, i.e., $f(y_n) \nrightarrow f(0)$ **(I)**

but $\|y_n\| = \left\|\frac{x_n}{n\|x_n\|}\right\| = \frac{\|x_n\|}{n\|x_n\|} = \frac{1}{n}$

as $n \to \infty$, we get $\|y_n\| \to 0$, and hence, $y_n \to \mathbf{0}_L$.

It follows that $f(y_n) \to \underbrace{f(\mathbf{0}_L) = \mathbf{0}'_L}_{\text{By Theorem 1.19(i)}}$ (Since f is a linear transformation)

This contradicts **(I)**, thus, f is bounded.

($\Rightarrow$) Assume that f is bounded to prove f is continuous for all $x \in L$. Let $x_0 \in L$ and $\epsilon > 0$, to find $\delta > 0$ such that

$\forall x \in L, \|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$.

$\|f(x) - f(x_0)\| = \|f(x - x_0)\|$ (f is linear transformation)

Since f is bounded, then $\exists k > 0$ s.t. $\|f(x)\| \leq k\|x\| \quad \forall x \in L$ **(I)**

Hence, $\|f(x) - f(x_0)\| = \underbrace{\|f(x - x_0)\| \leq k\|x - x_0\|}_{\text{By } \mathbf{(I)}}$

$$< k\delta \qquad (\text{Since } \|x - x_0\| < \delta)$$

$$= k.\tfrac{\epsilon}{k} \qquad (\text{By choosing } \delta = \tfrac{\epsilon}{k} = \epsilon)$$

Thus, $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$.

Hence, f is continuous at $x_0 \in L$. Since x_0 is an arbitrary, then f is cont. $\forall x \in L$. □

Theorem 2.65.

Let $(L, \| \|_L), (L', \| \|_L{}')$ be normed spaces and $f : L \to L'$ be a linear transformation. If L is a finite dimensional space then f is bounded (hence, continuous).

Example 2.66.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x, y) = x + y \quad \forall (x, y) \in \mathbb{R}^2$.

f is a linear transformation function (check!)

and $dim(\mathbb{R}^2) = 2$. Hence, f is bounded (hence, continuous).

2.9 Bounded Linear Transformation

Definition 2.67.
Let $(L, \| \|_L), (L', \| \|_L{}')$ be normed spaces over a field F. The set of all bounded linear transformation mappings from L to L' is defined as

$B(L, L') = \{T : T : L \to L' \text{ is a linear bounded (hence, cont.) trans.}\}$

Theorem 2.68.

Prove that $B(L, L')$ is a linear subspace (over a field F) of the space of linear transformation mappings with respect to usual addition and usual scalar multiplication.

Proof. Let $\alpha, \beta \in F$ and $T_1, T_2 \in B(L, L')$. To prove $\alpha T_1 + \beta T_2 \in B(L, L')$
Since T_1, T_2 are linear transformations, then by Theorem 1.30, $\alpha T_1, \beta T_2$ are linear trans.

Now, $\alpha T_1, \beta T_2$ are linear trans., by Theorem 1.30,$\alpha T_1 + \beta T_2$ is linear transformation.

Next, we show $\alpha T_1 + \beta T_2$ is bounded.

Since T_1, T_2 are bounded, then $\exists k_1, k_2 > 0$ such that $\forall x \in L$ we have

$\|T_1(x)\|'_L \leq k_1 \|x\|_L$ and $\|T_2(x)\|'_L \leq k_2 \|x\|_L$ **(I)**

Then, $\|(\alpha T_1 + \beta T_2)(x)\|'_L = \|(\alpha T_1)(x) + (\beta T_2)(x)\|'_L$

$$= \|\alpha.T_1(x) + \beta.T_2(x)\|'_L \qquad \text{(Definition of scalar multiplication)}$$

$$\leq \|\alpha.T_1(x)\|'_L + \|\beta.T_2(x)\|'_L$$

$$= |\alpha| \|T_1(x)\|'_L + |\beta| \|T_2(x)\|'_L$$

$$\leq |\alpha| k_1 \|x\|_L + |\beta| k_2 \|x\|_L$$

$$= (|\alpha| k_1 + |\beta| k_2) \|x\|_L = k \|x\|_L \qquad (k = |\alpha| k_1 + |\beta| k_2)$$

Hence, $\alpha T_1 + \beta T_2$ is bounded.

Since $\alpha T_1 + \beta T_2$ is bounded and linear transformation, then $\alpha T_1 + \beta T_2 \in B(L, L')$. □

Theorem 2.69.

Let $(L, \| \|_L), (L', \| \|_L{}')$ be normed space. Prove that $B(L, L')$ is a normed space such that $\forall T \in B(L, L')$ we have

$\|T\| = \sup\{\|T(x)\|_{L'} : x \in L, \|x\|_L \leq 1\}$

Proof. To prove $\| \|$ is a norm on $B(L, L')$

(1) since $\|T(x)\|_{L'} \geq 0 \;\; \forall x \in L, \|x\|_L \leq 1$, then $\|T\| \geq 0$.

(2) $\|T\| = 0 \iff \sup\{\|T(x)\|_{L'} : x \in L, \|x\|_L \leq 1\} = 0$

$$\iff \|T(x)\|_{L'} = 0 \;\; \forall x \in L, \|x\|_L \leq 1$$

$$\iff T(x) = 0 \;\; \forall x \in L, \|x\|_L \leq 1$$

$$\iff T = \hat{0}$$

(3) Let $T_1, T_2 \in B(L, L')$

$\|T_1 + T_2\| = \sup\{\|(T_1 + T_2)(x)\|_{L'} : x \in L, \|x\|_L \leq 1\}$

$$\leq \sup\{\|(T_1(x)\|_{L'} + \|T_2(x)\|_{L'} : x \in L, \|x\|_L \leq 1\}$$

$$\leq \sup_{x \in L}\{\|(T_1(x)\|_{L'} : \|x\|_L \leq 1\} + \sup_{x \in L}\{\|(T_2(x)\|_{L'} : \|x\|_L \leq 1\}$$

$$= \|T_1\| + \|T_2\|$$

(4) $\|\alpha T\| = \sup\{\|(\alpha . T(x)\|_{L'} : x \in L, \|x\|_L \leq 1\}$

$$= |\alpha| \sup\{\|T(x)\|_{L'} : x \in L, \|x\|_L \leq 1\}$$

$$= |\alpha| \, \|T\|$$

□