Chapter One

Fundamental Concepts

In this chapter, we introduce the following concepts:

1.1. Linear Spaces, Examples of Linear Spaces, General Properties of Linear Space and of Linear Subspaces.

1.2. Linear Combination, The Linearly Independent , Dimensional Linear Spaces.

1.3. The Convexity, Examples of Convex Sets, Some Properties About Convex Sets.

1.4. Linear Operator (Linear Transformation) and Linear Functional.

Linear (Vector) Space

A linear space (also called vector space), denoted by L or V, is a collection of objects called **vectors**, which may be added together and multiplied by numbers, called **scalars** which are taken from a field F. Before defining linear space, we first define an arbitrary field.

Definition (1.1):- Let F be a non-empty set and (+) and (.) be two binary operations on F. The ordered triple $(F, +, .)$ is called **field** if

1. $(F, +)$ is a commutative group
2. $(F - \{e\}, .)$ is a commutative group, where e is the identity with respect to (+).
3. (.) is distributed over (+) (from left and right)

Example (1.2):- Let (+) and (.) are **ordinary addition and multiplications**. Then

1. Each of $(\mathbb{R}, +, .)$, $(\mathbb{C}, +, .)$, and $(\mathrm{Q}, +, .)$ are examples of fields
2. $(Z, -, .)$ is not field (Definition 1.1(1) does not hold)

   and $(Z, +, .)$ is not field (Definition 1.1(2) does not hold)

Definition (1.3):- A ***vector space*** (or ***linear space***) over a field F is a nonempty set L of elements $x, y, \ldots$ (called vectors) with two algebraic operations, these operations are called vector addition (+) and multiplication of vectors by scalars (.) , then we say that $(L, +, .)$ is a vector (Linear) space over F.

1) Vectors addition satisfy :-

$(L, +)$ be a commutative group

2) Multiplication by scalars satisfy :-

a) $\alpha . x \in L \ , \forall \alpha \in F \ , x \in L$

b) $(\alpha\beta). x = \alpha\,(\beta . x) \quad , \forall x \in L \ and \ \alpha \ , \beta \in F$

c) $\alpha.(x + y) = \alpha . x + \alpha . y$

d) $(\alpha + \beta). x = \alpha . x + \beta . x$ Distributive laws

e) $1. x = x \ \ , \forall x \in L, \ 1 \ is \ the \ unity \ F$

Remarks(1.4):-

(1) The filed $F = \mathbb{C}$ or $\mathbb{R}$

(2) If $F = \mathbb{C}$ then we say that the $(L, +, .)$ is complex vector (complex linear) space

(3) If $F = \mathbb{R} \Rightarrow (L, +, .)$ is called real vector (linear)space

(4) The vectors addition is a mapping such that $+ : L \times L \to L$

(5) The multiplication by scalars (scalar multiplication) is a mapping such that $. : F \times L \to L$

(6) We can denote the zero vector by 0_L and the scalar by 0.

Examples of Linear (Vector) Space

Example (1.5):- The Euclidean space : $\mathbb{R}^n = \{ x = (x_1, x_2, \dots, x_n); x_i \in \mathbb{R}, i = 1, \dots, n\}$, with ordinary addition and multiplication. i.e ,

$$\left.\begin{array}{l} x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \\ \alpha . x = (\alpha . x_1, \alpha . x_2, \dots, \alpha . x_n) \end{array}\right\} \forall x, y \in \mathbb{R}^n, \ \forall \alpha \in \mathbb{R},$$

Then $(\mathbb{R}^n, +, .)$ is real Linear space over the field $F = \mathbb{R}$.

Solution: It is clear that the following conditions are satisfied

(1) To prove $(\mathbb{R}^n, +)$ is commutative group

a) **The closure**: $\forall\, x, y \in \mathbb{R}^n$ to prove $x + y \in \mathbb{R}^n$

$$x \in \mathbb{R}^n \to x = (x_1, x_2, \dots, x_n)$$

$$y \in \mathbb{R}^n \to y = (y_1, y_2, \dots, y_n)$$

$$x + y = (x_1 + y_1, \dots, x_n + y_n) \in \mathbb{R}^n$$

b) **Associative** : to prove $(x + y) + z = x + (y + z), \forall x, y, z \in \mathbb{R}^n$

$$(x + y) + z = ((x_1, x_2, \dots, x_n) + (y_1, \dots, y_n)) + (z_1, \dots, z_n)$$

$$= (x_1 + y_1, \dots, x_n + y_n) + (z_1, \dots, z_n)$$

$$= \big((x_1 + y_1) + z_1, \ \dots \ , (x_n + y_n) + z_n\big) \ (+ \text{ asso. on } \mathbb{R})$$

$$= (x_1 + (y_1 + z_1), \ \dots \ , (x_n + (y_n + z_n))$$

$$= (x_1, \dots, x_n) + (y_1 + z_1, \dots, y_n + z_n)$$

$$= (x_1, \dots, x_n) + ((y_1, \dots, y_n) + (z_1, \dots \dots, z_n))$$

$$= x + (y + z)$$

c) **Identity**: It is clear that, there exist a unique identity $e = (0,0, \dots ,0) = 0_L$

$$\forall x \in \mathbb{R}^n \;\; s.t. \;\; x + 0_L \;\; = 0_L \;\; + x = x$$

d) **Inverse**: $\forall\, x \in \mathbb{R}^n, \exists(-x)$the invers of x s.t. $x + (-x) = (-x) + x = 0_L$

e) **Commutative**: $\forall x, y \in \mathbb{R}^n$, we have $\quad x + y = y + x$

$\Longrightarrow (\mathbb{R}^n, +)$ is commutative group

(2) **Scalar Multiplication :**

a) To prove $\alpha . x \in \mathbb{R}^n$, $\forall x \in \mathbb{R}^n$and $\alpha \in \mathbb{R}$

$$\alpha.(x_1, x_2, \dots, x_n) = (\alpha.x_1, \dots, \alpha.x_n) \in \mathbb{R}^n (\text{since } \alpha.x_i \in \mathbb{R},$$

$$\forall\, i = 1,2,3, \dots \dots, n)$$

b) $(\alpha\beta).x = (\alpha\beta).(x_1, x_2, \dots, x_n)$

$$= ((\alpha\beta).x, \dots, (\alpha\beta).x_n)$$

$$= (\alpha.(\beta.x_1), \dots \;\alpha.(\beta.x_n))$$

$$= \;\alpha.(\beta.x_1, \dots, \beta.x_n)$$

$$= \alpha.(\beta.(x_1, \dots \dots, x_n)$$

$$= \alpha.(\beta.x)$$

c) $\alpha.(x + y) = \;\alpha.((x_1, \dots, x_n) + (y_1, \dots, y_n))$

$$= \;\alpha.((x_1 + y_1 \,, \dots, x_n + y_n))$$

$$= (\alpha.(x_1 + y_1), \dots, \alpha.(x_n + y_n)$$

$$= ((\alpha.x_1 + \alpha.y_1), \dots \dots, (\alpha.x_n + \alpha.y_n))$$

$$= (\alpha.x_1, \dots \dots, \alpha.x_n) + (\alpha.y_1, \dots \dots, \alpha.y_n)$$

$= \alpha.(x_1, \dots., x_n) + \alpha.(y_1, \dots \dots.., y_n)$

$= \alpha.x + \alpha.y$

d) $(\alpha + \beta).x = (\alpha + \beta).(x_1, \dots \dots, x_n)$

$= ((\alpha + \beta).x_1, \dots \dots (\alpha + \beta).x_n))$

$= (\alpha.x_1 + \beta.x_1, \dots..., \alpha.x_n + \beta.x_n)$

$= \alpha.(x_1, \dots \dots, x_n) + \beta.(x_1, \dots.., x_n)$

$= \alpha.x + \beta.x$

e) $1.x = 1.(x_1, \dots., x_n) = (1.x, \dots \dots, 1.x)$

$= (x_1, \dots., x_n) = x.$

Thus, $(\mathbb{R}^n, +, .)$ is linear space over $\mathbb{R}$.

Example (1.6):- Consider the space $\mathbb{C}^n$ with two operctions defined as in the previous example, then $(\mathbb{C}^n, +, .)$ is complex linear space over $\mathbb{C}$. **(H.W.)**

Example (1.7):- Show that the space $(l^2, +, .)$ is linear space over F where $l^2 = \{x = (x_1, x_2, x_3, \dots \dots), x_i \in F, \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$, with ordinary addition and multiplication. i.e ,

$x + y = (x_1, x_2, x_3, \dots.) + (y_1, y_2, y_3, \dots.) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 \dots \dots)$

$\alpha . x = \alpha.(x_1, x_2, x_3, \dots) = (\alpha.x_1, \alpha.x_2, \alpha.x_3 \dots.), \forall x, y \in l^2, \alpha \in F$?

Solution:

1. **To prove that $(l^2, +)$ is commutative group?**

❖ **The closure** : Let $x, y \in l^2$ to prove $x + y \in l^2$

$x \in l^2 \Longrightarrow x = (x_1, x_2, \dots);\ x_i \in F$ and $\sum_{i=1}^{\infty}|x_i|^2 < \infty$

$y \in l^2 \Longrightarrow y = (y_1, y, \dots);\ y_i \in F$ and $\sum_{i=1}^{\infty}|y_i|^2 < \infty$

$x + y = (x_1 + y_1, x_2 + y_2, \dots\quad);\ x_i + y_i \in F,$ for all $i = 1,2,3, \dots$

To prove $\quad \sum_{i=1}^{\infty}|x_i + y_i|^2 < \infty$.

$|x_i + y_i|^2 = |(x_i + y_i)^2|$

$= \left|x_i^2 + 2x_i y_i + y_i^2\right|$

$\leq |x_i|^2 + 2|x_i y_i| + |y_i|^2$

$\leq |x_i|^2 + |x_i|^2 + |y_i|^2 + |y_i|^2,$ because $\ (2|x_i y_i| \leq |x_i|^2 + |y_i|^2)$

$= 2|x_i|^2 + 2|y_i|^2$

Taking the sum to the both sides of the above inequality

$\sum_{i=1}^{\infty}|x_i + y_i|^2 \leq 2\sum_{i=1}^{\infty}|x_i|^2 + 2\sum_{i=1}^{\infty}|y_i|^2 < \infty + \infty = \infty$

- The associative: **(H.W.)**
- **The identity** : $\exists\ e = (0,0, \dots) \in l^2\ \ \forall x = (x_1, x_2, \dots\quad) \in l^2$

such that $e + x = x + e = x$

It is clear that $\sum_{i=1}^{\infty}|0|^2 < \infty$ **(H.W.)**

- **The inverse** : $\forall\ x = (x_1, x_2, \dots\quad) \in l^2,\ \exists - x = (-x_1, -x_2, \dots\quad) \in l^2$

such that $-x + x = x + -x = 0$. It is clear that $\sum_{i=1}^{\infty}|-x_i|^2 < \infty$ **(H.W.)**

2. **Multiplication by scalars :**

❖ To prove that $\alpha.x \in l^2$, $\forall x \in l^2, \alpha \in F$

$\alpha.x = (\alpha.x_1 , \alpha.x_2 , \dots)$ to prove $\sum_{i=1}^{\infty}|\alpha x_i|^2 < \infty$

$\sum_{i=1}^{\infty}|\alpha.x_i|^2 = \sum_{i=1}^{\infty}|\alpha|^2 |x_i|^2 = |\alpha|^2 \sum_{i=1}^{\infty}|x_i|^2 < \infty.$

The rest of the conditions are homework.

Remark (1.8):- In general the inequality holds , for any $1 \leq p < \infty$

$$|x_i + y_i|^p \leq 2^{p-1} [|x_i|^p + |y_i|^p].$$

Example (1.9):- Show that the space $(l^p , + , .)$ is linear space over F where $1 \leq p < \infty$ and $l^p = \{x = (x_1, x_2, x_3, \dots \dots), x_i \in F , \sum_{i=1}^{\infty}|x_i|^p < \infty\}$ with ordinary addition and multiplication ? **(H.W.)**

Example (1.10):- Consider the space $l^\infty = \{ x = (x_1, x_2, x_3 \dots \dots), x_i \in F, |x_i| < c_x , \ i = 1,2, \dots .\}$

where c_x is a real number which may depend on x but does not depend on i . Then $(l^\infty, + , .)$ is linear space over F when **+** and **.** are defined as in Example (1.7).

Solution:

To prove that $(l^\infty , +)$ **is commutative group?**

❖ **The closure** : Let $x, y \in l^\infty$ to prove $x + y \in l^\infty$

$x \in l^\infty \Longrightarrow x = (x_1, x_2, \dots);\ x_i \in F$ and $|x_i| < c_x \quad \forall i \in N$

$y \in l^{\infty} \Longrightarrow y = (y_1, y, \ldots);\ y_i \in F$ and $|y_i| < c_y \quad \forall i \in N$

$x + y = (x_1 + y_1, x_2 + y_2, \ldots \quad);\ x_i + y_i \in F,$ for all $i = 1,2,3, \ldots$

To prove $\quad |x_i + y_i| < c_q; c_q > 0.$

Now, $|x_i + y_i| \leq |x_i| + |y_i| \leq c_x + c_y = c_q.$

Thus, $x + y \in l^{\infty}$.

- ❖ The rest of the conditions are homework.

Example (1.11):- The space $(C^b[a,b], +, .)$ is linear space over $\mathbb{R}$ where + and defined as $(f+g)(x) = f(x) + g(x), \quad \forall f, g \in C^b[a,b], \quad x \in \mathbb{R}$

$$(\alpha f)(x) = \alpha f(x),\ \alpha \in \mathbb{R}$$

Solution:

The space $C^b[a,b] = \{ f: [a,b] \to \mathbb{R}$ continuous and bounded function $\}$

(1) To prove that prove $(C^b[a,b], +)$ is commutative group

(b) $\forall f, g \in C^b[a,b]$ T.p. $f + g \in C^b[a,b]$ —— f, g are continuous

f, g are bounded

$\therefore f$ and g are continuous functions $\Longrightarrow f + g$ also continuous function …..(i)

$\therefore f$ and g are bounded functions

$$\left.\frac{\exists k_1 \geq 0 \quad s.t \quad |f(x)| \leq k_1}{\text{and } \exists k_2 \geq 0 \;\; s.t\; |g(x)| \leq k_2}\right\} \forall x \in \mathbb{R}$$

Now , $|(f+g)(x)| = |f(x) + g(x)|$

$$\leq |f(x)| + |g(x)| \leq k_1 + k_2 = k\,, \qquad k \geq 0$$

$\Longrightarrow f + g$ is bounded function $\qquad\qquad \ldots\ldots\ (ii)$

From (i) & (ii) we have $f + g \in C^b[a,b]$

(b) $\forall f, g, h \in C^b[a,b]$ to prove that $(f+g)+h = f+(g+h)$

$$((f+g)+h)(x) = (f+g)(x) + h(x)$$

$$= \big(f(x) + g(x)\big) + h(x)$$

$$= f(x) + (g(x) + h(x))$$

$$= f(x) + (g+h)(x)$$

$$= (f + (g+h))(x)$$

(c) To prove that, there exist a unique function, for all $f \in C^b[a,b]$

Define $\breve{0}: [a,b] \to R \quad s.t \;\; \breve{0}(x) = 0\,, \forall x \in [a,b]$

s.t $f + \breve{0} = \breve{0} + f = 0\,, \forall f \in C^b[a,b]$

($\breve{0}$ is continuous and bounded since its constant) $\Longrightarrow \breve{0} \in C^b[a,b]$

(d) To prove that , $\forall f \in C^b[a,b]$, $\exists -f \in C^b[a,b]$

Such that $\quad f + (-f) = (-f) + f = \breve{0}$,

since $f \in C^b[a,b] \Longrightarrow f$ is continuous and bounded

$\Longrightarrow -f$ is also continuous (by previous proposition) and $-f$ is also bounded

since $(|-f(x)| = |f(x)| \leq k, \quad k \geq 0)$

$= -f \in C^b[a,b]$ and $\big(f + (-f)\big)(x) = f(x) + \big(-f(x)\big) = 0 = 0(x)$

(e) $\forall f, g \in C^b[a,b]$, to prove that $f + g = g + f$

$$(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$$

$\therefore C^b[a,b]$ is comm group

(2) Scalar Multiplication

(a) $\forall f \in C^b[a,b], \alpha \in \mathbb{R}$, to prove that $\alpha f \in C^b[a,b]$

$\therefore f \in C^b[a,b] \Longrightarrow f$ is also continuous $\Longrightarrow \alpha f$ also continuous .... (i)

$f \in C^b[a,b] \Longrightarrow f$ is bounded $\Longrightarrow |f(x)| \leq k, k \geq 0$

But $|(\alpha f)(x)| = |\alpha f(x)| \leq |\alpha| \ f(x) \leq |\alpha| k$

$\Longrightarrow \alpha f$ is bounded .... (ii)

From (i) and (ii) we get, $\alpha f \in C^b[a,b]$

(b) $\forall f \in C^b[a,b]$ *and* $\alpha, \beta \in \mathbb{R}$, to prove $(\alpha\beta)f = \alpha(\beta f)$

$$((\alpha\beta)f)(x) = (\alpha\beta)f(x) = \alpha\big(\beta f(x)\big) = \alpha(\beta f)(x)$$

$$\Longrightarrow (\alpha\beta)f = \alpha(\beta f)$$

(c) $\forall f, g \in C^b[a,b], \alpha \in \mathbb{R}$, to prove $\alpha(f+g) = \alpha f + \alpha g$

$$\big(\alpha(f+g)\big)(x) = \alpha(f+g)(x) = \alpha(f(x)+g(x))$$

$= \alpha f(x) + \alpha g(x) = (\alpha f + \alpha g)(x)$

(d) $\forall f \in C^b[a,b], \alpha,\beta \in \mathbb{R}$, to prove $\quad (\alpha+\beta)f = \alpha f + \beta f$

$$\big((\alpha+\beta)f\big)(x) = (\alpha+\beta)f(x) = \alpha f(x) + \beta\, f(x)$$

$$= (\alpha f)(x) + (\beta f)(x) = (\alpha f + \beta f)(x)$$

(e) Let $f \in C^b[a,b]$ *and* 1 is the unity of $\mathbb{R}$, then $(1f)(x) = 1\, f(x) = f(x)$.

General Properties of Linear Space (without prove)

Theorem(1.12):- Let $(L,+,.)$ be a linear space over F .Then

(1) $0.x = 0_L\ , \quad \forall x \in L$

(2) $\lambda.0_L = 0_L\ , \quad \lambda \in F$

(3) $(-\alpha.x) = (-\alpha).x = \alpha.(-x) \quad ,\forall x \in L,\ \alpha \in F$

(4) If $x,y \in L \Longrightarrow \exists!\, z \in L$ such that $x+z=y$

(5) $\alpha.(x-y) = \alpha.x - \alpha.y\ ,\ \forall x,y \in L\ ,\ \alpha \in F$

(6) If $\alpha.x = 0_L \Longrightarrow \alpha = 0$ or $x = 0_L$

(7) If $x \neq 0_L$ *and* $\alpha_1 x = \alpha_2 x \Longrightarrow \alpha_1 = \alpha_2$

(8) If $x \neq 0_L\ ,\ \alpha \neq 0\ , y \neq 0_L$ *and* $\alpha.x = \alpha.y \Longrightarrow x = y$

Linear subspace

Definition(1.13):- Let L be a linear space over F and $\emptyset \neq S \subseteq L$, then we say that S is **linear subspace** of L if S itself is a linear spase over F .

Theorem (1.14):- If L be a linear space over F and $\emptyset \neq S \subseteq L$, then S is linear subspace if satisfy the following conditions

(1) $x + y \in S$, $\forall x, y \in S$

(2) $\alpha . x \in S$, $\forall x \in S$ and $\alpha \in F$

Or satisfy the equivalent condition of two conditions above ,

$\alpha . x + \beta . y \in S, \forall x, y \in S$ and $\alpha, \beta \in F$

Remark(1.15):-

(1) A special subspace of L is improper subspace $S = L$

(2) Every other subspace of $L(\neq \{0\})$is called proper

(3) Another special subspace of any linear space L is S={0}

Example (1.16):- show that $S = \{(x, x_2) \in \mathbb{R}^2 \ ; \ x_2 = 3x_1\}$ is subspace of $\mathbb{R}^2$?

Solution :- It is clear that $S \subseteq \mathbb{R}^2$, and $S \neq \emptyset$ because $(0,0) = 0 \in S$

To prove that $\alpha . x + \beta . y \in S$, $\forall \alpha, \beta \in F = \mathbb{R}$, x , $y \in \mathbb{R}^2$

$x = (x_1, x_2), y = (y_1, y_2)$

$\alpha . x + \beta . y = (\alpha x_1, \alpha x_2) + (\beta y_1, \beta y_2)$

$= (\alpha x_1 + \beta y_1 \, , \alpha x_2 + \beta y_2)$

Now, $\alpha x_2 + \beta y_2 = \alpha(3x_1) + \beta(3y_1) = 3(\alpha x_1 + \beta y_1)$

$\Longrightarrow \alpha . x + \beta . y \in S$.

Example (1.17):- show that $S = \{\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} ; a, b \in \mathbb{R}\}$ is subspace of $M_{2\times2}(\mathbb{R})$?

Solution:- It is clear that $S \subseteq M_{2\times2}(\mathbb{R})$, and $S \neq \emptyset$ because $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in S$

$\forall x, y \in S \; and \; \alpha , \beta \in F.$

To prove that $\alpha . x + \beta . y = \alpha . \begin{pmatrix} 0 & a_1 \\ b_1 & 0 \end{pmatrix} + \beta . \begin{pmatrix} 0 & a_2 \\ b_2 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & \alpha a_1 \\ \alpha b_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \beta a_2 \\ \beta b_2 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & \alpha a_1 + \beta a_2 \\ \alpha b_1 + \beta b_2 & 0 \end{pmatrix} \in S$

Example (1.18):- show that $S = \{(x_1, x_2) \in \mathbb{R}^2; ax_1 + bx_2 = 0\}$ is subspace of $\mathbb{R}^2$? **(H.W.)**

Example (1.19):-The set $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \; ; \; x_1 = 1 + x_2\}$ is not subspace of $\mathbb{R}^3$?

Solution:

Consider $\alpha = 2 \; and \; x = (2,1,0) \in S,$ because $(2 = 1 + 1)$

$\alpha . x = 2.(2,1,0) = (4,2,0) \notin S,$ because$(4 \neq 1 + 2)$

Hence, S is not subspace of $\mathbb{R}^3$.

Theorem(1.20):-Let S_1 and S_2 be two subspaces of linear space L . then

(1) $S_1 \cap S_2$ is subspace of linear space L .

(2) $S_1 + S_2$ is subspace of linear space L

(3) $S_1 \subseteq S_1 + S_2$, $S_2 \subseteq S_1 + S_2$. **(H.W.)**

Exercise :

(1) Which of the following subsets of $\mathbb{R}^3$ be a subspace of $\mathbb{R}^3$

a) $S_1 = \{x = (x_1, x_2, x_3);\ x_1 = x_2 \text{ and } x_3 = 0\}$

b) $S_2 = \{(x_1, x_2, x_3); x_3 = x_2 + 1\}$

c) $S_3 = \{(x_1, x_2, x_3);\ \ x_1, x_2, x_3\ \geq 0\}$

d) $S_4 = \{(x_1, x_2, x_3);\ x_1 - x_2 + x_3 = k\}$

(2) If S_1 and S_2 are subspaces of linear space L , then $S_1 \cup S_2$ not necessary subspace of L (Give example)

(3) If $S \neq \emptyset$ is any subset of L show that span S is subspace of L .

(4) Show that the Cartesian product $L = L_1 \times L_2$ of two linear spaces over the same field becomes a vector space , we define the two algebraic operations by

$$\left.\begin{array}{c}(x_1,x_2)+(y_1,y_2)=(x_1+y_1,x_2+y_2)\\ \hline \alpha(x_1,x_2)=(\alpha x_1,\alpha x_2)\end{array}\right\} \quad \begin{array}{c}\left.\begin{array}{c}\forall\ x=(x_1,x_2)\\ \hline y=(y_1,y_2)\end{array}\right\} \in L\\ \hline \alpha\in F\end{array}$$

(5) Let M be a subspace of a linear space L . The coset of an element $x \in L$ with respect to M is denoted by $x + M$ where

$x + M = \{z, z = x + m, m \in M\}$. Show that $(\frac{L}{M}, +, .)$ is linear space over under algebraic operations defined as

$$(x+m)+(y+m)=(x+y)+m, \forall\, x+m, y+m \in \frac{L}{M}$$

$$\alpha.(x+m)=\alpha.x+m, \forall\, x+m \in \frac{L}{M}\,,\ \alpha \in F$$

Note : The space $(\frac{L}{M}, +, .)$ is called quotient space (or factor space) .

Dimensional Linear Spaces

Definition (1.21):- A **linear combination** of vectors $x_1, x_2, \dots, x_n$ of a linear space L is an expression of the from $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$, where $\alpha_1, \alpha_2, \dots, \alpha_n$ are any scalars

i.e., x is linear combination of $x_1, x_2, \dots, x_n$ if $\exists \alpha_1, \alpha_2, \dots, \alpha_n$ s.t.

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n.$$

Example (1. 22):- Let $S = \{(1,2,3), (1,0,2)\}$, Express $x = (-1,2,-1)$, as a linear combination of x_1 and x_2 .

Solution: We must find scalars $\alpha_1, \alpha_2 \in F$ such that $x = \alpha_1 . x_1 + \alpha_2 . x_2$

$$(-1,2,-1) = \alpha_1.(1,2,3) + \alpha_2.(1,0,2)$$

$$= (\alpha_1, 2\alpha_1, 3\alpha_1) + (\alpha_2, 0, 2\alpha_2)$$

So, $\alpha_1 + \alpha_2 = -1 \implies \alpha_2 = -\alpha_1 - 1$

$$2\alpha_1 + 0 = 2 \Longrightarrow 2\alpha_1 = 2 \Longrightarrow \alpha_1 = 1$$

and, $3\alpha_1 + 2\alpha_2 = -1$

$\therefore\ \alpha_2 = -1 - 1 = -2$.

Example (1. 23):-If $S = \{(1,2,3), (1,0,2)\}$. Show that $x = (-1,2,0)$, is not linear combination of x_1, x_2 .

Solution:

Let $\alpha_1, \alpha_2 \in F$ and $x_1, x_2 \in S$ such that $x = \alpha_1 . x_1 + \alpha_2 . x_2$, we have

$$\left.\frac{\alpha_1 \quad + \alpha_2 = -1\ 2\alpha_1 + \ 0 = \ 2}{3\alpha_1 + 2\alpha_2 = 0}\right\}$$

$$\left(\frac{1 \qquad\qquad 1 \ : -1}{2 \qquad 0 \ : \ 2\,3 \qquad\qquad 2 \ : \ 0}\right) \Longrightarrow \left(\frac{1 \quad 1 \ : -1\,0 \quad -2 \ :4}{0 \quad -1 :3}\right)$$

The system has no solution

$\therefore\ x$ not linear combination of x_1, x_2.

Example (1.24):- Let $S = \{x_1,\ x_2,\ x_3\}$ where $x_1 = (1,2),\ x_2 = (0,1)\ and\ x_3 = (1,1)$. Express (1,0) as a linear combination of x_1, x_2 and x_3.

Solution:

We must find scalars $\alpha_1, \alpha_2, \alpha_3 \in F$ such that $x = \alpha_1 . x_1 + \alpha_2 . x_2 + \alpha_3 . x_3$

$$(1,0) = \alpha_1 . (1,2) + \alpha_2 . (0,1) + \alpha_3 . (1,1)$$

$$(1,0) = (\alpha_1, 2\alpha_1) + (0, \alpha_2) + (\alpha_3, \alpha_3)$$

$$\alpha_1 + \alpha_3 = 1 \Longrightarrow \alpha_1 = 1 - \alpha_3$$

$$2\alpha_1 + \alpha_2 + \alpha_3 = 0 \Longrightarrow 2(1-\alpha_3) + \alpha_2 + \alpha_3 = 0 \Longrightarrow -\alpha_3 + \alpha_2 = -2$$

$$\alpha_2 = -2 - \alpha_3$$

This system has multiple solutions in this case there are multiple possibilities for the α_i.

Definition (1.25):- Let $\emptyset \neq M \subseteq L$ the smallest subspace of L contains M is called **subspace generated** by M and denoted by [M] or span M.

Remark(1.26):-

1. Let $\emptyset \neq M \subseteq L$, the set of all linear combinations of vectors of M is called span of M .

2. $M \subset$span (M).

3. Span (M) = the intersection of all subspace of L containing M.

Example (1.27):- Find span $\{x_1, x_2\}$ where $x_1 = (1,2,3)$ and $x_2 = (1,0,2)$?

Solution :- The span $\{x_1, x_2\}$ is the set of all vectors $(x, y, z) \in \mathbb{R}^3$ such that

$$(x, y, z) = \alpha_1.(1,2,3) + \alpha_2.(1,0,2)$$

We wish to know for what values of (x, y, z) does this system of equations have solutions for α_1, α_2

$\alpha_1.(1,2,3) + \alpha_2.(1,0,2) = (x, y, z)$

$(\alpha_1, 2\alpha_1, 3\alpha_1) + (\alpha_2, 0, 2\alpha_2) = (x, y, z)$

$\alpha_1 + \alpha_2 = x \Longrightarrow \alpha_2 = x - \alpha_1$

$2\alpha_1 = y \Longrightarrow \alpha_1 = \frac{1}{2}y$

$3\alpha_1 + 2\alpha_2 = z \Longrightarrow 6\alpha_1 + 4\alpha_2 - 2z = 0$

$6\left(\frac{1}{2}y\right) + 4\left(x - \frac{1}{2}y\right) - 2z = 0$

$3y + 4x - 2y - 2z = 0$

$4x + y - 2z = 0$

So, solutions when $4x + y - 2z = 0$

Thus span $\{x_1, x_2\}$ is the plane $4x + y - 2z = 0$

Example (1.28):-Show that $\{x_1, x_2\}$ span $\mathbb{R}^2$, when $x_1 = (1,1), x_2 = (2,1)$.

Solution : we being asked to show that any vectors in $\mathbb{R}^2$ can written as a linear combination of x_1, x_2. Let $(a, b) \in \mathbb{R}^2 and\ (a, b) = \alpha_1.(1,1) + \alpha_2.(2,1)$

$(\alpha_1, \alpha_1) + (2\alpha_2, \alpha_2) = (a, b)$

$\alpha_1 + 2\alpha_2 = a \Longrightarrow \alpha_1 = a - 2\alpha_2$

$\alpha_1 + \alpha_2 = b \Longrightarrow \alpha_2 = b - (a - 2\alpha_2)$

$-\alpha_2 = b - a \Longrightarrow \alpha_2 = a - b$

$\alpha_1 = a - 2(a-b) = 2b - a$ .Note that these two vectors span $\mathbb{R}^2$, that is every vector $\mathbb{R}^2$ can be expressed as a linear combination of them .

Example (1.29):-Show that $S = \{x_1, x_2, x_3\}\ span\ \mathbb{R}^2$, where $x_1 = (1,1), x_2 = (2,1), x_3 = (3,2)$. **(H.W.)**

Definition (1.30):- Let S = { $x_1, \dots, x_n$} be a subset of L , then S is called **linearly independent** if there exist $\alpha_1, \alpha_2, \ \dots, \alpha_n$ such that

if $\alpha_1.x_1 + \alpha_2.x_2 + \cdots + \alpha_n.x_n = 0$ then $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.

Definition (1.31):- Let S = { $x_1, x_2, \dots, x_n$} be a subset of L , then S is said to be **linearly dependent** if it is not linearly independent that is if

$\alpha_1.x_1 + \alpha_2.x_2 + \cdots + \alpha_n.x_n = 0$ but the $\alpha_1, \alpha_2, \ \dots, \alpha_n$ not all zero .

Example (1.32):- Determine $S = \{x_1, x_2\}$ is linearly dependent or independent where $x_1 = (1,2,3),\ x_2 = (1,0,2)$.

Solution : Let $\alpha_1, \alpha_2 \in F$

$\alpha_1(1,2,3) + \alpha_2(1,0,2) = (0,0,0)$, only solution is trivial solution $\alpha_1 = \alpha_2 = 0$. Thus, S is linearly independent.

Example (1.33):-Determine $S = \{x_1, x_2\}$ is linearly dependent or independent where $x_1 = (1,1,1), x_2 = (2,2,2)$?

Solution: Let $\alpha_1, \alpha_2 \in F$

$\alpha_1(1,1,1) + \alpha_2(2,2,2) = (0,0,0)$

$\alpha_1 + 2\alpha_2 = 0 \Longrightarrow \alpha_1 = 2\alpha_2$

So, S is linearly dependent

Theorem (1.34):- **(without prove)**

(1) Every m vectors set in $\mathbb{R}^n$, if $m > n$ then , the set is linearly dependent

(2) A linearly independent set in $\mathbb{R}^n$ has at most n vectors .

Remark (1.35):- Let L linear space over F , $S \subseteq L$ and $x_0 \in L$, then

(1) If $0_L \in S \Longrightarrow S$ is linear dependent . i.e., every subspace is linear dependent set

(2) If $x_0 \neq 0_2 \Longrightarrow \{x_0\}$ is linearly independent

Definition (1.36):- Let L be a linear space over F .A subset B of L is a **basis** if it is linearly independent and spans L i.e,

(1) B is linearly independent

(2) Span (B) = L L تولد الـ B

The number of elements in a basis for L is called the **dimension** of L and is denoted by dim (L)

Example(1.37):- Consider the linear space $(\mathbb{R}^3, +, ..)$

The dimension of L is 3. i.e., $dim\,(\mathbb{R}^3) = 3$

Since $B = \{(1,0,0), (0,1,0), (0,0,1)\}$ is basis for $\mathbb{R}^3$

Remark (1.38):-

(1) Every linear space $L \neq \{0\}$ has a basis

(2) If $L = \{0\} \Longrightarrow dim\,(L) = 0$

(3) If L finite dimension linear space and S be a subspace of L, then $dim\, S \leq dim\, L$.

(4) If $dim\, S = \dim L \Longrightarrow S = L$

(5) If S $= \{\, x_1, \ldots, x_n\}$ be a linearly independent in L then there exists $c > 0$ such that $\|x\| = \| \sum_{i=1}^{n} \alpha_i x_1 \| \geq c \sum_{i=1}^{n} |\alpha_i|$.

(6) The dimension of quotient space (or factor space) is called codimension of M and denoted by $codm(M) = dim\,(\frac{L}{M})$.

The Convexity

Definition (1.39):- Let A be a subset of linear space L then , we say that A is **convex set** if satisfy the following condition :

$\forall x, y \in A\,, \lambda \in [0,1]$ then $\lambda x + (1-\lambda) y \in A$.

Examples About Convex Sets

Example(1.40):- If $A = (a, b) \subset \Longrightarrow A$ is convex set

Solution :

Let $x, y \in A\,, \lambda \in [0,1]$

$$x \in (a,b) \Rightarrow a < x < b \ \Rightarrow \lambda a \ < \lambda x < \lambda b \quad \ldots (1)$$

$$y \in (a,b) \Rightarrow a < y < b \ \Rightarrow (1-\lambda)a < (1-\lambda)y < (1-\lambda)b \quad (2)$$

By summing up (1) and (2)

$$\lambda a + (1-\lambda)a < \lambda x + (1-\lambda)y < \lambda b + (1-\lambda)b$$

$$a < \lambda x + (1-\lambda)y < b.$$

Example(1.41):- Every linear subspace is convex, but the converse is not true in general

Solution :

Let M be a subspace of linear space L $\Longrightarrow \alpha x + \beta y \in M$, $\forall x, y \in M$, $\alpha, \beta \in F$. Put $\alpha = \lambda, \beta = 1 - \lambda$, $0 \leq \lambda \leq 1$

$$\Longrightarrow \lambda x + (1-\lambda)y \in M\,,\ \forall x, y \in M\,, 0 \leq \ \lambda \ \leq 1 \quad \Longrightarrow M \text{ is convex set}$$

For the converse , consider the following example

Let $L = \mathbb{R}^2, M = \{(x,y) \in \mathbb{R}^2, x \geq 0\,, y \geq 0\}$, then M is convex set but not subspace.

❖ To prove M is convex set.

Let $(x_1, y_1), (x_2, y_2) \in M$ and $0 \leq \lambda \leq 1$. To prove $\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) \in M$.

$(x_1, y_1) \in M \Rightarrow\ x_1 \geq 0\,, y_1 \geq 0$ and $(x_2, y_2) \in M \Rightarrow\ x_2 \geq 0\,, y_2 \geq 0$.

Thus, $\lambda x_1 + (1-\lambda)y_1 \geq 0$ and $\lambda x_2 + (1-\lambda)y_2 \geq 0$.

Then , $\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) = (\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2) \in M.$

❖ Show that M is not a subspace **(H.W.)**

Theorem(1.42): (Some Properties About Convex Sets)

1. The singleton set is convex set
2. The intersection of convex set is convex.
3. The empty set and the whole space are convex. **(H.W.)**
4. If A is convex set $\Longrightarrow$ αA also convex $\alpha \in F$. **(H.W.)**
5. If A and B are convex set $\Longrightarrow A + B$ also convex set.

Proof (1) Let $A = \{x\}$ to prove A is convex set.

Take $x \in A$ and $\lambda \in [0,1]$ then $\lambda x + (1-\lambda)x = x \in M$.

Proof (2) Let A and B are convex sets To prove $A \cap B$ is convex set.

Let $x, y \in A \cap B$ and $0 \leq \lambda \leq 1$ to prove $\lambda x + (1-\lambda)y \in A \cap B$.

$x, y \in A$ and A is convex $\Rightarrow$ $x + (1-\lambda)y \in A$ …. (1)

$x, y \in B$ and B is convex $\Rightarrow$ $x + (1-\lambda)y \in B$ …. (2)

From (1)&(2) we get $x + (1-\lambda)y \in A \cap B$. Then, $A \cap B$ is convex set.

Proof (5) Let $a_1 + b_1, a_2 + b_2 \in A + B$, then $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

To prove $\lambda\,(a_1 + b_1) + (1-\lambda)(a_2 + b_2) \in A + B$, $\forall \lambda \in [0,1]$

Since A is convex set and $a_1, a_2 \in A \Rightarrow \lambda\, a_1 + (1-\lambda)a_2 \in A \;\; \forall\lambda \in [0,1]$…(1)

Since B is convex set and $b_1, b_2 \in B \Rightarrow \lambda\, b_1 + (1-\lambda)b_2 \in B \;\; \forall\lambda \in [0,1]$…(2)

By summing up (1) and (2) we get

$\lambda\, a_1 + (1-\lambda)a_2 + \lambda\, b_1 + (1-\lambda)b_2 \in A + B$

i.e., $\lambda\,(a_1 + b_1) + (1-\lambda)(a_2 + b_2) \in A + B$. Thus, $A + B$ is a convex set.

Linear operator and linear functional

Definition(1.43):- Let L and L' are linear spaces over the same field F . A mapping $T: L \rightarrow L'$ is called **Linear operator** or (**Linear transformation**) if

$$T(\alpha.x + \beta.y) = \alpha T(x) + \beta T(y)\,, \forall x, y \in L \text{ and } \alpha, \beta \in F$$

Note : The linear operator $T: L \rightarrow F$ is said to be **linear functional**.

Examples of Linear Functional

Example (1.44):- Let $T: \mathbb{R} \rightarrow \mathbb{R}$ such that $T(x) = mx\,, m \in \mathbb{R}$ the T is linear functional

Solution : Let $x, y \in \mathbb{R}\,, \alpha, \beta \in \mathbb{R}$

$$T(\alpha x+\beta y)=m(\alpha x+\beta y)=m(\alpha x)+m(\beta y)$$
$$=\alpha(mx)+\beta(my)$$
$$=\ \alpha T(x)+\beta T(y)$$

Example(1.45):- Let $T:\mathbb{R}\to\mathbb{R}$ s.t. $T(x)=mx+b\ ,\ \ m,b\neq 0\in\mathbb{R}$ then T is not linear functional ?

Solution: Let $x\ _,\ y\in\mathbb{R}$

$$T(\alpha x+\beta y)=m(\alpha x+\beta y)+b=(m\alpha)x+(m\beta)y+b$$
$$=\alpha(mx)+\beta(my)+b\ \ldots..\qquad (1)$$

Now, $\alpha T(x)+\beta T(y)=\ \alpha(mx+b)+\beta(my+b)$

$$=\alpha mx+\beta my+(\alpha+\beta)b\ \ldots\ldots\quad (2)$$

Let $\alpha=1,\beta=2\ ,\ we\ get\ 1\neq 2$

Example(1.46):- Let $T:C[a,b]\to\mathbb{R}\ such\ that\ T\ (f)=\int_a^b f(x)dx$

Show that T is linear functional.

Solution : Let $f,g\in C[a,b]\ ,\alpha,\beta\in\mathbb{R}$. To prove that

$$T(\alpha f+\beta g)=\alpha T(f)+\beta T(g)$$

$$T(\alpha f+\beta g)=\int_a^b(\alpha f+\beta g)(x)dx$$
$$=\ \int_a^b\alpha f(x)\ dx+\ \int_a^b\beta g(x)dx$$
$$=\ \alpha\int_a^b f(x)dx+\ \beta\int_a^b g(x)dx=\ \alpha T(f)+\beta T(g).$$

Example(1.47):- Show that $T:\mathbb{R}^2\to\mathbb{R}^2$ such that

$T(x_1, x_2) = (x_1 + x_2, x_1 - x_2 + 1)$ is not linear operator

Solution : $T(0) = T((0,0)) = (0,1) \neq (0,0) \Longrightarrow T$ not linear functional.

Exercise

(1) If $T: \mathbb{R}^2 \to \mathbb{R}$ s.t $T(x_1, x_2) = {x^2}_1 + {x^2}_2$. Show that T is not linear functional.

(2) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t $T(x_1, x_2) = (x_1 + x_2, 0)$. Is T linear transformation ?

(3) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t $T(x_1, x_2) = (\alpha x_1, x_2), \alpha \in \mathbb{R}$. Is T linear transformation ?

(4) If $T: \mathbb{R}^3 \to \mathbb{R}^3$ s.t $T(x_1, x_2, x_3) = (x_2, x_1 - x_2, 0)$. Is T linear transformation ?

Some Theorems About Linear operators

Theorem(1.48):- Let L, L', L'' are linear spaces over the same field F such that $T: L \to L'$ and g: $L' \to L''$ linear operators then

(1) $T(0_L) = 0_{L'}$ and $g(0_{L'}) = 0_{L''}$

(2) $T(x - y) = T(x) - T(y)$

(3) goT is linear operator .

Theorem(1.49):- Let L, L' are linear spaces over the same field F and $T_1, T_2 : L \to L'$ are two linear operators. Define $+$ and $.$ as follows:

$$(T_1 + T_2)(x) = T_1(x) + T_2(x), \forall x \in L$$

$$(\lambda T_1)(x) = \lambda . T_1(x), \quad \forall x \in L, \ \lambda \in F$$

Then, $T_1 + T_2$ and αT_1 are linear operators

Proof :- Let $x, y \in L \quad and \ \alpha, \beta \in F$ then

(1) $(T_1 + T_2)(\alpha . x + \beta . y) = T_1(\alpha . x + \beta . y) + T_2(\alpha . x + \beta . y)$

$= \alpha . T_1(x) + \beta . T_1(y) + \alpha . T_2(x) + \beta . T_2(y)$

$= \alpha . (T_1(x) + T_2(x)) + \beta . (T_1(y) + T_2(y))$

$= \alpha . (T_1 + T_2)(x) + \beta . (T_2 + T_2)(y)$

(2) $\lambda T_1(\alpha . x + \beta . y) = \lambda . [\alpha . T_1(x) + \beta . T_1(y)]$

$= (\lambda \alpha) . T_1(x) + (\lambda \beta) . T_1(y)$

$= \alpha . \big(\lambda . T_1(x)\big) + \beta \big(\lambda . T_1(y)\big)$

$= \alpha . (\lambda T_1)(x) + \beta . (\lambda T_1)(y).$