Ch.5 Inventory Models

The inventory deals with stocking an item to meet fluctuations in demand. The inventory problem involves placing and receiving orders of given sizes periodically. The basis for the decision is a model that balances the cost of capital resulting from holding too much inventory against the penalty cost resulting from inventory shortage. The problem reduces to controlling the inventory level by devising an ***inventory policy*** that answers two questions:

1. How much to order?
2. When to order?

The basis for answering these questions is the minimization of the following inventory cost function:

$$\begin{pmatrix} Total \\ inventory \\ cost \end{pmatrix} = \begin{pmatrix} Purchasing \\ cost \end{pmatrix} + \begin{pmatrix} Setup \\ cost \end{pmatrix} + \begin{pmatrix} Holding \\ cost \end{pmatrix} + \begin{pmatrix} Shortage \\ cost \end{pmatrix}$$

1. ***Purchasing cost*** is the price per unit of an inventory item. At times the item is offered at a discount if the order size exceeds a certain amount, which is a factor in deciding how much to order.

2. ***Setup cost*** represents the fixed charge incurred when an order is placed regardless of its size. This includes salaries, transportation cost, insurance, etc.

3. ***Holding cost*** represents the cost of maintaining inventory in stock. It includes the interest on capital, the cost of storage, maintenance, and handling.

4. ***Shortage cost*** is the penalty incurred when we run out of stock. It includes potential loss of income, disruption in production, and the more subjective cost of loss in customer's goodwill.

An inventory system may be based on ***periodic review*** (e.g., ordering every week or every month). Alternatively, the system may be based on ***continuous review***, where a new order is placed when the inventory level drops to a certain level, called the ***reorder point***.

5.1 Role of Demand in the Development of Inventory Models

In general, the analytic complexity of inventory models depends on whether the demand for an item is deterministic or probabilistic. Within either category, the demand may or may not vary with time. For example, the consumption of natural gas used in heating homes is seasonal. Though this seasonal pattern repeats itself annually, the same-month consumption may vary from year to year, depending, for example, on the severity of weather.

In practical situations the demand pattern in an inventory model may assume one of four types:

1. Deterministic and constant (static) with time.
2. Deterministic and variable (dynamic) with time.
3. Probabilistic and stationary over time.
4. Probabilistic and non-stationary over time.

This categorization assumes the availability of data that are representative of future demand. Demand is usually probabilistic, but in some cases the simpler deterministic approximation may be acceptable. The complexity of the inventory problem does not allow the development of a general model that covers all possible situations.

5.2 Static Economic-Order-Quantity (EOQ) Models

5.2.1 Classic EOQ Model (Constant-Rate Demand, no Shortage)

The simplest of the inventory models involves constant-rate demand with instantaneous order replenishment and no shortage. Define:

y = Order quantity (number of units)

D = Demand rate (units per unit time)

t_0 = Ordering cycle length (time units)

The inventory level follows the pattern explained if Figure (5.1). When the inventory reaches zero level, an order of size y units is received instantaneously. The stock is then depleted uniformly at the constant demand rate D.

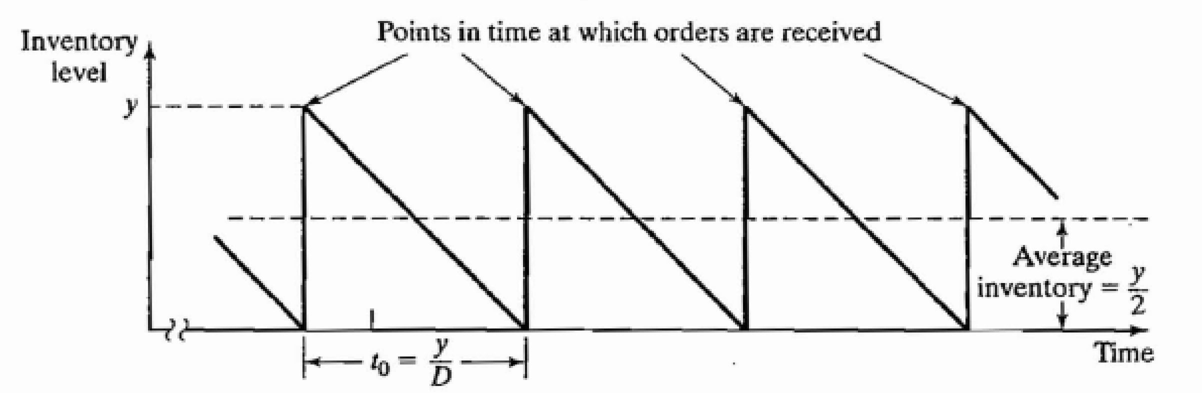

Figure (5.1)

The ordering cycle for this pattern is:

$$t_0 = \frac{y}{D} \ time\ units$$

The cost model requires two cost parameters:

K = Setup cost associated with the placement of an order (monetary units per order)

h = Holding cost (monetary units per inventory unit per unit time)

Given that the average inventory level is $\frac{y}{2}$, the total ***cost per unit time*** (***TCU***) is thus computed as

$TCU(y)$ = Setup cost per unit time + Holding cost per unit time

$$= \frac{Setup\ cost + Holding\ cost\ per\ cycle\ t_0}{t_0}$$

$$= \frac{K+h(\frac{y}{2})t_0}{t_0} = \frac{K}{(\frac{y}{D})} + h\left(\frac{y}{2}\right) = \frac{KD}{y} + \frac{hy}{2}$$

The optimum value of the order quantity y is determined by minimizing (y). Assuming y is continuous, a necessary condition for optimality is:

$$\frac{dTCU(y)}{dy} = -\frac{KD}{y^2} + \frac{h}{2} = 0$$

The condition is also sufficient because $TCU(y)$ is convex.

The solution of the equation yields the EOQ y^* as

$$y^* = \sqrt{\frac{2KD}{h}}$$

Thus, the optimum inventory policy for the proposed model is

$$Order\ y^* = \sqrt{\frac{2KD}{h}}\ units\ every\ t_o^* = \frac{y^*}{D}\ time\ units$$

Actually, a new order need not be received at the instant it is ordered. Instead, a positive ***lead time***, L, may occur between the placement and the receipt of an order. In this case, the ***reorder point*** occurs when the inventory level drops to LD units. Sometimes, it is assumed that the lead time L is less than the cycle length t_0^* ,which may not be the case in general. To account for this situation, we define the ***effective lead time*** as

$$L_e = L - nt_0^*$$

where n is the largest integer not exceeding $\frac{L}{t_0^*}$. The reorder point occurs at L_eD units, and the inventory policy can be restated as:

Order the quantity y^* whenever the inventory level drops to L_eD units

Example (5.1):

Neon lights on the U of A campus are replaced at the rate of 100 units per day. The physical plant orders the neon lights periodically. It costs 100 $ to initiate a purchase order. A neon light kept in storage is estimated to cost about 0.02 $ per day. The lead time between placing and receiving an order is 12 days. Determine the optimal inventory policy for ordering the neon lights.

Solution:

From the data of the problem, we have:

D = 100 units per day

K = 100 $ per order

h = 0.02 $ per unit per day

L = 12 days

Thus,

$$y^* = \sqrt{\frac{2KD}{h}} = \sqrt{\frac{2\times 100\times 100}{0.02}} = 1000 \text{ neon light}$$

The associated cycle length is:

$$t_0^* = \frac{y^*}{D} = \frac{1000}{100} = 10\ days$$

Because the lead time $L = 12$ days exceeds the cycle length t_0^* (= 10 days) , we must compute L_e. The number of integer cycles included in L is

$$n = \left(Largest\ integer\ \leq \frac{L}{t_0^*}\right) = \left(Largest\ integer\ \leq \frac{12}{10}\right) = 1$$

Thus,

$$L_e = L - nt_0^* = 12 - 1\times 10 = 2 \text{ days}$$

The reorder point thus occurs when the inventory level drops to

$$L_eD = 2\ X\ 100 = 200 \text{ neon lights}$$

The inventory policy for ordering the neon lights is:

Order 1000 units whenever the inventory level drops to 200 units.

The daily inventory cost associated with the proposed inventory policy is:

$$TCU(y) = \frac{KD}{y} + \frac{hy}{2} = \frac{100\times 100}{1000} + 0.02\left(\frac{1000}{2}\right) = 20\ \$/\text{ day}$$

Exercise 5.1 (in addition to text book exercises)

A carpenter orders 48000 unit of an item yearly. The order costs 800$ and the holding cost is 10 cents per item monthly. The lead time between placing and receiving an order is 4 month. Determine the optimal inventory policy.

5.2.2 Manufacturing Model, no Shortage

In previous discussed models we have assumed that the replenishment time is zero and the items are procured in one lot. But in real practice, particularly in manufacturing model, items are produced on a machine at a finite rate per unit of time; hence we cannot say the replenishment time as zero. Here we assume that the replenishment rate is finite say at the rate of α units per unit of time. Let:

y = Order quantity (number of units)

D = Demand quantity (units per unit time)

P = Production quantity (units per unit time) ($P > D$)

t_0 = Ordering cycle length (time units)

K = Setup cost associated with the placement of an order (monetary units per order)

h = Holding cost (monetary units per inventory unit per unit time)

Figure (5.2) shows variation of inventory with time

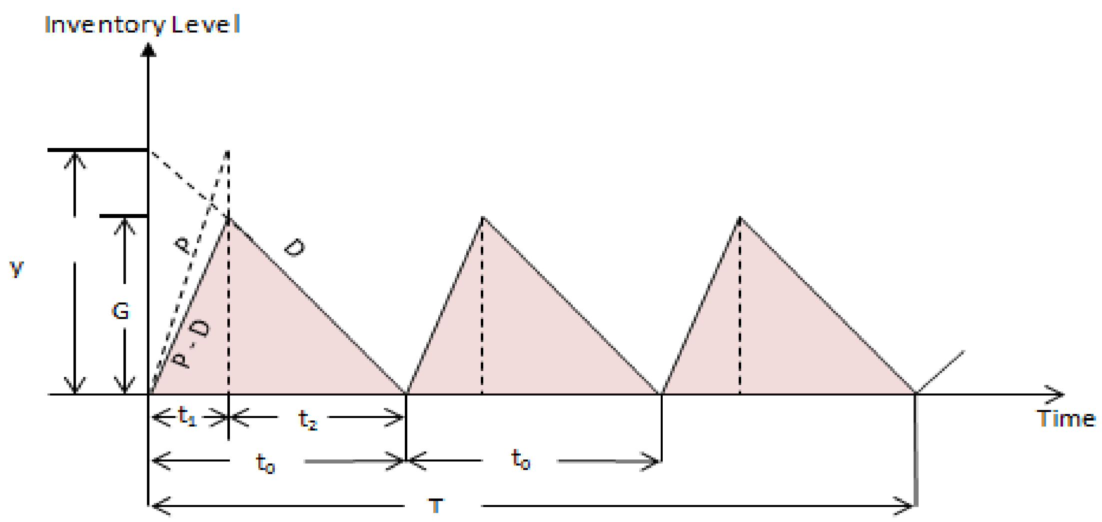

Figure (5.2)

Here each production run of length t consists of two parts t_1 and t_2 , where:

i) t_1 is the time during which the stock is building up at a rate $P - D$ units per unit time.

ii) t_2 is the time during which there is no production (for supply or replenishment) and inventory is decreasing at a constant demand rate D per unit time.

Let $G =$ the maximum inventory available at the end of time t_1 which is expected to be consumed during the remaining period t_2 at the demand rate D.

$TCU(y)/circle$ $=$ Setup cost per unit time + Holding cost per unit time

$$= K + h\frac{G}{2}t_1 + h\frac{G}{2}t_2 = K + h\frac{G}{2}(t_1 + t_2)$$

Since $t_0 = t_1 + t_2$, then:

$$TCU(y)/circle = K + h\frac{G}{2}t_0$$

From right-angled triangle: $t_1 = \frac{G}{P-D} \Longrightarrow G = t_1(P - D)$

$\Longrightarrow G = \frac{y}{P}(P - D) = y(1 - \frac{D}{P})$

Let: $b = 1 - \frac{D}{P}$, then $G = yb$

$\Longrightarrow TCU(y)/circle = K + h\frac{yb}{2}t_0$

$$\Longrightarrow TCU(y) = \frac{K}{t_0} + h\frac{yb}{2} = \frac{KD}{y} + h\frac{yb}{2}$$

The optimum value of the order quantity y is determined by minimizing (y) . Assuming y is continuous, a necessary condition for optimality is*:*

$$\frac{dTCU(y)}{dy} = -\frac{KD}{y^2} + \frac{hb}{2} = 0$$

Then: $y^* = \sqrt{\frac{2KD}{hb}}$

Thus, the optimum inventory policy for the proposed model is

$Order\ y^* = \sqrt{\frac{2KD}{hb}}$ *units whenever the inventory level drops to* $G^* = y^* b$

Example (5.2):

A manufacturer must supply 10000 units of an item to a car factory daily. He can produce 25000 units daily; the holding cost of each unit is 2 cents per year and the fixed cost of production is 18 $. Determine the optimal number of produced items (no shortage) then find the total inventory cost for a year and the optimum inventory policy.

Solution:

From the data of the problem, we have:

D=10000 units per day

P=25000 units per day

h=0.02/360 $ per day

K=18 $ per cycle

$$b = 1 - \frac{D}{P} = 1 - \frac{10000}{25000} = \frac{3}{5}$$

$$y^* = \sqrt{\frac{2KD}{hb}} = \sqrt{\frac{2\times 18\times 10000}{\frac{0.02}{360}\times\frac{3}{5}}} = 10400\,unit$$

$$cost = \frac{KD}{y^*} + h\frac{y^* b}{2} = \frac{18\times 10000}{10400} + \frac{0.02}{360}\times\frac{10400\times\frac{3}{5}}{2} = 1224\text{ \$/day}$$

$$Then\ cost\ per\ year = 1224\times 360 = 440640\text{ \$}$$

$G = y^* b = 10400\times\frac{3}{5} = 6240\ unit$. Then the optimal inventory policy is:

Produce 10400 units when the inventory level drops to 6240 unit.

Exercise 5.2 (in addition to text book exercises)

A company has a demand of 12000 units / year for an item and it can produce 2000 such items per month. The cost of one setup is 400 $ and the holding cost / unit / month is 0.15 $. Find the optimum lot size and the total cost per year.

5.3 Probabilistic inventory models

The models previously discussed are only artificial since in practical situations demand is hardly known precisely. In most situations demand is probabilistic since only probability distribution of future demand, rather than the exact value of demand itself, is known. The probability distribution of future demand is usually determined from the data collected from past experience. In such situations we choose policies that minimize the expected costs rather than the actual costs.

5.3.1 Instantaneous Demand, Setup Cost Zero, Stock Levels Discrete and Lead Time Zero

This model deals with the inventory situation of items that require one time purchase only. Perishable items such that cut flowers, cosmetics, spare parts, seasonal items such as calendars and diaries, etc. fall under this category.

In this model the item is ordered at the beginning of the period to meet the demand during that period, the demand being instantaneous as well as discrete in nature. At the end of the period, there are two types of cost involved: over-stocking cost and under-stocking cost. They represent opportunity losses incurred when the number of units stocked is not exactly equal to the number of units actually demanded. Let:

D = Discrete demand rate with probability P_D

y_m = Discrete stock level for time interval t_0

t_0 = Ordering cycle length

C_1 = Over-stocking cost (over-ordering cost). This is opportunity loss associated with each unit left unsold.

$= C + C_h - V$

C_2 = Under-stocking cost (under-ordering cost). This is opportunity loss due to not meeting the demand.

$= S - C - C_h/2 + C_s$

Where C is the unit cost price, C_h the unit carrying (holding) cost, C_s the unit shortage cost, S the unit selling price and V is the salvage value. If value of any parameter is not given, it is taken as zero.

Production is assumed to be instantaneous and lead time is negligibly small. The problem is to determine the optimal inventory level y_m, where $D \le y_m$(there is no shortage) or $D > y_m$ (shortage occur).

Then the optimal order quantity y_m^* is determined when value of cumulative probability distribution exceeds the ratio $\frac{C_2}{C_1+C_2}$ by computing:

$$P_{D \le y_m - 1} \le \frac{C_2}{C_1 + C_2} \le P_{D \le y_m}$$

Example (5.3):

A trader stocks a particular seasonal product at the beginning of the season and cannot reorder: the item costs him 25 \$ and he sells it at 50 \$ each. For any item that cannot be met on demand, the trader has estimated a goodwill cost of 15 \$. Any item unsold will have a salvage value of 10 \$. Holding cost during the period is estimated to be 10 % of the price. The probability of demand is as follows:

Units stocked	2	3	4	5	6
Probability of demand	0.35	0.25	0.20	0.15	0.05

Determine the optimal number of items to be stocked.

Solution:

Here: $= 25\$, S = 50\$, C_h = 0.10 \times 25 = 2.5\$, C_s = 15\$, V = 10\$$.

$\therefore\ C_1 = C + C_h - V = 25 + 2.5 - 10 = 17.5\ \$$

$C_2 = S - C - \frac{C_h}{2} + C_s = 50 - 25 - \frac{2.5}{2} + 15 = 38.75\ \$$

Cumulative probability of demand is now calculated:

Units stocked	2	3	4	5	6
Probability of demand	0.35	0.25	0.20	0.15	0.05
Cumulative probability of demand $\sum_{D=0}^{y_m} P_D$	0.35	0.60	0.80	0.95	1.00

Now: $\frac{C_2}{C_1+C_2} = \frac{38.75}{17.5+38.75} = 0.69.$

Since $0.60 \ < 0.69 < 0.80,$ then $3 < y_m < 4$. Then $y_m^* = 4$ units.

Example (5.4):

A newspaper boy buys papers for 5 ¢ each and sells them for 6 ¢ each. He cannot return unsold newspapers. Daily demand D for newspapers follows the distribution:

D	10	11	12	13	14	15	16
P_D	0.05	0.15	0.40	0.20	0.10	0.05	0.05

If each day's demand is independent of the previous day's, how many papers should be ordered each day?

Solution:

Here: $= 0.05\$, S = 0.06\$, C_h = 0\ \$, C_s = 0\ \$, V = 0\ \$$.

$\therefore\ C_1 = C + C_h - V = 0.05\ \$$

$C_2 = S - C - \frac{C_h}{2} + C_s = 0.06 - 0.05 = 0.01\ \$$

Cumulative probability of demand is now calculated:

D	10	11	12	13	14	15	16
P_D	0.05	0.15	0.40	0.20	0.10	0.05	0.05
$\sum_{D=0}^{y_m} P_D$	0.05	0.20	0.60	0.80	0.90	0.95	1.00

Now: $\frac{C_2}{C_1+C_2} = \frac{0.01}{0.01+0.05} = \frac{1}{6} = 0.167.$

Since $0.05 \ < 0.167 < 0.20,$ then $10 < y_m < 11$. Then $y_m^* = 11$ newspapers.

5.3.2 Instantaneous Demand, Setup Cost Zero, Stock Levels Continuous and Lead Time Zero

In this model, all conditions are the same as model in 5.3.1 except that the stock levels are continuous. Therefore, probability $f(D)dD$ will be used instead of P_D, where f(D) is the probability density function of the demand rate D.

Then the optimal order quantity y_m^* is determined when value of cumulative probability distribution exceeds the ratio $\frac{C_2}{C_1+C_2}$ by computing:

$$\int_{D=0}^{y_m} f(D)dD = \frac{C_2}{C_1 + C_2}$$

Example (5.5):

A baking company sells one of its types of cakes by weight. It makes profit of 95 ¢ a pound on every pound of cake sold on the day it is baked. It disposes all cakes not sold on the day they are baked at loss of 15 ¢ a pound. If demand is known to have a probability density function:

$$f(D) = 0.03 - 0.0003\,D$$

Find the optimum amount of cake the company should bake daily.

Solution:

Penalty cost / unit of oversupply, $C_1 = 0.15$ $

Penalty cost / unit of undersupply, $C_2 = 0.95$ $

Using the relation: $\int_{D=0}^{y_m} f(D)dD = \frac{C_2}{C_1+C_2}$, we get:

$\int_0^{y_m}(0.03 - 0.0003D)dD = \frac{0.95}{0.15+0.95} = \frac{0.95}{1.1} = 0.8636$

$0.03y_m - 0.00015y_m^2 = 0.8636 \qquad (\times 10^5)$

$3000y_m - 15y_m^2 = 86360 \qquad (\div 15)$

$200y_m - y_m^2 = 5757$

$y_m^2 - 200y_m + 5757 = 0,$

$y_m = \frac{200 \pm \sqrt{(200)^2 - 4\times 5757}}{2} = 165.15$ or 34.84 pounds

$y_m = 165.15$ pounds is not feasible since the given probability distribution of D is not applicable above 100 pounds.

$\therefore \quad y_m^* = 34.85$ pounds

Exercise 5.3 (in addition to text book exercises)

1: The probability distribution of monthly sales of certain item is as follows:

Monthly sales	0	1	2	3	4	5	6
Probability	0.01	0.06	0.25	0.35	0.20	0.03	0.10

The cost of carrying inventory is 30 $ per unit per month and the cost of unit shortage is 70 $ per month. Determine the optimum stock level which minimizes the total expected cost.

2: A baking company sells one of its types of cakes by weight. It makes profit of 50 ¢ a pound on every pound of cake sold on the day it is baked. It disposes all cakes not sold on the day they are baked at loss of 12 ¢ a pound. If the demand is known to be rectangular between 2000 and 3000 pounds, determine the optimum daily amount baked.(In a rectangular(or uniform) distribution all values within a range between a and b are equally likely. The probability density is: 1 / (b - a))