$$S.t. \quad 2x_1 + 4x_2 \leq 80$$
$$6x_1 + 2x_2 \geq 60$$
$$8x_1 + 6x_2 \geq 120$$
$$x_1, x_2 \geq 0$$

2.9 The Simplex Method

The graphical method cannot be applied when the number of variables in the LPP is more than three, or rather two, since even with three variables the graphical solution becomes tedious as it involves intersection of planes in three dimensions. The simplex method can be used to solve any LPP (for which the solution exists) involving any number of variables and constraints.

The computational procedure in the simplex method is based on a fundamental property that the optimal solution to an LPP, if it exists, occurs only at one of the corner points of the feasible region. The simplex method is an iterative method starts with initial basic feasible solution at the origin, i.e. Z=0. If the solution is not optimal, we move to the adjacent corner, until after a finite number of trials, the optimal solution, if it exists, is obtained.

The steps of the simplex method are as follows:

Step 1: Convert the given problem into the standard form. The Right Hand Side (RHS) of each constraint must be non-negative. Write the objective function in the form: $Z - \sum_{j=1}^{n} c_j x_j \ = 0$

Step 2: Set $x_1 = x_2 = \ldots = x_n = 0$, i.e. $x_1, x_2, \ldots, x_n$ are non-basic variables, thus $s_1, s_2, \ldots, s_m$ are the basic-variables.

Step 3: Construct the initial ***simplex table*** (or ***tableau***) with all slack variables in the BVS. The simplex table for the general LPP (in 2.4) is:

Basic variables	x_1	x_2	…	x_n	s_1	s_2	…	s_m	Solution
s_1	a_{11}	a_{12}	…	a_{1n}	1	0	…	0	b_1
s_2	a_{21}	a_{22}	…	a_{2n}	0	1	…	0	b_2
⋮	⋮	⋮	…	⋮	⋮	⋮	…	⋮	⋮
s_m	a_{m1}	a_{m2}	…	a_{mn}	0	0	…	1	b_m
Z	$-c_1$	$-c_2$	…	$-c_n$	0	0	…	0	*0*

Table (2.1)

The coefficients a_{ij} in the constraints (written under non-basic variables $x_1, x_2, \ldots, x_n$) is called the ***body matrix*** (or ***coefficient matrix***). The last column of the table (2.1) is called ***solution value column*** (briefly ***solution column***) or ***quantity column*** or ***b-column*** or ***RHS column***.

Step 4: Check the optimality of the current solution: In maximization (minimization) problem the simplex table is optimal, if in the Z-row there are non-negative (non-positive) coefficients in any NBV's. If the table is optimal the algorithm terminates, and the optimal value and decision can be read from the BV and RHS columns.

Step 5: If the current solution is not optimal, then determine which non-basic variable should become a basic variable (***entering variable***) and which basic variable should become a non-basic variable (***leaving variable***) to find a new BFS with a better objective function value.

i) In maximization (minimization) problem, the entering variable will correspond to the variable with the most negative (positive) coefficient in the objective function. The column of this variable is called the ***pivot*** (***key***) ***column***.

ii) The mechanics of determining the leaving variable from the simplex table calls for computing the non-negative ratio of the b-column to the corresponding coefficients in the pivot column (since solutions must satisfy the non-negativity condition). The minimum non-negative ratio identifies the leaving variable; its row is called the ***pivot*** (***key***) ***row***. The rule associated with this ratio is called the ***feasibility condition***.

iii) Update the solution by preparing the new simplex table. This is done by performing ***Gauss-Jordan row operations***. The intersection of the pivot row and the pivot column is called the ***pivot*** (***key***) ***element***.

The Gauss-Jordan computations needed to produce the new BFS includes:

a) Pivot row:
   1- Replace leaving variable in the Basic variables column with the entering variable.
   2- New pivot row=Current pivot row ÷ Pivot element

b) All other rows, including Z:
   New row= Current row – its pivot column coefficient × New pivot row

Step 6: Repeat steps 4 and 5 until, after a finite number of steps, an optimal solution, if it exists, is reached.

Example (2.16):

Find the optimal solution of the following LPP:

$$\begin{aligned} max \quad & Z = 12x_1 + 15x_2 + 14x_3 \\ S.t. \quad & -x_1 + x_2 \le 0 \\ & -x_2 + 2x_3 \le 0 \\ & x_1 + x_2 + x_3 \le 100 \end{aligned}$$

$x_1, x_2, x_3 \geq 0$

Solution:

The standard form of the LPP (with modification of the objective function) is:

$max \quad Z - 12x_1 - 15x_2 - 14x_3 = 0$

$S.t. \quad -x_1 + x_2 + s_1 = 0$

$-x_2 + 2x_3 + s_2 = 0$

$x_1 + x_2 + x_3 + s_3 = 100$

$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

Set $x_1 = x_2 = x_3 = 0$in the constraints yield the following initial basic feasible solution: $s_1 = 0, s_2 = 0, s_3 = 100, Z = 0$. The simplex table is:

Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	Solution	
s_1	−1	$\boxed{1}$	0	1	0	0	0	0/1=0
s_2	0	−1	2	0	1	0	0	
s_3	1	1	1	0	0	1	100	100/1=100
Z	− 12	−15	−14	0	0	0	0	

Since some elements in Z row are negative then the initial solution is not optimal, then:

B.V.	x_1	x_2	x_3	s_1	s_2	s_3	Solution
x_2	−1	1	0	1	0	0	0
s_2	−1	0	2	1	1	0	0
s_3	$\boxed{2}$	0	1	−1	0	1	100
Z	−27	0	−14	15	0	0	0

$x_1 = 0, s_2 = 0, s_3 = 100, Z = 0$ and it is not optimal, then:

B.V.	x_1	x_2	x_3	s_1	s_2	s_3	Solution	
x_2	0	1	1/2	1/2	0	1/2	50	50/(1/2)=100
s_2	0	0	$\boxed{5/2}$	1/2	1	1/2	50	50/(5/2)=20
x_1	1	0	1/2	−1/2	0	1/2	50	50/(1/2)=100
Z	0	0	−1/2	3/2	0	27/2	1350	

$x_1 = 50, s_2 = 50, x_1 = 50, Z = 1350$ and it is not optimal, then:

B.V.	x_1	x_2	x_3	s_1	s_2	s_3	Solution
x_2	0	1	0	2/5	−1/5	2/5	40
x_3	0	0	1	1/5	2/5	1/5	20
x_1	1	0	0	−3/5	−1/5	2/5	40
Z	0	0	0	8/5	1/5	68/5	1360

Optimal solution is: $x_1 = 40, x_2 = 40, x_3 = 20, Z_{max} = 1360.$

Example (2.17):

Find the optimal solution of the following LPP:

$$min \quad Z = x_1 - 3x_2 + 3x_3$$
$$S.t. \quad 3x_1 - x_2 + 2x_3 \leq 7$$
$$2x_1 + 4x_2 \geq -12$$
$$-4x_1 + 3x_2 + 8x_3 \leq 10$$
$$x_1, x_2, x_3 \geq 0$$

Solution:

The RHS of the second constrain is negative, it is made positive by multiplying both side of the constraint by -1. Thus, the constraint takes the form:

$$-2x_1 - 4x_2 \leq 12$$

The standard form of the LPP (with modification of the objective function) is:

$$min \quad Z - x_1 + 3x_2 - 3x_3 = 0$$
$$S.t. \quad 3x_1 - x_2 + 2x_3 + s_1 = 7$$
$$-2x_1 - 4x_2 + s_2 = 12$$
$$-4x_1 + 3x_2 + 8x_3 + s_3 = 10$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

Set $x_1 = x_2 = x_3 = 0$ in the constraints yield the following initial basic feasible solution: $s_1 = 7, s_2 = 12, s_3 = 10, Z = 0$. This solution and further improved solutions are given in the following tables:

B.V.	x_1	x_2	x_3	s_1	s_2	s_3	Solution
s_1	3	-1	2	1	0	0	7
s_2	-2	-4	0	0	1	0	12
s_3	-4	$\boxed{3}$	8	0	0	1	10
Z	-1	3	-3	0	0	0	0
s_1	$\boxed{5/3}$	0	14/3	1	0	1/3	31/3
s_2	$-22/3$	0	32/3	0	1	4/3	76/3
x_2	$-4/3$	1	8/3	0	0	1/3	10/3
Z	3	0	-11	0	0	-1	-10
x_1	1	0	14/5	3/5	0	1/5	31/5
s_2	0	0	156/5	22/5	1	14/5	354/5
x_2	0	1	32/5	12/5	0	3/5	58/5
Z	0	0	$-97/5$	$-9/5$	0	$-8/5$	$-143/5$

Optimal solution is: $x_1 = \frac{31}{5}, x_2 = \frac{58}{5}, x_3 = 0, Z_{min} = -143/5$.

Exercises 2.3 (In addition to the text book exercises)

Solve the following problems by the simplex method:

1. $max \quad Z = 6x_1 + 3x_2$

$S.t. \quad 3x_1 + 6x_2 \leq 30$

$3\, x_1 + 3x_2 \leq 18$

$3x_1 - 3x_2 \leq 6$

$3x_1 - 6x_2 \leq 3$

$x_1\, , x_2, \geq 0$

2. $min \quad Z = 2x_1 + x_2 - 3x_3 + 5x_4$

$S.t. \quad x_1 + 7x_2 + 3x_3 + 7x_4 \leq 46$

$3x_1 - x_2 + x_3 + 2x_4 \leq 8$

$2x_1 + 3x_2 - x_3 + x_4 \leq 10$

$x_1\, , x_2,\ x_3,\ x_4 \geq 0$

2.10 The M-method (Big M-method)

If the LPP has any contain of ($\geq$) or ($=$) type, then the slack variables cannot provide an initial basic feasible solution. In such cases, we introduce another type of variables called ***artificial variables***. These variables have no physical meaning; they are only a device to get the starting BFS so that the simplex algorithm is applied as usual to get optimal solution. This method consists of the following steps:

Step 1: Express the LPP in standard form, add slack variables to the constraints of ($\leq$) type and subtract them to the constraints of ($\geq$) type.

Step 2: Add non-negative variables to the left-hand-side of all constraints of ($\geq$) or ($=$) type. These variables are called ***artificial variables***. In order to get rid of the artificial variables in the final optimum iteration, we assign a very large penalty $-$M (M) in maximization (minimization) problem to the artificial variables.

Step 3: Solve the modified LPP by simplex method. While making iterations by this method, one of the following three cases may arise:

1. If no artificial variable remains in the basis, and the optimal condition is satisfied, then the current solution is an optimal BFS.
2. If at least one artificial variable appears in the basis zero level (with zero value in the solution column), and the optimality condition is satisfied, then the current solution is optimal BFS (though degenerate).

3. If at least one artificial variable appears in the basis zero level (with positive value in the solution column), and the optimality condition is satisfied, then the original problem has no feasible solution. The solution satisfies the constraints but does not optimize the objective function because it contains a very large penalty M and is termed as the ***pseudo optimal solution***.

While applying the simplex method, whenever an artificial variable happens to leave the basis, we drop artificial variable, and omit all the entries corresponding to its column from the simplex table.

Step 4: Application of simplex method until, either an optimal BFS is obtained or there is an indication of the existence of an unbounded solution to the given LPP.

Remark (2.4):

1. For computer solutions, some specific value must be assigned to M.
2. Variables, other than the artificial variables, once driven out in an iteration, may re-enter in a subsequent iteration. But, an artificial variable, once driven out, can never re-enter because of the large penalty coefficient M associated with it.

Example (2.18):

Find the optimal solution of the following LPP:

$$\begin{aligned} max \quad & Z = 3x_1 - x_2 \\ S.t. \quad & x_1 - 2x_2 \geq 8 \\ & x_1 + x_2 \leq 16 \\ & x_1 \geq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution:

The standard form of the LPP after adding the artificial variables is:

$$\begin{aligned} max \quad & Z = 3x_1 - x_2 - MR_1 - MR_2 \\ S.t. \quad & x_1 - 2x_2 - s_1 + R_1 = 8 \\ & x_1 + x_2 + s_2 = 16 \\ & x_1 - s_3 + R_2 = 8 \\ & x_1, x_2, s_1, s_2, s_3, R_1, R_2 \geq 0 \end{aligned}$$

From the first and the third constraints:

$$R_1 = 8 - x_1 + 2x_2 + s_1$$

$$R_2 = 8 - x_1 + s_3$$

Substitute R_1 and R_2 in Z-equation:

$Z = 3x_1 - x_2 - M(8 - x_1 + 2x_2 + s_1) - M(8 - x_1 + s_3)$

$Z = 3x_1 - x_2 - 8M + Mx_1 - 2Mx_2 - Ms_1 - 8M + Mx_1 - Ms_3$

$Z - (3 + 2M)x_1 + (1 + 2M)x_2 + Ms_1 + Ms_3 = -16M$

The standard form of LPP (with modification of the objective function) is:

$$
\begin{aligned}
max \quad & Z - (3 + 2M)x_1 + (1 + 2M)x_2 + Ms_1 + Ms_3 = -16M \\
S.t. \quad & x_1 - 2x_2 - s_1 + R_1 = 8 \\
& x_1 + x_2 + s_2 = 16 \\
& x_1 - s_3 + R_2 = 8 \\
& x_1, x_2, s_1, s_2, s_3, R_1, R_2 \geq 0
\end{aligned}
$$

Let $x_1 = x_2 = s_1 = s_3 = 0$,then $R_1 = 8, s_2 = 16, R_2 = 8, Z = -16M$. The simplex table is:

B.V.	x_1	x_2	s_1	s_2	s_3	R_1	R_2	Solution	
R_1	$\boxed{1}$	−2	−1	0	0	1	0	8	8/1=8
s_2	1	1	0	1	0	0	0	16	16/1=16
R_2	1	0	0	0	−1	0	1	8	8/1=8
Z	−3−2M	1+2M	M	0	M	0	0	−16M	

The current solution is not optimal, then:

B.V.	x_1	x_2	s_1	s_2	s_3	R_2	Solution	
x_1	1	−2	−1	0	0	0	8	
s_2	0	3	1	1	0	0	8	8/3=2.7
R_2	0	$\boxed{2}$	1	0	−1	1	0	0/2=0
Z	0	−5−2M	−3−M	0	M	0	24	

$x_1 = 8, s_2 = 8, R_2 = 0, Z = 24$, the current solution is not optimal, further improved solutions are given in the following tables

B.V.	x_1	x_2	s_1	s_2	s_3	Solution
x_1	1	0	0	0	−1	8
s_2	0	0	−1/2	1	$\boxed{3/2}$	8
x_2	0	1	1/2	0	−1/2	0
Z	0	0	−1/2	0	−5/2	24
x_1	1	0	−1/3	2/3	0	40/3
s_3	0	0	−1/3	2/3	1	16/3
x_2	0	1	$\boxed{1/3}$	1/3	0	8/3
Z	0	0	−4/3	5/3	0	112/3

x_1	1	1	0	1	0	16
s_3	0	1	0	1	1	8
s_1	0	3	1	1	0	8
Z	0	4	0	3	0	48

The optimal solution is: $x_1 = 16, x_2 = 0, Z_{max} = 48$

Example (2.19):

Find the optimal solution of the following LPP:

$min \quad Z = 3x_1 + 8x_2 + x_3$

$S.t. \quad 6x_1 + 2x_2 + 6x_3 \geq 6$

$6x_1 + 4x_2 = 12$

$2x_1 - 2x_2 \leq 2$

$x_1, x_2, \; x_3 \geq 0$

Solution:

The standard form of the LPP after adding the artificial variables is:

$min \quad Z = 3x_1 + 8x_2 + x_3 + MR_1 + MR_2$

$S.t. \quad 6x_1 + 2x_2 + 6x_3 - s_1 + R_1 = 6$

$6x_1 + 4x_2 + R_2 = 12$

$2x_1 - 2x_2 + s_2 = 2$

$x_1, x_2, s_1, s_2, R_1, R_2 \geq 0$

From the first and the second constraints:

$R_1 = 6 - 6x_1 - 2x_2 - 6x_3 + s_1$

$R_2 = 12 - 6x_1 - 4x_2$

Substitute R_1 and R_2 in Z-equation:

$Z = 3x_1 + 8x_2 + x_3 + M(6 - 6x_1 - 2x_2 - 6x_3 + s_1) + M(12 - 6x_1 - 4x_2)$

$Z = 3x_1 + 8x_2 + x_3 + 6M - 6Mx_1 - 2Mx_2 - 6Mx_3 + Ms_1 + 12M - 6Mx_1 - 4Mx_2$

$Z + (-3 + 12M)x_1 + (-8 + 6M)x_2 + (-1 + 6M)x_3 - Ms_1 = 18M$

The LPP (with modification of the objective function) is:

$min \quad Z + (-3 + 12M)x_1 + (-8 + 6M)x_2 + (-1 + 6M)x_3 - Ms_1 = 18M$

$S.t. \quad 6x_1 + 2x_2 + 6x_3 - s_1 + R_1 = 6$

$6x_1 + 4x_2 + R_2 = 12$

$2x_1 - 2x_2 + s_2 = 2$

$x_1, x_2, s_1, s_2, R_1, R_2 \geq 0$

Let $x_1 = x_2 = x_3 = s_1 = 0$, then $R_1 = 6, s_2 = 2, R_2 = 12, Z = 18M$. The simplex table is:

B.V.	x_1	x_2	x_3	s_1	s_2	R_1	R_2	Solution	
R_1	6	2	6	−1	0	1	0	6	6/6=1
R_2	6	4	0	0	0	0	1	12	12/6=2
s_2	2	−2	0	0	1	0	0	2	2/2=1
Z	−3+12M	−8+6M	−1+6M	−M	0	0	0	18M	

The current solution is not optimal, then:

B.V.	x_1	x_2	x_3	s_1	s_2	R_2	Solution	
x_1	1	1/3	1	−1/6	0	0	1	1/(1/3)=3
R_2	0	2	−6	1	0	1	6	6/2=3
s_2	0	−8/3	−2	1/3	1	0	0	
Z	0	−7+2M	2 − 6M	(−1/2)+M	0	0	3+6M	

$x_1 = 1, s_2 = 0, R_2 = 6, Z = 3 + 6M,$ the current solution is not optimal, further improved solutions are given in the following tables:

B.V.	x_1	x_2	x_3	s_1	s_2	Solution	
x_1	1	0	2	−1/3	0	0	
x_2	0	1	−3	1/2	0	3	3/(1/2)=6
s_2	0	0	−10	5/3	1	8	8/(5/3)=4.8
Z	0	0	−19	3	0	24	
x_1	1	0	0	0	1/5	8/5	
x_2	0	1	0	0	−3/10	3/5	
s_1	0	0	−6	1	3/5	24/5	
Z	0	0	−1	0	−9/5	48/5	

The optimal solution is: $x_1 = \frac{8}{5}, x_2 = \frac{3}{5}, x_3 = 0, Z_{min} = 48/5$

Exercises 2.4 (In addition to the text book exercises)

Find the optimal solution of the following LPP:

1. $min \quad Z = 9x_1 + 6x_2 + 3x_3$

$S.t. \quad 3x_1 + 12x_2 + 9x_3 \geq 150$

$6x_1 + 3x_2 + 3x_3 \geq 90$

$-9x_1 - 6x_2 - 3x_3 \leq -120$

$x_1 , x_2, \ x_3 \geq 0$

2. $max \quad Z = -12x_1 - 3x_2$

$S.t. \quad 9x_1 + 3x_2 = 9$

$12x_1 + 9x_2 \geq 18$

$3\,x_1 + 6x_2 \leq 9$

$$x_1, x_2 \geq 0$$

2.11 Definition of the Dual Problem

The ***dual*** problem is an LPP defined directly and systematically from the ***primal*** (or ***original***) LP model. The two problems are so closely related that the optimal solution of one problem automatically provides the optimal solution to the other. If the primal problem contains a large number of constraints and a smaller number of variables, the computational procedure can be considerably reduced by converting it into dual and then solving it.

2.12 Dual Problem Characteristics

1. If the primal contains *n* variables and *m* constraints, the dual will contain m variables and n constraints.
2. The maximization problem in the primal becomes a minimization problem in the dual and vice versa.
3. Constraints of $(\leq)\ type$ in the primal become of $(\geq)\ type$ in the dual and vice versa.
4. The coefficient matrix of the constraints of the dual is the transpose of the coefficient matrix in the primal and vice versa.
5. A new set of variables appear in the dual.
6. The constants $c_1, c_2, \dots, c_n$ in the objective function of the primal appear in the right-hand-side of the constraints of the dual.
7. The constants $b_1, b_2, \dots, b_m$ in the constraints of the primal appear in the objective function of the dual.
8. The variables of both problems are non-negative.
9. For each constraint in the primal there is an associated variable in the dual.

Example (2.20):

Construct the dual of the primal problem

$$\begin{aligned} min \quad & Z = 3x_1 - 2x_2 + 4x_3 \\ S.t. \quad & 3x_1 + 5x_2 + 4x_3 \geq 7 \\ & 6x_1 + x_2 + 3x_3 \geq 4 \\ & 7x_1 - 2x_2 - x_3 \leq 10 \\ & x_1 - 2x_2 + 5x_3 \geq 3 \\ & 4x_1 + 7x_2 - 2x_3 \geq 2 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution:

All the constraints must be of the same type. Multiplying the third constraint by (-1) on both sides, we get:

$-7x_1 + 2x_2 + x_3 \geq -10$

The dual of the given problem is:

$$max \quad W = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5$$
$$S.t. \quad 3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \leq 3$$
$$5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \leq -2$$
$$4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \leq 4$$
$$y_1, y_2, y_3, y_4, y_5 \geq 0$$

Example (2.21):

Construct the dual of the primal problem

$$max \quad Z = 3x_1 + 5x_2$$
$$S.t. \quad 2x_1 + 7x_2 = 12$$
$$-9x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Solution:

The first constraint is of equality form, which is equivalent to:

$2x_1 + 7x_2 \leq 12 \; and \; 2x_1 + 7x_2 \geq 12$

The primal problem can be expressed as:

$$max \quad Z = 3x_1 + 5x_2$$
$$S.t. \quad 2x_1 + 7x_2 \leq 12$$
$$-2x_1 - 7x_2 \leq -12$$
$$-9x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Let y_1', y_1'' and y_2 be the dual variables associated with the first, second, and third constraints. Then the dual problem is:

$$min \quad W = 12y_1' - 12y_1'' + 4y_2$$
$$S.t. \quad 2y_1' - 2y_1'' - 9y_2 \geq 3$$
$$7y_1' - 7y_1'' + y_2 \geq 5$$
$$y_1', y_1'', y_2 \geq 0$$

Or equivalently:

$$min \quad W = 12(y_1' - y_1'') + 4y_2$$
$$S.t. \quad 2(y_1' - y_1'') - 9y_2 \geq 3$$
$$7(y_1' - y_1'') + y_2 \geq 5$$
$$y_1', y_1'', y_2 \geq 0$$

If we put $y_1 = y_1' - y_1''$, then the new variable y_1, which is the difference between two non-negative variables, become unrestricted in sign and the dual problem becomes:

$$\begin{aligned} min \quad & W = 12y_1 + 4y_2 \\ S.t. \quad & 2y_1 - 9y_2 \geq 3 \\ & 7y_1 + y_2 \geq 5 \\ & y_1 \text{ unrestricted}, y_2 \geq 0 \end{aligned}$$

This example leads to the following remark.

Remark (2.5):

The dual variable which corresponds to an equality constraint must be unrestricted in sign. Conversely, when a primal variable is unrestricted in sign, its dual constraint must be in equality form.

2.13 Some Duality Theorems

Theorem (2.1):

If either the primal or the dual problem has an unbounded solution, then the solution to the other problem is infeasible.

Theorem (2.2) (Fundamental Theorem of Duality):

If both the primal and the dual problems have feasible solutions, then both have optimal solutions and $max\, Z = min\, W$ (and $min\, Z = max\, W$).

Remark (2.6):

Values of the decision variables of the primal are given by the Z-row of the solution under the slack variables (if there are any) in the dual, neglecting the –ve sign if any.

Example (2.22):

Use duality to solve the following LPP:

$$\begin{aligned} min \quad & Z = 36x + 60y + 45z \\ S.t. \quad & x + 2y + 2z \geq 40 \\ & 2x + y + 5z \geq 25 \\ & x + 4y + z \geq 50 \\ & x, y, z \geq 0 \end{aligned}$$

Solution:

The dual problem of the LPP is:

$$\begin{aligned} max \quad & W = 40y_1 + 25y_2 + 50y_3 \\ S.t. \quad & y_1 + 2y_2 + y_3 \leq 36 \\ & 2y_1 + y_2 + 4y_3 \leq 60 \end{aligned}$$

$$2y_1 + 5y_2 + y_3 \leq 45$$
$$y_1, y_2, y_3 \geq 0$$

Adding slack variables s_1, s_2, and s_3, we get:

$max \quad W - 40y_1 - 25y_2 - 50y_3 = 0$

$S.t. \quad y_1 + 2y_2 + y_3 + s_1 = 36$

$$2y_1 + y_2 + 4y_3 + s_2 = 60$$
$$2y_1 + 5y_2 + y_3 + s_3 = 45$$
$$y_1, y_2, y_3, s_1, s_2, s_3 \geq 0$$

The initial basic feasible solution of the dual is: $y_1 = y_2 = y_3 = 0, s_1 = 36, s_2 = 60, s_3 = 45, W = 0$. This solution and further improved solutions are given in the following tables:

B.V.	y_1	y_2	y_3	s_1	s_2	s_3	Solution	
s_1	1	2	1	1	0	0	36	36/1=36
s_2	2	1	$\boxed{4}$	0	1	0	60	60/4=15
s_3	2	5	1	0	0	1	45	45/1=45
W	−40	−25	−50	0	0	0	0	
s_1	1/2	7/4	0	1	−1/4	0	21	21/(1/2)=42
y_3	1/2	1/4	1	0	1/4	0	15	15/(1/2)=30
s_3	$\boxed{3/2}$	19/4	0	0	−1/4	1	30	30/(3/2)=20
W	−15	−25/2	0	0	25/2	0	750	
s_1	0	1/6	0	1	−1/6	−1/3	11	
y_3	0	−4/3	1	0	1/3	−1/3	5	
y_1	1	19/6	0	0	−1/6	2/3	20	
W	0	35	0	0	10	10	1050	

The optimal solution of the primal is $x = 0, y = 10, z = 10, and\ Z_{min} = W_{max} = 1050$.

Example (2.23):

Use the duality to find the optimal solution of the LPP in example (2.19).

$min \quad Z = 3x_1 + 8x_2 + x_3$

$S.t. \quad 6x_1 + 2x_2 + 6x_3 \geq 6$

$$6x_1 + 4x_2 = 12$$
$$2x_1 - 2x_2 \leq 2$$
$$x_1, x_2, x_3 \geq 0$$

Solution:

Multiplying the third constraint by (-1) on both sides, we get:

$-2x_1 + 2x_2 \geq -2$

The dual problem of the LPP is:

$max \quad W = 6y_1 + 12y_2 - 2y_3$

$S.t. \quad 6y_1 + 6y_2 - 2y_3 \leq 3$

$2y_1 + 4y_2 + 2y_3 \leq 8$

$6y_1 \leq 1$

$y_1, y_3 \geq 0, y_2$ is unrestrected

Adding slack variables $s_1, s_2,$ and s_3, we get:

$max \quad W - 6y_1 - 12y_2 + 2y_3 = 0$

$S.t. \quad 6y_1 + 6y_2 - 2y_3 + s_1 = 3$

$2y_1 + 4y_2 + 2y_3 + s_2 = 8$

$6y_1 + s_3 = 1$

$y_1, y_3, s_1, s_2, s_3 \geq 0, y_2$ is unrestrected

The initial basic feasible solution of the dual is: $y_1 = y_2 = y_3 = 0, s_1 = 3, s_2 = 8, s_3 = 1, W = 0$. This solution and further improved solutions are given in the following tables:

B.V.	y_1	y_2	y_3	s_1	s_2	s_3	Solution	
s_1	6	$\boxed{6}$	−2	1	0	0	3	3/6=1/2
s_2	2	4	2	0	1	0	8	8/4=2
s_3	6	0	0	0	0	1	1	
W	−6	−12	2	0	0	0	0	
y_2	1	1	−1/3	1/6	0	0	1/2	
s_2	−2	0	$\boxed{10/3}$	−2/3	1	0	6	
s_3	6	0	0	0	0	1	1	
W	6	0	−2	2	0	0	6	
y_2	4/5	1	0	1/10	−1/10	0	11/10	
y_3	−3/5	0	1	−1/5	3/10	0	9/5	
s_3	6	0	0	0	0	1	1	
W	24/5	0	0	8/5	3/5	0	48/5	

The optimal solution of the primal is $x_1 = 8/5, x_2 = 3/5, x_3 = 0, and\ Z_{min} = W_{max} = 48/5$.

Exercises 2.5 (In addition to the text book exercises)

Use the duality to solve the following LPP:

1. $min \quad Z = 10x_1 + 15x_2 + 30x_3$

$S.t. \quad x_1 + 3x_2 + x_3 \geq 90$

$2x_1 + 5x_2 + 3x_3 \geq 120$

$$x_1 + x_2 + x_3 \geq 60$$
$$x_1, x_2, x_3 \geq 0$$

2. $max \quad Z = 10x_1 + 24x_2 + 8x_3$

$S.t. \quad 2x_1 + 4x_2 + 2x_3 \leq 10$
$$4x_1 - 2x_2 + 6x_3 = 4$$
$$x_1, x_2, x_2 \geq 0$$

2.14 The Dual Simplex Method

The dual simplex method starts with a solution that satisfies the optimality condition but infeasible. To start the LP optimal and infeasible, two requirements must be met:

1. The objective function must satisfy the optimality condition of the regular simplex method.
2. All the constraints must be of the type ($\leq$).

The dual simplex method consists of the following steps:

Step 1: Convert the ($\geq$) type constraint to a ($\leq$) type constraint by multiplying both sides by (-1). If the LPP includes an equality constraint, the equation can be replaced by two inequalities, then convert the constraint of ($\geq$) type into a constraint of ($\leq$) type.

Step 2: Convert the LPP into the standard form and express the problem information in the form of a table known as the ***dual simplex table***.

Step 3: Three cases arises:

a) If the Z- row satisfies the optimality condition and all $b_i \geq 0$, then the current solution is optimal basic feasible solution.
b) If at least one element in the Z-row doesn't satisfy the optimality condition, the method fails.
c) If the Z- row satisfies the optimality condition and at least one $b_i \leq 0$, then proceed to step 4.

Step 4: Select the row that contains the most negative b_i. Ties are broken arbitrarily. This row is called the ***pivot*** (***key***) ***row***. The corresponding variable leaves the basis. This is called the ***dual feasibility condition***.

Step 5: Look at the elements of the pivot row:

a) If all elements are non-negative, the problem does not have a feasible solution.
b) If at least one element is negative, divide the elements of the Z-row to the corresponding negative elements in the pivot row. Choose the smallest of

these ratios. Ties are broken arbitrarily. The corresponding column is the ***key column*** and the associated variable is the ***entering variable***. This is called ***dual optimality condition***. Mark the ***pivot*** (***key***) element.

Step 6: Make the key element unity. Perform as in regular simplex method and repeat iterations until an optimal feasible solution is obtained in a finite number of steps or there is an indication of the non-existence of a feasible solution.

Example (2.24):

Find the optimal solution of the following LPP

$$\begin{aligned} max \quad & Z = -3x_1 - 2x_2 - x_3 \\ S.t. \quad & 2x_1 + x_2 + x_3 \geq 4 \\ & 3x_1 + x_2 + 3x_3 \geq 10 \\ & -x_1 + 2x_2 - x_3 \geq 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution:

First we convert constraints of ($\geq$) type into constraints of ($\leq$) type , so the LPP will be:

$$\begin{aligned} max \quad & Z = -3x_1 - 2x_2 - x_3 \\ S.t. \quad & -2x_1 - x_2 - x_3 \leq -4 \\ & -3x_1 - x_2 - 3x_3 \leq -10 \\ & x_1 - 2x_2 + x_3 \leq -1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

The standard form (with modification in the objective function) is:

$$\begin{aligned} max \quad & Z + 3x_1 + 2x_2 + x_3 = 0 \\ S.t. \quad & -2x_1 - x_2 - x_3 + s_1 = -4 \\ & -3x_1 - x_2 - 3x_3 + s_2 = -10 \\ & x_1 - 2x_2 + x_3 + s_3 = -1 \\ & x_1, x_2, x_3, s_1, s_2, s_3 \geq 0 \end{aligned}$$

Let $x_1 = x_2 = x_3 = 0$, then $s_1 = -4, s_2 = -10,$ and $s_3 = -1$. Since $s_1, s_2,$ and s_3 are negative, then solution is infeasible. The dual simplex table is:

BV	x_1	x_2	x_3	s_1	s_2	s_3	b
s_1	-2	-1	-1	1	0	0	-4
s_2	-3	-1	$\boxed{-3}$	0	1	0	$\boxed{-10}$
s_3	1	-2	1	0	0	1	-1
Z	3	2	1	0	0	0	0

$(\left|\frac{3}{-3}\right| = 1, \left|\frac{2}{-1}\right| = 2, \left|\frac{1}{-3}\right| = 1/3)$

BV	x_1	x_2	x_3	s_1	s_2	s_3	b
s_1	−1	−2/3	0	1	−1/3	0	−2/3
x_3	1	1/3	1	0	−1/3	0	10/3
s_3	0	$\boxed{-7/3}$	0	0	1/3	1	$\boxed{-13/3}$
Z	2	5/3	0	0	1/3	0	−10/3
s_1	−1	0	0	1	−3/7	−2/7	4/7
x_3	1	0	1	0	−2/7	1/7	19/7
x_2	0	1	0	0	−1/7	−3/7	13/7
Z	2	0	0	0	4/7	5/7	−45/7

The optimal solution is : $x_1 = 0, x_2 = \frac{13}{7}, x_3 = \frac{19}{7}$, and $Z_{max} = -45/7$

Example (2.25):

Use the dual simplex method to find the optimal solution of the LPP in example (2.19).

$$\begin{aligned} min \quad & Z = 3x_1 + 8x_2 + x_3 \\ S.t. \quad & 6x_1 + 2x_2 + 6x_3 \geq 6 \\ & 6x_1 + 4x_2 = 12 \\ & 2x_1 - 2x_2 \leq 2 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution:

Replace the second constraint by the following two constraints:

$6x_1 + 4x_2 \leq 12$ and $6x_1 + 4x_2 \geq 12$

Then convert each constraint of ($\geq$) type into a constraint of ($\leq$) type. The LPP will be:

$$\begin{aligned} min \quad & Z = 3x_1 + 8x_2 + x_3 \\ S.t. \quad & -6x_1 - 2x_2 - 6x_3 \leq -6 \\ & 6x_1 + 4x_2 \leq 12 \\ & -6x_1 - 4x_2 \leq -12 \\ & 2x_1 - 2x_2 \leq 2 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

The standard form (with modification in the objective function) is:

$$\begin{aligned} min \quad & Z - 3x_1 - 8x_2 - x_3 = 0 \\ S.t. \quad & -6x_1 - 2x_2 - 6x_3 + s_1 = -6 \\ & 6x_1 + 4x_2 + s_2 = 12 \end{aligned}$$

$$-6x_1 - 4x_2 + s_3 = -12$$
$$2x_1 - 2x_2 + s_4 = 2$$
$$x_1, x_2, x_3, s_1, s_2, s_3, s_4 \geq 0$$

Let $x_1 = x_2 = x_3 = 0$, then $s_1 = -6, s_2 = 12$, and $s_3 = -12, s_4 = 2$. Since s_1 and s_3 are negative, then solution is infeasible. The dual simplex table is:

BV	x_1	x_2	x_3	s_1	s_2	s_3	s_4	b
s_1	−6	−2	−6	1	0	0	0	−6
s_2	6	4	0	0	1	0	0	12
s_3	$\boxed{-6}$	−4	0	0	0	1	0	$\boxed{-12}$
s_4	2	−2	0	0	0	0	1	2
Z	−3	−8	−1	0	0	0	0	0
s_1	0	2	−6	1	0	−1	0	6
s_2	0	0	0	0	1	1	0	0
x_1	1	2/3	0	0	0	−1/6	0	2
s_4	0	$\boxed{-10/3}$	0	0	0	1/3	1	$\boxed{-2}$
Z	0	−6	−1	0	0	−1/2	0	6
s_1	0	0	−6	1	0	−4/5	3/5	24/5
s_2	0	0	0	0	1	1	0	0
x_1	1	0	0	0	0	−1/10	1/5	8/5
x_2	0	1	0	0	0	−1/10	−3/10	3/5
Z	0	0	−1	0	0	−11/10	−9/5	48/5

The optimal solution is : $x_1 = \frac{8}{5}, x_2 = \frac{3}{5}, x_3 = 0$, and $Z_{min} = 48/5$

Exercises 2.6 (In addition to the text book exercises)

Use dual simplex method to find the optimal solution of the following LPP:

1. $min \quad Z = 3x_1 + 6x_2 + 9x_3$

$S.t. \quad 6x_1 - 3x_2 + 3x_3 \geq 12$

$3x_1 + 3x_2 + 6x_3 \leq 24$

$3x_2 - 6x_3 \geq 6$

$x_1, x_2, x_3 \geq 0$

2. $max \quad Z = -6x_1 - 3x_3$

$S.t. \quad 3x_1 + 3x_2 - 3x_3 \geq 15$

$3x_1 - 6x_2 + 12x_3 \geq 24$

$x_1, x_2, x_3 \geq 0$