Ch. 4: Transportation Problem

4.1 Definition of the Transportation Problem

The ***transportation problem*** is a special case of linear programming in which the objective is to transport a homogeneous commodity from various origins or factories to different destinations or markets at a total minimum cost.

Suppose that there are m sources and n destinations. Let a_i be the number supply units available at source i $(i = 1, 2, \dots, m)$ and b_j be the number demand units required at destination j $(j = 1, 2, \dots, n)$. Let c_{ij} represent the unit transportation cost for transporting the units from source i to destination j. The objective is to determine the number of units to be transported from source i to destination j so that the total transportation cost is minimum. In addition, the supply limits at the source and the demand requirements at the destination must be satisfied exactly.

If x_{ij} $(x_{ij} \geq 0)$ is the number of units shipped from source I to destination j, the equivalent LP model will be:

Find $x_{ij}(i = 1,2, \dots, m\, ; j = 1,2, \dots, n)$ in order to

$$min \quad Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

$$S.t. \quad \sum_{j=1}^{n} x_{ij} = a_i \qquad i = 1,2, \dots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad j = 1,2, \dots, n$$

$$x_{ij} \geq 0 \qquad i = 1,2, \dots, n\, ;\; j = 1,2, \dots, m$$

The two sets of constraints will be ***consistent***, i.e., the system will be in ***balance*** if:

$$\sum_{i=1}^{m} a_{i=} \sum_{j=1}^{n} b_j$$

The consistency condition is necessary and sufficient condition for a transportation problem to have a feasible solution. The above information can be put in the form of a general matrix shown below. This table is called the ***transportation matrix***. In the table (4.1), $c_{ij}, i = 1,2, \dots, m; j = 1,2, \dots, n$, is the unit shipping cost from the ith origin (source) to the jth destination, x_{ij} is the quantity shipped from the ith origin to the jth destination, a_i is the supply available at origin i and b_j is the demand at destination j.

		Destinations						
		1	**2**	**…**	$\boldsymbol{j}$	**…**	$\boldsymbol{n}$	**Supply**
Sources(or Origins)	**1**	c_{11} x_{11}	c_{12} x_{12}	…	c_{1j} x_{1j}	…	c_{1n} x_{1n}	a_1
	2	c_{21} x_{21}	c_{22} x_{22}	…	c_{2j} x_{2j}	…	c_{2n} x_{2n}	a_2
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	$\boldsymbol{i}$	c_{i1} x_{i1}	c_{ij} x_{i2}	…	c_{ij} x_{ij}	…	c_{in} x_{in}	a_i
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	$\boldsymbol{m}$	c_{m1} x_{m1}	c_{m2} x_{m2}	…	c_{mj} x_{mj}	…	c_{mn} x_{mn}	a_m
	Demand	b_1	b_2	…	b_j	…	b_n	

Table (4.1)

Definition (4.1):

An allocation is said to satisfy the ***rim requirements***, *i.e.*, it must satisfy availability constraints and requirement constraints.

4.2 Solution of the Transportation Model

The steps to solve transportation problem are:

Step 1: Make a transportation model.

Step 2: Find a basic feasible solution.

Step 3: Perform optimality test.

Step 4: Iterate toward an optimal solution.

Step 5: Repeat steps 3-4 until optimal solution is reached.

Step 1: Make a Transportation Model

This consists in expressing supply from origins, requirements at destinations and cost of shipping from origins to destinations in the form of a cost matrix.

A check is made to find if the problem is balanced, if not add a dummy origin or destination to balance the supply and demand.

Example (4.1): A dairy firm has three plants located throughout a state. Daily milk production at each plant is as follows:

Plant 1: 6 million liters
Plant 2: 1 million liters, and
Plant 3: 10 million liters

Each day the firm must fulfill the needs of its four distribution centers. Milk requirements at each center are as follows:

Distribution center 1: 7 million liters
Distribution center 2: 5 million liters
Distribution center 3: 3 million liters, and
Distribution center 4: 2 million liters

Cost of shipping of one million liter of milk from each plant to each distribution center is given in the following table in hundreds of Iraqi dinars:

		Distribution centers			
		1	**2**	**3**	**4**
Plants	**1**	2	3	11	7
	2	1	0	6	1
	3	5	8	15	9

a) Construct the cost table.
b) Formulate the mathematical model of the problem.

Solution:

a) The cost table is:

		Distribution centers				
		1	**2**	**3**	**4**	**Supply**
Plants	**1**	2	3	11	7	6
	2	1	0	6	1	1
	3	5	8	15	9	10
Requirement		7	5	3	2	17

$\sum_{i=1}^{3} a_{i=}\ 6+1+10=17$,$\sum_{j=1}^{4} b_j = 7+5+3+2=17$,i.e. the constraints are consistent.

b) Let $x_{ij}, i = 1,2,3; j = 1,2,3,4$ denotes the quantity of units to be transported from each origin to each destination (i.e., x_{ij} are decision variables), then the objective function is to minimize the cost of transportation.

i.e. $\min \quad Z = 2x_{11} + 3x_{12} + 11x_{13} + 7x_{14} + x_{22} + 6x_{23} + x_{24} + 5x_{31} + 8x_{32} + 15x_{33} + 9x_{34}$

In general, if c_{ij} is the unit cost of shipping from the ith source to the jth destination, the mathematical LP model is:

$$\begin{aligned}
&min \quad && Z = \textstyle\sum_{i=1}^{3}\sum_{j=1}^{4} \; c_{ij}\; x_{ij} \\
&S.t. && x_{11} + x_{12} + x_{13} + x_{14} = 6 \\
& && x_{21} + x_{22} + x_{23} + x_{24} = 1 \\
& && x_{31} + x_{32} + x_{33} + x_{34} = 10 \\
& && x_{11} + x_{21} + x_{31} = 7 \\
& && x_{12} + x_{22} + x_{32} = 5 \\
& && x_{13} + x_{23} + x_{33} = 3 \\
& && x_{14} + x_{24} + x_{34} = 2 \\
& && x_{ij} \geq 0 \qquad\qquad i = 1,2,3\,; \;\; j = 1,2,3,4
\end{aligned}$$

Step 2: Find a Basic Feasible Solution

There are many methods for finding the basic feasible solution; three of them are described below:

4.2.1 North-West Corner Method (NWCM)

This rule may be stated as follows:

a) Start in the north-west corner of the transportation matrix framed in step 1, i.e. cell (1,1). Compare a_1 and b_1:
   - i) If $a_1 < b_1$, set $x_{11} = a_1$, compute the balance supply and demand and proceed to cell (2,1) (i.e. proceed vertically).
   - ii) If $b_1 < a_1$, set $x_{11} = b_1$, compute the balance supply and demand and proceed to cell (1,2) (i.e. proceed horizontally).
   - iii) If $a_1 = b_1$, set $x_{11} = a_1 = b_1$, compute the balance supply and demand and proceed to cell (2,2) (i.e. proceed diagonally). Also make a zero allocation to the least cost cell in a_1/b_1.

b) Continue in the same manner, step by step, away from the north-west corner until, finally, a value is reached in the south-east corner.

Example (4.2):

For the transportation problem in example (4.1):

		Distribution centers				
		1	**2**	**3**	**4**	**Supply**
Plants	**1**	2 6	3	11	7	~~6~~
	2	1 1	0 0	6	1	~~1~~
	3	5	8 5	15 3	9 2	~~10~~ ~~5~~ ~~2~~
Requirement		~~7~~ ~~1~~	~~5~~	~~3~~	~~2~~	17

$Z = (6 \times 2 + 1 \times 1 + 5 \times 8 + 3 \times 15 + 2 \times 9) \times 100 = 11600\ ID$

4.2.2 Least-Cost Method

This method consists in allocating as much as possible in the lowest cost cell/cells and then further allocation is done in the cell/cells with second lowest cost and so on. In case of tie among the cost, select the cells where allocation of more number of units can be made.

Example (4.3):

For the transportation problem in example (4.1):

		Distribution centers				
		1	**2**	**3**	**4**	**Supply**
Plants	**1**	2 6	3	11	7	~~6~~
	2	1	0 1	6	1	~~1~~
	3	5 1	8 4	15 3	9 2	~~10~~ ~~9~~ ~~5~~ ~~3~~
Requirement		~~7~~ ~~1~~	~~5~~ ~~4~~	~~3~~	~~2~~	17

$Z = (6 \times 2 + 1 \times 5 + 4 \times 8 + 3 \times 15 + 2 \times 9) \times 100 = 11200\ ID$

4.2.3 Vogel's Approximation Method (VAM)

Vogel's approximation method (or penalty method) makes effective use of the cost information and yields a better initial solution than obtained by other methods. This method consists of the following sub-steps:

i) Write down the cost matrix. Enter the difference between the smallest and the second smallest element in each column below the corresponding column and the difference between the smallest and the second smallest element in each row to the right of the row.

ii) Select the row or column with the greatest difference and allocate as much as possible within the restriction of the rim condition to the lowest cost cell in the row or column selected.

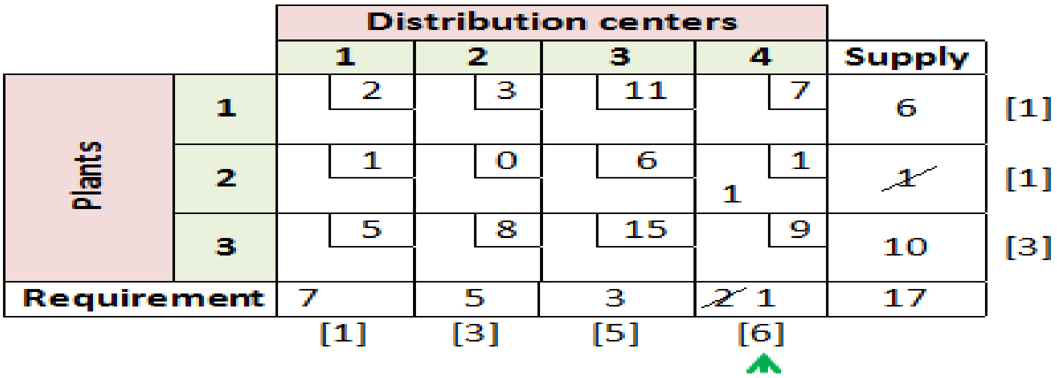
In case of a tie among the highest penalties, select the row or column having minimum cost. In case of tie in the minimum cost also, select the cell which can have maximum allocation. If there is a tie among maximum allocation cells also, select the cell arbitrarily for allocation. Following these rules yields the best possible initial basic feasible solution and reduces the number of iterations required to reach the optimal solution.

iii) Cross out the row or column completely satisfied by the allocation just made.

iv) Repeat steps i to iii until all assignments have been made.

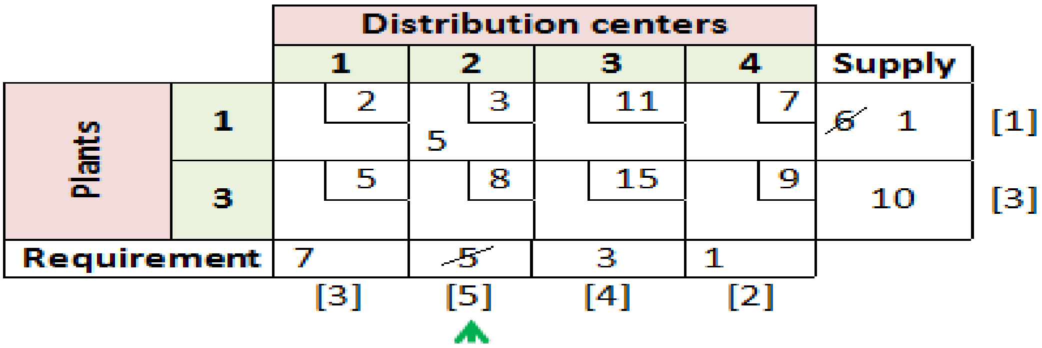
Example (4.4):

For the transportation problem in example (4.1), the cost matrix with penalties is shown below:



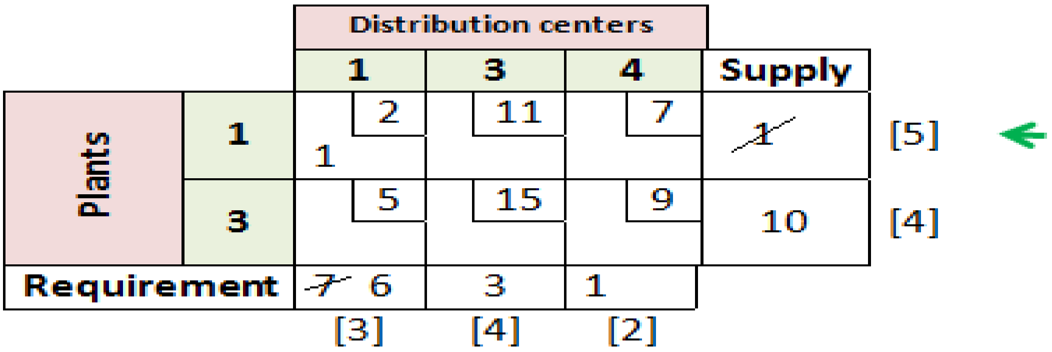
		Distribution centers					
		1	2	3	4	Supply	
Plants	1	2	3	11	7	6	[1]
	2	1	0	6	1 1	~~1~~	[1]
	3	5	8	15	9	10	[3]
Requirement		7	5	3	~~2~~ 1	17	
		[1]	[3]	[5]	[6]		
					▲		

The greatest penalty is [6], so we choose the 4th column and allocate as much as possible (i.e. 1) to cell (2,4)(the cell with smallest cost in the 4th column). Supply of plant 2 is completely satisfied, so row 2 is crossed out and the shrunken matrix with penalties and allocation is as below:



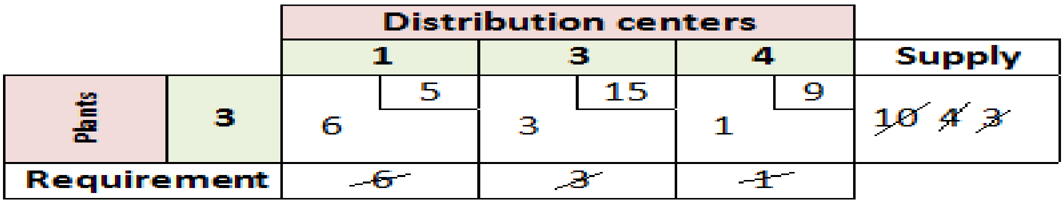
		Distribution centers					
		1	2	3	4	Supply	
Plants	1	2	3 5	11	7	~~6~~ 1	[1]
	3	5	8	15	9	10	[3]
Requirement		7	~~5~~	3	1		
		[3]	[5]	[4]	[2]		
			▲				

The greatest penalty is [5], so we choose the 2nd column and allocate as much as possible (i.e. 5) to cell (1,4)(the cell with smallest cost in the 2nd column). Requirement of distribution center 2 is completely satisfied, so column 2 is crossed out and the shrunken matrix with penalties and allocation is as below:



		Distribution centers					
		1	3	4	Supply		
Plants	1	2 1	11	7	~~1~~	[5]	←
	3	5	15	9	10	[4]	
Requirement		~~7~~ 6	3	1			
		[3]	[4]	[2]			

The greatest penalty is [5], so we choose the 1st row and allocate as much as possible (i.e. 1) to cell (1,1)(the cell with smallest cost in the 1st row). Supply of plant 1 is completely satisfied, so row 1 is crossed out and the shrunken matrix is as below:



		Distribution centers			
		1	3	4	Supply
Plants	3	5 6	15 3	9 1	~~10~~ ~~4~~ ~~3~~
Requirement		~~6~~	~~3~~	~~1~~	

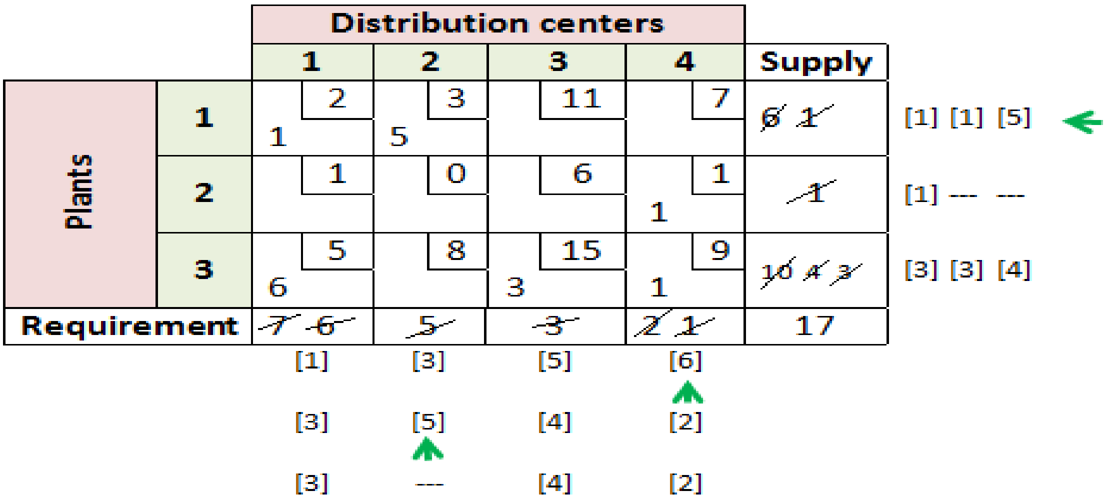
It is possible to find row difference, but it is not possible to find column difference. Therefore, the remaining allocations are made by following the least-cost method. The transportation matrix will be:

		Distribution centers				
		1	2	3	4	Supply
Plants	1	2 1	3 5	11	7	6
	2	1	0	6	1 1	1
	3	5 6	8	15 3	9 1	10
Requirement		7	5	3	2	17

$$Z = (1 \times 2 + 5 \times 3 + 1 \times 1 + 3 \times 15 + 6 * 5 + 1 \times 9) \times 100 = 10200\ ID$$

Remark (4.1):

It is possible to all the previous steps in one table as follows:



		Distribution centers					
		1	2	3	4	Supply	
Plants	1	2 1	3 5	11	7	~~6~~ ~~1~~	[1] [1] [5] ←
	2	1	0	6	1 1	~~1~~	[1] --- ---
	3	5 6	8	15 3	9 1	~~10~~ ~~4~~ ~~3~~	[3] [3] [4]
Requirement		~~7~~ ~~6~~	~~5~~	~~3~~	~~2~~ ~~1~~	17	
		[1]	[3]	[5]	[6] ↑		
		[3]	[5] ↑	[4]	[2]		
		[3]	---	[4]	[2]		

Remark (4.2):

Vogel method yields the best initial solution, and the north-west corner method yields the worst.

Step 3: Perform Optimality Test

Make an optimality test to find whether the obtained feasible solution is optimal or not. An optimality test can be performed only on that feasible solution in which:

1) Number of allocations is $m+n-1$, where m is the number of rows and n is the number of columns.
2) These $m+n-1$ allocations should be in independent positions, i.e. it is impossible to increase or decrease any allocation without either changing the position of allocation or violating the row and column restrictions. A simple rule for allocations to be in independent position is that it is impossible to travel from any allocation, back to itself by a series of horizontal and vertical jumps from one occupied cell to another, without a direct reversal of route, or simply do not form a closed loop.

To check optimality we must find empty cells evaluation, if there is at least one cell with negative evaluation, and then the current solution is not optimal.

4.2.4 The Stepping-Stone Method

Starting from the chosen empty cell, trace a path in the matrix consisting of a series of alternate horizontal and vertical lines. The path begins and terminates in the chosen cell. All other corners of the path lie in the cells for which allocations have been made. The path may skip over any number of occupied or vacant cells. Mark the corner of the path in the chosen vacant cell as positive and other corners of the path alternately $-ve, +ve, -ve$ and so on.

Allocate 1 unit to the chosen cell; subtract and add 1 unit from the cells at the corners of the path, maintaining the row and column requirements. The net change in the total cost resulting from this adjustment is called the ***evaluation*** of the chosen empty cell. In a transportation problem involving m rows and n columns, the total number of empty cells will be $m.n-(m+n-1)=(m-1)(n-1)$. Therefore, there are $(m-1)(n-1)$ evaluations which must be calculated.

Example (4.5):

We will check optimality of basic feasible solution obtained by Vogel's method in example (4.4):

		Distribution centers				
		1	**2**	**3**	**4**	**Supply**
Plants	**1**	2 1	3 5	11	7	6
	2	1	0	+1 6	−1 1 1	1
	3	5 6	8	−1 15 3	9 1 +1	10
Requirement		7	5	3	2	17

1) Number of allocations=6, $m=3, n=4, m+n-1=3+4-1=6$ =number of allocations.
2) These allocations are independent in positions.

To find the evaluation of the empty cell (2,3) for example, the closed path the begins and end in cell (2,3) is explained in the table above. To allocate 1 unit in cell (2,3), we must subtract, add, subtract 1 unit from cells (2,4), (3,4), and (3,3) respectively. For each empty cell, the closed path that start and end with the empty cell and whose other corners are allocated cells and the evaluation of the empty cell(in hundreds of dinars) are as follows:

Cell (1, 3): $(1,3)\rightarrow(3,3)\rightarrow(3,1)\rightarrow(1,1)$

Evaluation of cell (1, 3) $=c_{13}-c_{33}+c_{31}-c_{11}=11-15+5-2=-1$

Cell (1, 4): $(1,4)\rightarrow(3,4)\rightarrow(3,1)\rightarrow(1,1)$

Evaluation of cell (1, 4) $=c_{14}-c_{34}+c_{31}-c_{11}=7-9+5-2=+1$

Cell (2, 1): $(2,3)\rightarrow(2,4)\rightarrow(3,4)\rightarrow(3,1)$

Evaluation of cell (2, 1) $=c_{23}-c_{24}+c_{34}-c_{31}=1-1+9-5=+4$

Cell (2, 2): $(2,2)\rightarrow(2,4)\rightarrow(3,4)\rightarrow(3,1)\rightarrow(1,1)\rightarrow(1,2)$

Evaluation of cell (2, 2) $= c_{22} - c_{24} + c_{34} - c_{31} + c_{11} - c_{12} = 0 - 1 + 9 - 5 + 2 - 3 = +2$

Cell (2, 3): $(2,3) \rightarrow (2,4) \rightarrow (3,4) \rightarrow (3,3)$

Evaluation of cell (2, 3) $= c_{23} - c_{24} + c_{34} - c_{33} = 6 - 1 + 9 - 15 = -1$

Cell (3, 2): $(3,2) \rightarrow (3,1) \rightarrow (1,1) \rightarrow (1,2)$

Evaluation of cell (3, 3) $= c_{32} - c_{31} + c_{11} - c_{12} = 8 - 5 + 2 - 3 = +2$

Since the evaluation of cells (1,3) and (2,3) are negative, then the current solution is not optimal.

4.2.5 The Modified Distribution (MODI) Method

It is also called the ***$u - v$ method***. This method calculates cell evaluation of all unoccupied cells simultaneously. Thus it provides considerable time saving over the stepping-stone method. It consists of the following sub-steps:

Sub-step 1: Set-up a cost matrix containing the unit costs associated with the cells for which allocations have been made.

Sub-step 2: Introduce dual variables corresponding to the supply and demand constraints. If there are m origins and n distinations, there will be $m + n$ dual variables. Let $u_i (i = 1,2, \dots, m)$ and $v_j (j = 1,2, \dots, n)$ be the dual variables corresponding to supply and demand constraints. Variables $u_i$and v_j are such that $u_i + v_j = c_{ij}$. Therefore, enter a set of numbers $u_i (i = 1,2, \dots, m)$ along the left of the matrix and $v_j (j = 1,2, \dots, n)$ across the top of the matrix so that their sums equal the costs entered in sub-step 1. Assume one of them equal to zero and find their values.

Sub-step 3: Fill the vacant cells with the sum of u_i and v_j.

Sub-step 4: Subtract the cell values of the matrix of sub-step 3 from the original cost matrix. The resulting matrix is called the ***cell evaluation matrix*** (***CEM***).

Sub-step 5: Signs of the values in the cell evaluation matrix indicates wether optimal solution has been obtained or not. The sign have the following significance:

a) A negative value in an unoccupied cell indicates that a better solution can be obtained by allocating units to this cell.

b) A positive value in an unoccupied cell indicates that a poorer solution will result by allocating units to this cell.

c) A zero value in an unoccupied cell indicates that another solution of the same value can be obtained by allocating units to this cell.

Example (4.6):

We will check optimality of basic feasible solution obtained by Vogel's method in example (4.4). Since number of allocations= $6 = m + n - 1$ and they are in independent positions, then we can check optimality. The cost matrix of allocated cells is:

2	3		
			1
5		15	9

$\Longrightarrow$ Entering $u_i (i = 1,2,3)$ and $v_j (j = 1,2,3,4)$ such that:

$u_1 + v_1 = 2$, $u_1 + v_2 = 3$, $u_2 + v_4 = 1$, $u_3 + v_1 = 5$, $u_3 + v_3 = 15$, $u_3 + v_4 = 5$. Let $v_1 = 0$, then: $u_1 = 2$, $v_2 = 1$, $u_3 = 5$, $v_3 = 10$, $v_4 = 4$, $u_2 = -3$.

	v_j	v_1	v_2	v_3	v_4
	u_i	0	1	10	4
u_1	2	2	3		
u_2	-3				1
u_3	5	5		15	9

$\Longrightarrow$

v_j / u_i	0	1	10	4
2	☒	☒	12	6
−3	−3	−2	7	☒
5	☒	6	☒	☒

$u_i + v_j$

$\Rightarrow$

☒	☒	11-12	7-6
1+3	0+2	6-7	☒
☒	8-6	☒	☒

$c_{ij} - (u_i + v_j)$

$\Longrightarrow$

☒	☒	−1	1
4	2	−1	☒
☒	2	☒	☒

C.E.M.

Since the evaluation of cells (1, 3) and (2, 3) is –ve , then the current solution is not optimal.

Step 4: Iterate Toward an Optimal Solution

This involves the following sub-steps:

Sub-step 1: From the cell evaluation matrix, identify the cell with the most negative evaluation. This is the rate by which total transportation cost can be reduced if one unit is allocated to this cell. This cell is called the ***identified cell***. In case of tie in the cell evaluation, the cell which maximum allocation can be made is selected.

Sub-step 2: Write down again the initial basic feasible solution, check mark (√) the identified cell.

Sub-step 3: Trace a closed path in the matrix. This closed path consists of vertical and horizontal lines (not diagonal) begin and terminate in the identified cell and all other corners of the path lie in the allocated cell only.

Sub-step 4: Mark the identified cell as positive and each occupied cell at the corners of the path alternately –ve, +ve, —ve and so on.

Sub-step 5: Make a new allocation in the identified cell by entering the smallest allocation on the path that has been assigned a –ve sign. Add and subtract this new allocation from the cells at the corners of the path, maintaining the row and column requirements.

Step 5: Repeat Steps 3-4 Until Optimal Solution is Reached

Repeat steps 3 and 4 until an optimal solution is reached.

Example (4.7):

After we check optimality of basic feasible solution obtained by Vogel's method in example (4.4) and find that this solution is not optimal. Then we proceed to find the optimal solution. Choose cell (1, 3) as the identified cell, then:

		Distribution centers				
		1	**2**	**3**	**4**	**Supply**
Plants	**1**	– 1	5	+ √		6
	2				1	1
	3	+ 6		3 –	1	10
Requirement		7	5	3	2	17

The smallest element in the corners with negative sign is 1, so add and subtract 1 from the cells at the corners of the path. The matrix will be:

		Distribution centers				
		1	**2**	**3**	**4**	**Supply**
Plants	**1**	2	3 5	11 1	7	6
	2	1	0	6	1 1	1
	3	5 7	8	15 2	9 1	10
Requirement		7	5	3	2	17

Since number of allocations$= 6 = m + n - 1$ and they are in independent positions, then we can check optimality.

v_j / u_i	1	3	11	5
0		3	11	
−4				1
4	5		15	9

v_j / u_i	1	3	11	5
0	1	☒	☒	5
−4	−3	−1	7	☒
4	☒	7	☒	☒

$u_i + v_j$

⇒

2−1	☒	☒	7−5
1+3	0+1	6−7	☒
☒	8−7	☒	☒

$c_{ij} - (u_i + v_j)$

⇒

1	☒	☒	2
4	1	−1	☒
☒	1	☒	☒

C.E.M.

The new solution is not optimal, Choose cell (2, 3) as the identified cell, then:

		Distribution centers				
		1	**2**	**3**	**4**	**Supply**
Plants	**1**		5	1		6
	2			+ √	1 −	1
	3	7		− 2	1 +	10
Requirement		7	5	3	2	17

The smallest element in the corners with negative sign is 1, so add and subtract 1 from the cells at the corners of the path. The matrix will be:

		Distribution centers				
		1	**2**	**3**	**4**	**Supply**
Plants	**1**	$\boxed{2}$	$\boxed{3}$ 5	$\boxed{11}$ 1	$\boxed{7}$	6
	2	$\boxed{1}$	$\boxed{0}$	$\boxed{6}$ 1	$\boxed{1}$	1
	3	$\boxed{5}$ 7	$\boxed{8}$	$\boxed{15}$ 1	$\boxed{9}$ 2	10
Requirement		7	5	3	2	17

Since number of allocations$= 6 = m + n - 1$ and they are in independent positions, then we can check optimality.

$$\begin{array}{c|c|c|c|c|}
{}^{v_j}_{u_i} & 0 & 2 & 10 & 4 \\ \hline
1 & & 3 & 11 & \\ \hline
-4 & & & 6 & \\ \hline
5 & 5 & & 15 & 9 \\ \hline
\end{array}$$

$$\underset{u_i+v_j}{\begin{array}{c|c|c|c|c|}
{}^{v_j}_{u_i} & 0 & 2 & 10 & 4 \\ \hline
1 & 1 & \boxtimes & \boxtimes & 5 \\ \hline
-4 & -4 & -2 & \boxtimes & 0 \\ \hline
5 & \boxtimes & 7 & \boxtimes & \boxtimes \\ \hline
\end{array}} \Rightarrow \underset{c_{ij}-(u_i+v_j)}{\begin{array}{|c|c|c|c|}
\hline
2-1 & \boxtimes & \boxtimes & 7-5 \\ \hline
1+4 & 0+2 & \boxtimes & 1+0 \\ \hline
\boxtimes & 8-7 & \boxtimes & \boxtimes \\ \hline
\end{array}} \Rightarrow \underset{\text{C.E.M.}}{\begin{array}{|c|c|c|c|}
\hline
1 & \boxtimes & \boxtimes & 2 \\ \hline
5 & 2 & \boxtimes & 1 \\ \hline
\boxtimes & 1 & \boxtimes & \boxtimes \\ \hline
\end{array}}$$

Since all the elements of the cell evaluation matrix are positive, then the optimal solution is:

		Distribution centers				
		1	**2**	**3**	**4**	**Supply**
Plants	**1**	2	3 5	11 1	7	6
	2	1	0	6 1	1	1
	3	5 7	8	15 1	9 2	10
Requirement		7	5	3	2	17

And the transportation cost is:

$$Z = (5*3+1*11+1*6+7*5+1*15+2*9)*100 = 10000 \quad \text{ID}$$

4.3 The Unbalanced Transportation Problem

In many real life situations, the total availability may not be equal to the total demand, i.e. $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$, such problems are called ***Unbalanced Transportation Problem***. In these problems either some resources will remain unused or some requirements will remain unfilled. Since a feasible solution exists only for a balanced problem, it is necessary that the total availability be made equal to the total demand.

1) If $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j$: we add a dummy resource, the costs of this resource are set equal to zero.
2) If $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$: we add a dummy destination, the costs of this destination are set equal to zero.

The supply (demand) of the dummy resource (destination) is: $\left|\sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j\right|$.

Example (4.8):

Find the optimal solution of the following transportation problem:

		Stores				
		1	**2**	**3**	**4**	**Supply**
Factories	**1**	4	6	8	13	50
	2	13	11	10	8	70
	3	14	4	10	13	30
	4	9	11	13	8	50
Requirement		25	35	105	20	

Solution:

$\sum_{i=1}^{4} a_i = 50 + 70 + 30 + 50 = 200$, $\sum_{j=1}^{4} b_j = 25 + 35 + 105 + 20 = 185$, then the problem is unbalanced. Therefore, we create a dummy destination. The associated cost coefficients are taken as zero. The cost matrix becomes

		Stores					
		1	**2**	**3**	**4**	**d**	**Supply**
Factories	**1**	4	6	8	13	0	50
	2	13	11	10	8	0	70
	3	14	4	10	13	0	30
	4	9	11	13	8	0	50
Requirement		25	35	105	20	15	200

$\sum_{i=1}^{4} a_i = 200$, $\sum_{j=1}^{5} b_j = 200$, then the problem is balanced. We shall use Vogel's approximation method to find the initial feasible solution. The cost matrix with penalties is shown below:

		Stores						
		1	2	3	4	d	Supply	
Factories	1	4	6	8	13	0	50	[4]
	2	13	11	10	8	0	70	[8]
	3	14	4	10	13	0	30	[4]
	4	9	11	13	8	0 15	~~50~~ 35	[8] ←
Requirement		25	35	105	20	~~15~~ 0	200	
		[5]	[2]	[2]	[0]	[0]		

The greatest penalty is [8], so we choose the 4th row and allocate as much as possible (i.e. 15) to cell (4, 5)(the cell with smallest cost in the 4th row). Requirement of store d is completely satisfied, so column d is crossed out and the shrunken matrix with penalties and allocation is as below:

		Stores					
		1	2	3	4	Supply	
Factories	1	4	6	8	13	50	[2]
	2	13	11	10	8	70	[2]
	3	14	4 30	10	13	~~30~~ 0	[6] ←
	4	9	11	13	8	35	[1]
Requirement		25	~~35~~ 5	105	20	200	
		[5]	[2]	[2]	[0]		

The greatest penalty is [6], so we choose the 3rd row and allocate as much as possible (i.e. 30) to cell (3, 2)(the cell with smallest cost in the 3rd row). Supply of factory 3 is completely satisfied, so row 3 is crossed out and the shrunken matrix with penalties and allocation is as below:

		Stores					
		1	2	3	4	Supply	
Factories	1	4 25	6	8	13	~~50~~ 25	[2]
	2	13	11	10	8	70	[2]
	4	9	11	13	8	35	[1]
Requirement		~~25~~ 0	5	105	20	200	
		[5]	[5]	[2]	[0]		
		↑					

The greatest penalty is [5], so we choose the 1st column and allocate as much as possible (i.e. 25) to cell (1, 1)(the cell with smallest cost in the 1st column). Requirement of store 1 is completely satisfied, so column 1 is crossed out and the shrunken matrix with penalties and allocation is as below:

		Stores				
		2	3	4	Supply	
Factories	1	6 5	8	13	~~25~~ 20	[2]
	2	11	10	8	70	[2]
	4	11	13	8	35	[3]
Requirement		~~5~~ 0	105	20	200	
		[5] ↑	[2]	[0]		

The greatest penalty is [5], so we choose the 2nd column and allocate as much as possible (i.e. 5) to cell (1, 2)(the cell with smallest cost in the 2nd column). Requirement of store 2 is completely satisfied, so column 2 is crossed out and the shrunken matrix with penalties and allocation is as below:

		Stores			
		3	4	Supply	
Factories	1	8 20	13	~~20~~ 0	[5]←
	2	10	8	70	[2]
	4	13	8	35	[5]
Requirement		~~105~~ 85	20	200	
		[2]	[0]		

The greatest penalty is [5], so we choose the 1st row and allocate as much as possible (i.e. 20) to cell (1, 3)(the cell with smallest cost in the 1st row). Supply of factory 1 is completely satisfied, so row 1 is crossed out and the shrunken matrix with penalties and allocation is as below:

		Stores			
		3	4	Supply	
Factories	2	10	8	70	[2]
	4	13	8 20	~~35~~ 15	[5] ←
Requirement		85	~~20~~ 0	200	
		[3]	[0]		

The greatest penalty is [5], so we choose the 4th row and allocate as much as possible (i.e. 20) to cell (4, 4)(the cell with smallest cost in the 4th row). Requirement of store 4 is completely satisfied, so column 4 is crossed out and the shrunken matrix with allocation is as below(according to least cost):

		Stores	
		3	**Supply**
Factories	**2**	10 70	~~70~~ 0
	4	13 15	~~15~~ 0
Requirement		~~85~~ ~~15~~ 0	200

That is the initial feasible solution is:

		Stores					
		1	**2**	**3**	**4**	**d**	**Supply**
Factories	**1**	4 25	6 5	8 20	13	0	50
	2	13	11	10 70	8	0	70
	3	14	4 30	10	13	0	30
	4	9	11	13 15	8 20	0 15	50
Requirement		25	35	105	20	15	200

Since number of allocations$= 8 = m + n - 1$ and they are in independent positions, then we can check optimality. The sub-steps are:

v_j / u_i	0	2	4	−1	−9
4	4	6	8		
6			10		
2		4			
9			13	8	0

⇒

v_j / u_i	0	2	4	−1	−9
4	☒	☒	☒	3	−5
6	6	8	☒	5	−3
2	2	☒	6	1	−7
9	9	11	☒	☒	☒

$u_i + v_j$

⇒

☒	☒	☒	10	5
7	3	☒	3	3
12	☒	4	12	7
0	0	☒	☒	☒

C.E.M ($c_{ij} - (u_i + v_j)$)

Since cell values are positive, then the first feasible solution is optimal and

$$Z = 25 * 4 + 5 * 6 + 20 * 8 + 70 * 10 + 30 * 4 + 15 * 13 + 20 * 8 + 15 * 0$$
$$= 1465$$

4.4 Degeneracy in Transportation Problem

In transportation problem with m origins and n destinations if a basic feasible solution has less than $m + n - 1$ allocations (occupied cells), the problem is said to be a degenerate transportation problem. Degeneracy can occur in the initial solution or during some subsequent iteration.

In this case we allocate an infinitesimally but positive value ϵ to vacant cell (cells) with least cost so that there are exactly $m+n-1$ allocated cells in independent positions and the procedure can then be continued in the usual manner. Subscripts are used when more than one such letter is required (f.e. $\epsilon_1, \epsilon_2,$ etc.). These ϵ's are treated like any other positive basic variable and are kept in the transportation matrix until temporary degeneracy is removed or until the optimal solution is reached, whichever occurs first. At this point we set each $\epsilon = 0$. Notice that ϵ is infinitesimally small and hence its effect can be neglected when it is added to or subtracted from a positive value (f.e. $10 + \epsilon = 10, 5 - \epsilon = 5, \epsilon + \epsilon = 2\epsilon, \epsilon - \epsilon = 0$). Consequently, they do not alter the physical nature of the original set of allocations but do help in carrying out further computations such as optimality test.

Example (4.9):

Find the optimal solution of the following transportation problem.

		Destinations						
		1	**2**	**3**	**4**	**5**	**6**	**Supply**
Origins	**1**	9	12	9	6	9	10	5
	2	7	3	7	7	5	5	6
	3	6	5	9	11	3	11	2
	4	6	8	11	2	2	10	9
Requirement		4	4	6	2	4	2	

Solution:

$\sum_{i=1}^{4} a_i = 5+6+2+9 = 22, \sum_{j=1}^{6} b_j = 4+4+6+2+4+2 = 22$, then the system is balanced. The initial basic feasible solution by using Vogel's approximation method is:

		Destinations							
		1	2	3	4	5	6	Supply	
Origins	1	9	12	9 5	6	9	10	~~5~~ 0	[3] [3] [0] [0] [0] [0]
	2	7	3 4	7	7	5	5 2	~~6~~ ~~4~~ 0	[2] [2] [2] [4]← -- --
	3	6 1	5	9 1	11	3	11	~~2~~ ~~1~~ 0	[2] [2] [2] [1] [3] [3]←
	4	6 3	8	11	2 2	2 4	10	~~9~~ ~~7~~ ~~3~~ 0	[0] [0] [4]← [2] [5]← --
Requirement		~~4~~ ~~1~~ 0	~~4~~ 0	~~6~~ ~~1~~ 0	~~2~~ 0	~~4~~ 0	~~2~~ 0		
		[0]	[2]	[2]	[4]	[1]	[5]		
		[0]	[2]	[2]	[4]	[1]	↑		
		[0]	[2]	[2]	↑	[1]			
		[0]	[2]	[2]	--	--	--		
		[0]	--	[0]	--	--	--		
		[3]	--	[0]	--	--	--		

Since number of allocations $= 8 \neq 9 = m+n-1 (m=4, n=6)$, then we must select unoccupied cell with least cost and set an infinitesimal allocation to it. The unoccupied cell (3, 5) has the least cost, but this cell form a closed loop with cells (3, 1), (4, 1), and (4, 5). There are two next higher cost cell ((2, 5) and (3, 2)), allocation in either of these cells does not result a closed loop. Let us choose cell (2, 5) and allocate ϵ to it. The matrix will be:

		Destinations						
		1	2	3	4	5	6	Supply
Origins	1	9	12	9 5	6	9	10	5
	2	7	3 4	7	7	5 ϵ	5 2	6
	3	6 1	5	9 1	11	3	11	2
	4	6 3	8	11	2 2	2 4	10	9
Requirement		4	4	6	2	4	2	

Since number of allocations$= 9 = m+n-1$ and they are in independent positions, then we can check optimality. The sub-steps are:

v_j / u_i	0	−6	3	−4	−4	−4
6			9			
9		3			5	5
6	6		9			
6	6			2	2	

$\Rightarrow$

v_j / u_i	0	−6	3	−4	−4	−4
6	6	0	☒	2	2	2
9	9	☒	12	5	☒	☒
6	☒	0	☒	2	2	2
6	☒	0	9	☒	☒	2

$u_i + v_j$

$\Longrightarrow$

	1	2	3	4	5	6
1	3	12	☒	4	7	8
2	−2	☒	−5	2	☒	☒
3	☒	5	☒	9	1	9
4	☒	8	2	☒	☒	8

C.E.M$(c_{ij} - (u_i + v_j))$

Since cells (2, 1) and (2, 3) have negative values, then the current feasible solution is not optimal. Cell (2, 3) has the most negative value, then:

	1	2	3	4	5	6
1			5			
2		4	+ $\surd$		− ϵ	2
3	+ 1		− 1			
4	− 3			2	+ 4	

$\Longrightarrow$

	1	2	3	4	5	6
1			5			
2		4	ϵ		$\epsilon - \epsilon$	2
3	$1 + \epsilon$		$1 - \epsilon$			
4	$3 - \epsilon$			2	$4 + \epsilon$	

$\Longrightarrow$

	1	2	3	4	5	6
1			5			
2		4	ϵ			2
3	$1 + \epsilon$		$1 - \epsilon$			
4	$3 - \epsilon$			2	$4 + \epsilon$	

Since number of allocations$= 9 = m + n - 1$ and they are in independent positions, then we can check optimality. The sub-steps are:

v_j u_i	0	−1	3	−4	−4	1
6			9			
4		3	7			5
6	6		9			
6	6			2	2	

$\Longrightarrow$

v_j u_i	0	−1	3	−4	−4	1
6	6	5	☒	2	2	7
4	4	☒	☒	0	0	☒
6	☒	5	☒	2	2	7
6	☒	5	9	☒	☒	7

$u_i + v_j$

$\Longrightarrow$

	1	2	3	4	5	6
1	3	7	☒	4	7	3
2	3	☒	☒	7	5	☒
3	☒	0	☒	9	1	4
4	☒	3	2	☒	☒	3

C.E.M$(c_{ij} - (u_i + v_j))$

Since all the elements of the cell evaluation matrix are positive, then the optimal solution is (considering $\epsilon = 0$):

		Destinations						
		1	**2**	**3**	**4**	**5**	**6**	**Supply**
Origins	**1**	9	12	9 5	6	9	10	5
	2	7	3 4	7	7	5	5 2	6
	3	6 1	5	9 1	11	3	11	2
	4	6 3	8	11	2 2	2 4	10	9
Requirement		4	4	6	2	4	2	

The cost is:

$$Z = 5*9+4*3+2*5+1*6+1*9+3*6+2*2+4*2 = 112 \text{ units}$$

Exercises 4.1 (In addition to the text book exercises)

Find the optimal solution of the following transportation problems:

1:

		Destinations				
		1	**2**	**3**	**4**	**Supply**
Origins	**1**	90	90	100	110	200
	2	50	70	130	85	100
Requirement		75	100	100	30	

2:

	1	**2**	**3**	**4**	**5**	**Supply**
1	8	6	2	4	12	80
2	10	4	6	8	10	60
3	6	10	12	6	4	40
4	4	8	8	10	6	20
Requirement	60	60	30	40	10	