Operations Research II

Second Semester

For the 3rd class students / Mathematics Department / College of Science for Women

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Preface

These lecture notes are for the course “Operations Research II” for the 3rd grade- second semester in Mathematics Department / College of Science for Women /Baghdad University.

The author claims no originality. These lecture notes are collected from references listed in the “Bibliography”.

Contents

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Ch. 1: Assignment Problem

The ***assignment problem*** may be defined as follows: Given n facilities and n jobs and given the effectiveness of each facility for each job, the problem is to assign each facility to one and only one job so as to optimize the given measures of effectiveness. The assignment problem is a special case of transportation problem.

Table(1.1) represents the assignment of n facilities(machines) to n jobs, c_{ij} is the cost of assigning ith facility to jth job and x_{ij} represents the assignment of ith facility to jth job. If ith facility can be assigned to jth job, $x_{ij} = 1$, otherwise zero. The matrix is called the ***cost matrix***.

		Jobs				
		1	**2**	**…**	**n**	**a_i (Supply)**
Facilities	**1**	c_{11}	c_{12}	…	c_{12}	1
	2	c_{21}	c_{22}	…	c_{2n}	1
	⋮	⋮	⋮	…		⋮
	n	c_{n1}	c_{n2}	…	c_{nn}	1
	b_j(Demand)	1	1	…	1	

Table (1.1)

1.1 Mathematical Representation of the Assignment Model

Mathematically, the assignment model can be expressed as follows:

Let

$$x_{ij} = \begin{cases} 0, if\ the\ ith\ facility\ is\ not\ assigned\ to\ jth\ job \\ 1, if\ the\ ith\ facility\ is\ assigned\ to\ jth\ job \end{cases}$$

Then, the model is given by:

$$min \quad Z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\, x_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}\, x_{ij}$$

$$S.t. \quad \sum_{j=1}^{n} x_{ij} = 1 \qquad i = 1,2,\dots,n$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad j = 1,2,\dots,n$$

$$x_{ij} = 0\ \ or\ 1 \qquad i = 1,2,\dots,n;\ \ j = 1,2,\dots,n \ \ (\text{or } x_{ij} = {x_{ij}}^2)$$

The technique used for solving assignment model makes use of two theorems:

Theorem (1.1)

In an assignment problem, if we add or subtract a constant to every element of a row (or column) in the cost matrix, then an assignment which minimizes the total cost on one matrix also minimizes the total cost on the other matrix.

Theorem (1.2)

If all $c_{ij} \geq 0$ and we can find a set $x_{ij} = x_{ij}^*$ such that $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \, x_{ij}^* = 0$, then this solution is optimal.

The above two theorems indicates that if one can create a new c_{ij} matrix with zero entries, and if these zero elements, or a subset thereof, contains feasible solution, then this feasible solution is the optimal solution.

1.2 The Hungarian Method

The ***Hungarian method*** (or ***reduced matrix method***) was developed by D. König, a Hungarian mathematician. The method consists of the following steps:

Step 1: Prepare a square matrix. Add dummy rows (columns) if needed (rows (columns) with zero cost).

Step 2: Reduce the matrix. Subtract the smallest element of each row from all the elements of the row. So there will be at least one zero in each row. Examine if there is at least one zero in each column. If not, subtract the smallest element of the column(s) not containing zero from all the elements of the column. This step reduces the elements of the matrix until zeros, ***called zero opportunity costs***, are obtained in each column.

Step 3: Check whether an optimal assignment can be made in the reduced matrix or not. For this:

a) Examine rows successively until a row with exactly one unmarked zero is obtained. Make an assignment to this single zero by marking square (□) around it. Cross (X) all other zeros in the same column as they will not be considered for making any more assignment in that column. Proceed in this way until all rows have been examined.

b) Now examine columns successively until a column with exactly one unmarked zero is found. Make an assignment there by marking square (□) around it and cross (X) any other zeros in the same row. Proceed in this way until all columns have been examined.

In case there is no row or column containing single unmarked zero (they contain more than one unmarked zero), mark square (□) around it arbitrarily and cross

(X) all other zeros in its row and column. Proceed in this manner till there is no unmarked zero left in the cost matrix.

Repeat sub-steps (a) and (b) till one of the following two cases occur:

i) There is one assignment in each row and in each column. In this case the optimal assignment can be made in the current solution. The minimum number of lines crossing all zeros is n, the order of the matrix.

ii) There is some row and/or column without assignment. In this case optimal assignment cannot be made in the current solution. The minimum number of lines crossing all zeros has to be obtained in this case by following step 4.

Step 4: Find the minimum number of lines crossing all zeros. This consists of the following sub-steps:

a) Mark (√) the rows that do not have assignments.

b) Mark (√) the columns (not already marked) that have zeros in marked rows.

c) Mark (√) the rows (not already marked) that have assignment in the marked columns.

d) Repeat sub-steps (b) and (c) till no more rows or columns can be marked.

e) Draw straight lines through all unmarked rows and marked columns. This gives the minimum number of lines crossing all zeros.

Step 5: Iterate towards the optimal solution. Examine the uncovered elements. Select the smallest element and subtract it from all the uncovered elements. Add this smallest element to every element that lies at the intersection of two lines. Leave the remaining elements of the matrix without change. This yields a new basic feasible solution.

Step 6: Repeat steps 3 through 5 successively until the minimum number of lines crossing all zeros becomes equal to n, the order of the matrix. In such a case every row and column will have one assignment. This indicates that an optimal solution has been obtained. The total cost associated with this solution is obtained by adding the original costs of the assigned cells.

Example (1.1):

A machine tool company decides to make four subassemblies through four contractors. Each contractor is to receive only one subassembly. The cost of each subassembly is determined by the bids submitted by each contractor and is shown in the following table in millions of Iraqi dinars.

1) Formulate the mathematical model for the problem.

2) Assign the different subassemblies to contractors to minimize the total cost.

		Contractors			
		1	**2**	**3**	**4**
Subass-emblies	**1**	15	13	14	17
	2	11	12	15	13
	3	13	12	10	11
	4	15	17	14	16

Solution:

1) Let $x_{ij} = \begin{cases} 0, if\ the\ ith\ subassembly\ \ is\ not\ assigned\ to\ jth\ contractor \\ 1, if\ the\ ith\ subassembly\ is\ assigned\ \ to\ jth\ contractor \end{cases}$

Then, the model is given by:

$$min \quad Z = \sum_{i=1}^{4}\sum_{j=1}^{4} c_{ij}\ x_{ij} = \sum_{j=1}^{4}\sum_{i=1}^{4} c_{ij}\ x_{ij}$$

S.t.

$$\left.\begin{aligned} x_{11} + x_{12} + x_{13} + x_{14} &= 1 \\ x_{21} + x_{22} + x_{23} + x_{24} &= 1 \\ x_{31} + x_{32} + x_{33} + x_{34} &= 1 \\ x_{41} + x_{42} + x_{43} + x_{44} &= 1 \end{aligned}\right\} \text{Constraints on subassemblies}$$

$$\left.\begin{aligned} x_{11} + x_{21} + x_{31} + x_{41} &= 1 \\ x_{12} + x_{22} + x_{32} + x_{42} &= 1 \\ x_{13} + x_{23} + x_{33} + x_{43} &= 1 \\ x_{14} + x_{24} + x_{34} + x_{44} &= 1 \end{aligned}\right\} \text{Constraints on contractors}$$

$$x_{ij} = 0 \ \ or\ 1 \qquad i = 1,2,3,4\ ;\ \ j = 1,2,3,4 \ \ (\text{or } x_{ij} = {x_{ij}}^2)$$

2) We will reduce the matrix; the smallest element in the first row is 13, so we subtract 14 from all elements of the first row. Similarly for the remaining three rows. This gives the following matrix:

	1	**2**	**3**	**4**
1	2	0	1	4
2	0	1	4	2
3	3	2	0	1
4	1	3	0	2

Each row contains at least one zero. The last column does not contain any zero , the we subtract the smallest element in that column (which is 1) from all the elements of the column. This gives the following matrix:

	1	**2**	**3**	**4**
1	2	0	1	3
2	0	1	4	1
3	3	2	0	0

4	1	3	0	1

The assignment is given in the following matrix:

		Contractors			
		1	**2**	**3**	**4**
Subassemblies	**1**	2	$\boxed{0}$	1	3
	2	$\boxed{0}$	1	4	1
	3	3	2	$\cancel{0}$	$\boxed{0}$
	4	1	3	$\boxed{0}$	1

Since there is one assignment in each row and in each column, the optimal assignment can be made in the current solution. The optimal assignment is:

Subassembly 1 is assigned to contractor 2

Subassembly 2 is assigned to contractor 1

Subassembly 3 is assigned to contractor 4

Subassembly 4 is assigned to contractor 3

And the minimum total cost is:

$$Z_{min} = (13 + 11 + 11 + 14) \times 10^6 = 49000000 \text{ ID}$$

Example (1.2):

Four different jobs can be done on four different machines. The matrix below gives the cost in dolars of producing job i on machine j.

		Machines			
		M_1	**M_2**	**M_3**	**M_4**
Jobs	**J_1**	5	7	11	6
	J_2	8	5	9	6
	J_3	4	7	10	7
	J_4	10	4	8	3

How should the jobs be assigned to the machines so that the total cost is minimized?

Solution:

Reducing the matrix involves the following steps:

	M_1	M_2	M_3	M_4
J_1	5	7	11	6
J_2	8	5	9	6
J_3	4	7	10	7
J_4	10	4	8	3

$\Rightarrow$

	M_1	M_2	M_3	M_4
J_1	0	2	6	1
J_2	3	0	4	1
J_3	0	3	6	3
J_4	7	1	5	0

The third column does not contain a zero, then we subtract 4 (the smallest element of the third column) from all the elements of that column. This gives the following matrix:

	M_1	M_2	M_3	M_4
J_1	0	2	2	1
J_2	3	0	0	1
J_3	0	3	2	3
J_4	7	1	1	0

The assignment is given in the following matrix:

	M_1	M_2	M_3	M_4
J_1	$\boxed{0}$	2	2	1
J_2	3	$\boxed{0}$	$\cancel{0}$	1
J_3	$\cancel{0}$	3	2	3
J_4	7	1	1	$\boxed{0}$

Row 3 and column 3 are without any assignment; hence we proceed as follows to find the minimum number of lines crossing all zeros:

	M_1	M_2	M_3	M_4	
J_1	$\boxed{0}$	2	2	1	√
J_2	3	$\boxed{0}$	$\cancel{0}$	1	
J_3	$\cancel{0}$	3	2	3	√
J_4	7	1	1	$\boxed{0}$	
	√				

The minimum number of lines crossing all zeros is $3 \neq n (n = 4 \text{ here})$. Hence the optimal assignment is not possible in the current solution. The smallest element in the cells that do not have a line through is 1. By applying step 5, the matrix will be:

	M_1	M_2	M_3	M_4

J_1	0	1	1	0
J_2	4	0	0	1
J_3	0	2	1	2
J_4	8	1	1	0

The assignment is given in the following matrix:

	M_1	**M_2**	**M_3**	**M_4**
J_1	$\cancel{0}$	1	1	$\cancel{0}$
J_2	4	$\boxed{0}$	$\cancel{0}$	1
J_3	$\boxed{0}$	2	1	2
J_4	8	1	1	$\boxed{0}$

Row 1 and column 3 are without any assignment; hence we proceed as follows to find the minimum number of lines crossing all zeros:

	M_1	**M_2**	**M_3**	**M_4**	
J_1	$\cancel{0}$	1	1	$\cancel{0}$	√
J_2	4	$\boxed{0}$	$\cancel{0}$	1	
J_3	$\boxed{0}$	2	1	2	√
J_4	8	1	1	$\boxed{0}$	√
	√			√	

The minimum number of lines crossing all zeros is $3 \neq n (n = 4 \text{ here})$. Hence the optimal assignment is not possible in the current solution. The smallest element in the cells that do not have a line through is 1. By applying step 5, the matrix will be:

	M_1	**M_2**	**M_3**	**M_4**
J_1	0	0	0	0
J_2	5	0	0	2
J_3	0	1	0	2
J_4	8	0	0	0

The assignment is given in the following matrix:

	M_1	M_2	M_3	M_4
J_1	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$
J_2	5	$\boxed{0}$	$\cancel{0}$	2
J_3	$\cancel{0}$	1	$\boxed{0}$	2
J_4	8	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$

Since there is one assignment in each row and in each column, the optimal assignment can be made in the current solution. The optimal assignment is:

J_1 is assigned to M_1

J_2 is assigned to M_2

J_3 is assigned to M_3

J_4 is assigned to M_4

And the minimum total cost is: $Z_{min} = 5 + 5 + 10 + 3 = 23\ \$$

1.3 Variations of the Assignment Problem

1.3.1 Non-square Matrix (Unbalanced Assignment Problem)

Such a problem is found when the number of facilities is not equal to the number of jobs. Since the Hungarian method of solution requires a square matrix, dummy facilities or jobs may be added and zero costs is assigned to the corresponding cells of the matrix. These cells are then treated the same way as the real cost cells during the solution procedure.

Example (1.3):

A company has one surplus truck in each of the cities A, B, C, D and E and one dificit truck in each of the cities 1, 2, 3, 4, 5 and 6. The distance between the cities in kilometres is shown in the matrix below. Find the assignment of trucks from cities in surplus to cities in deficit so that the total distance covered by vehicles is minimum.

	1	2	3	4	5	6
A	12	10	15	22	18	8
B	10	18	25	15	16	12
C	11	10	3	8	5	9
D	6	14	10	13	13	12
E	8	12	11	7	13	10

Solution:

The matrix is non-square, so we add a dummy city with surplus vehicle.Since there is no distance associated with it, the corresponding cell values are made all zeros.

	1	2	3	4	5	6
A	12	10	15	22	18	8
B	10	18	25	15	16	12
C	11	10	3	8	5	9
D	6	14	10	13	13	12
E	8	12	11	7	13	10
d	0	0	0	0	0	0

	1	2	3	4	5	6
A	12	10	15	22	18	8
B	10	18	25	15	16	12
C	11	10	3	8	5	9
D	6	14	10	13	13	12
E	8	12	11	7	13	10
d	0	0	0	0	0	0

$\Rightarrow$

	1	2	3	4	5	6
A	4	2	7	14	10	0
B	0	8	15	5	6	2
C	8	7	0	5	2	6
D	0	8	4	7	7	6
E	1	5	4	0	6	3
d	0	0	0	0	0	0

The assignment is given in the following matrix:

	1	2	3	4	5	6
A	4	2	7	14	10	$\boxed{0}$
B	$\boxed{0}$	8	15	5	6	2
C	8	7	$\boxed{0}$	5	2	6
D	$\cancel{0}$	8	4	7	7	6
E	1	5	4	$\boxed{0}$	6	3
d	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$

Row 4 and column 5 are without any assignment; hence we proceed as follows to find the minimum number of lines crossing all zeros:

	1	2	3	4	5	6	
A	4	2	7	14	10	$\boxed{0}$	
B	$\boxed{0}$	8	15	5	6	2	√
C	8	7	$\boxed{0}$	5	2	6	
D	$\cancel{0}$	8	4	7	7	6	√
E	1	5	4	$\boxed{0}$	6	3	
d	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	
	√						

The minimum number of lines crossing all zeros is $5 \neq n(n = 6 \text{ here})$. Hence the optimal assignment is not possible in the current solution. The smallest element in the cells that do not have a line through is 2. By applying step 5 the matrix will be:

	1	2	3	4	5	6
A	6	2	7	14	10	0
B	0	6	13	3	4	0
C	10	7	0	5	2	6
D	0	6	2	5	5	4
E	3	5	4	0	6	3
d	2	0	0	0	0	0

The assignment is given in the following matrix:

	1	2	3	4	5	6
A	6	2	7	14	10	$\boxed{0}$
B	$\boxed{0}$	6	13	3	4	$\cancel{0}$
C	10	7	$\boxed{0}$	5	2	6
D	$\cancel{0}$	6	2	5	5	4
E	3	5	4	$\boxed{0}$	6	3
d	2	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$

Row 4 and column 5 are without any assignment; hence we proceed as follows to find the minimum number of lines crossing all zeros:

	1	2	3	4	5	6	
A	6	2	7	14	10	$\boxed{0}$	√
B	$\boxed{0}$	6	13	3	4	$\cancel{0}$	√
C	10	7	$\boxed{0}$	5	2	6	
D	$\cancel{0}$	6	2	5	5	4	√
E	3	5	4	$\boxed{0}$	6	3	
d	2	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	
	√					√	

The minimum number of lines crossing all zeros is $5 \neq n(n = 6 \text{ here})$. Hence the optimal assignment is not possible in the current solution. The smallest element in the cells that do not have a line through is 2. By applying step 5 the matrix will be:

	1	2	3	4	5	6
A	6	0	5	12	8	0
B	0	4	11	1	2	0
C	12	7	0	5	2	8
D	0	4	0	3	3	4
E	5	5	4	0	6	5
d	4	0	0	0	0	2

The assignment is given in the following matrix:

	1	2	3	4	5	6
A	6	$\boxed{0}$	5	12	8	$\cancel{0}$
B	$\cancel{0}$	4	11	1	2	$\boxed{0}$
C	12	7	$\boxed{0}$	5	2	8
D	$\boxed{0}$	4	$\cancel{0}$	3	3	4
E	5	5	4	$\boxed{0}$	6	5
d	4	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	2

Since there is one assignment in each row and in each column, the optimal assignment can be made in the current solution. The optimal assignment is:

City A should supply the vehicle to city 2
City B should supply the vehicle to city 6
City C should supply the vehicle to city 3
City D should supply the vehicle to city 1
City E should supply the vehicle to city 4
Minimum distance traveled $= 10 + 12 + 3 + 6 + 7 = 38\ km$

No truck supplid to city 5

Example (1.4):

Solve the following assignment problem for minimal optimal cost:

	1	2	3	4
I	9	14	19	15
II	7	17	20	19
III	9	18	21	18
IV	10	12	18	19
V	10	15	21	16

Solution:

The matrix is non-square, so we add a dummy job with cell values are made all zeros.

	1	2	3	4	d
I	9	14	19	15	0
II	7	17	20	19	0
III	9	18	21	18	0
IV	10	12	18	19	0
V	10	15	21	16	0

$\Rightarrow$

	1	2	3	4	d
I	2	2	1	0	0
II	0	5	2	4	0
III	2	6	3	3	0
IV	3	0	0	4	0
V	3	3	3	1	0

The assignment is given in the following matrix:

	1	2	3	4	d
I	2	2	1	$\boxed{0}$	$\cancel{0}$
II	$\boxed{0}$	5	2	4	$\cancel{0}$
III	2	6	3	3	$\boxed{0}$
IV	3	$\boxed{0}$	$\cancel{0}$	4	$\cancel{0}$
V	3	3	3	1	$\cancel{0}$

Row 5 and column 3 are without any assignment; hence we proceed as follows to find the minimum number of lines crossing all zeros:

	1	2	3	4	d	
I	2	2	1	$\boxed{0}$	$\cancel{0}$	
II	$\boxed{0}$	5	2	4	$\cancel{0}$	
III	2	6	3	3	$\boxed{0}$	√
IV	3	$\boxed{0}$	$\cancel{0}$	4	$\cancel{0}$	
V	3	3	3	1	$\cancel{0}$	√
					√	

The minimum number of lines crossing all zeros is $4 \neq n(n = 5 \text{ here})$. Hence the optimal assignment is not possible in the current solution. The smallest element in the cells that do not have a line through is 1. By applying step 5 the matrix will be:

	1	2	3	4	d
I	2	2	1	0	1
II	0	5	2	4	1
III	1	5	2	2	0
IV	3	0	0	4	1
V	2	2	2	0	0

The assignment is given in the following matrix:

	1	2	3	4	d
I	2	2	1	$\boxed{0}$	1
II	$\boxed{0}$	5	2	4	1
III	1	5	2	2	$\boxed{0}$
IV	3	$\boxed{0}$	$\cancel{0}$	4	1
V	2	2	2	$\cancel{0}$	$\cancel{0}$

Row 5 and column 3 are without any assignment; hence we proceed as follows to find the minimum number of lines crossing all zeros:

	1	2	3	4	d	
I	2	2	1	$\boxed{0}$	1	√
II	$\boxed{0}$	5	2	4	1	
III	1	5	2	2	$\boxed{0}$	√
IV	3	$\boxed{0}$	$\cancel{0}$	4	1	
V	2	2	2	$\cancel{0}$	$\cancel{0}$	√
				√	√	

The minimum number of lines crossing all zeros is $4 \neq n(n = 5 \text{ here})$. Hence the optimal assignment is not possible in the current solution. The smallest element in the cells that do not have a line through is 1. By applying step 5 the matrix will be:

	1	2	3	4	d
I	1	1	0	0	1
II	0	5	2	5	2
III	0	4	1	2	0
IV	3	0	0	5	2
V	1	1	1	0	0

The assignment is given in the following matrix:

	1	2	3	4	d
I	1	1	$\boxed{0}$	$\cancel{0}$	1
II	$\boxed{0}$	5	2	5	2
III	$\cancel{0}$	4	1	2	$\boxed{0}$
IV	3	$\boxed{0}$	$\cancel{0}$	5	2
V	1	1	1	$\boxed{0}$	$\cancel{0}$

Since there is one assignment in each row and in each column, the optimal assignment can be made in the current solution. The optimal assignment is:

I is assigned to 3

II is assigned to 1

IV is assigned to 2

V is assigned to 4

The minimum cost $= 19 + 7 + 12 + 16 = 54$ units. III is not assigned.

1.3.2 Maximization Problem

Sometimes the assignment problem may deal with the maximization of the objective function. The maximization problem has to be changed to minimization before the Hungarian method may be applied. This transformation may be done in either of the following two ways:

a) By subtracting all the elements from the largest element of the matrix.

b) By multiplying the matrix elements by -1.

The hungarian method can then be applied to this equivalent minimization problem to obtain the optimal solution.

Example (1.5):

A company has a team of four salesmen and there are four districts where the company wants to start business. After taking into account the capabilities of salesmen and the nature of districts, the company estimates that the profit per day in hundreds of thousends of dinars for each salesman in each district is as below:

		District			
		1	2	3	4
Slesman	A	16	10	14	11
	B	14	11	15	15
	C	15	15	13	12
	D	13	12	14	15

Find the assignment of salesmen to various districts which will yield maximum profit.

Solution:

As the given problem is of a maximization type, it has to be changed to minimization type before solving it by the Hungarian method. This is achieved by subtracting all the elements of the matrix from the largest element (16), the equivalent matrix is:

		District			
		1	2	3	4
Slesman	A	0	6	2	5
	B	2	5	1	1
	C	1	1	3	4
	D	3	4	2	1

The Hungarian method can now be applied, the reduced matrix is:

	1	2	3	4
A	0	6	2	5
B	1	4	0	0
C	0	0	2	3
D	2	3	1	0

The assignment is given in the following matrix:

	1	2	3	4
A	$\boxed{0}$	6	2	5
B	1	4	$\boxed{0}$	$\cancel{0}$
C	$\cancel{0}$	$\boxed{0}$	2	3
D	2	3	1	$\boxed{0}$

Since there is one assignment in each row and in each column, the optimal assignment can be made in the current solution. The optimal assignment is:

A is assigned to 1

B is assigned to 3

C is assigned to 2

D is assigned to 4

The maximum profit $= (16 + 15 + 15 + 15) \times 10^5 = 6100000$ ID.

Example (1.6):

Solve the following assignment problem for maximal optimal profit:

	I	II	III	IV	V

1	40	40	35	25	50
2	42	30	16	25	27
3	50	48	40	60	50
4	20	19	20	18	25
5	58	60	59	55	53
6	45	52	38	50	49

Solution:

Making the matrix a square matrix, then subtract all the elements of the matrix from the largest element (60). The following tables are obtained:

	I	II	III	IV	V	d
1	40	40	35	25	50	0
2	42	30	16	25	27	0
3	50	48	40	60	50	0
4	20	19	20	18	25	0
5	58	60	59	55	53	0
6	45	52	38	50	49	0

$\Rightarrow$

	I	II	III	IV	V	d
1	20	20	25	35	10	60
2	18	30	44	35	33	60
3	10	12	20	0	10	60
4	40	41	40	42	35	60
5	2	0	1	5	7	60
6	15	8	22	10	11	60

The reduced matrix is:

	I	II	III	IV	V	d
1	10	10	15	25	0	50
2	0	12	26	17	15	42
3	10	12	20	0	10	60
4	5	6	5	7	0	25
5	2	0	1	5	7	60
6	7	0	14	2	3	52

$\Rightarrow$

	I	II	III	IV	V	d
1	10	10	14	25	0	25
2	0	12	25	17	15	17
3	10	12	19	0	10	35
4	5	6	4	7	0	0
5	2	0	0	5	7	35
6	7	0	13	2	3	27

The assignment is given in the following matrix:

	I	II	III	IV	V	d
1	10	10	14	25	$\boxed{0}$	25
2	$\boxed{0}$	12	25	17	15	17
3	10	12	19	$\boxed{0}$	10	35
4	5	6	4	7	$\cancel{0}$	$\boxed{0}$
5	2	$\cancel{0}$	$\boxed{0}$	5	7	35
6	7	$\boxed{0}$	13	2	3	27

Since there is one assignment in each row and in each column, the optimal assignment can be made in the current solution. The optimal assignment is:

1 is assigned to V
2 is assigned to I
3 is assigned to IV
5 is assigned to III
6 is assigned to II. 4 is not assigned.

The maximum profit $= 50 + 42 + 60 + 59 + 52 = 263$ units .

1.3.3 Restrictions on Assignment

Sometimes technical, space, legal or other restrictions do not permit the assignment of a particular facility to a particular job. Such problems can be solved by assigning a very heavy cost (infinite cost) to the corresponding cell. Such a job will then be automatically excluded from further consideration (making assignment).

Example (1.7):

Four new machines M_1, M_2, M_3 and M_4 are to be placed in a machine shop. There are five vacant places A, B, C, D and E available. Because of the limited space, machine M_2 cannot be placed at C and M_3 cannot be placed at A.The assignment cost of machine i to place j in thousands of dolars is shown below:

	A	B	C	D	E
M_1	4	6	10	5	6
M_2	7	4	--	5	4
M_3	--	6	9	6	2
M_4	9	3	7	2	3

Find the optimal assignment schedule.

Solution:

As the given matrix is non-square, we add a dummy machine and associate zero cost with the corresponding cells. As machine M_2 cannot be placed at C and M_3 cannot be placed at A, we assign infinite cost (∞) in cells (M_2,C) and (M_3,A), resulting the following matrix:

	A	B	C	D	E
M_1	4	6	10	5	6
M_2	7	4	∞	5	4
M_3	∞	6	9	6	2
M_4	9	3	7	2	3
d	0	0	0	0	0

The reduced matrix is:

	A	B	C	D	E
M_1	0	2	6	1	2
M_2	3	0	∞	1	0
M_3	∞	4	7	4	0
M_4	7	1	5	0	1
d	0	0	0	0	0

The assignment is given in the following matrix:

	A	B	C	D	E
M_1	$\boxed{0}$	2	6	1	2
M_2	3	$\boxed{0}$	∞	1	$\cancel{0}$
M_3	∞	4	7	4	$\boxed{0}$
M_4	7	1	5	$\boxed{0}$	1
d	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$

Since there is one assignment in each row and in each column, the optimal assignment can be made in the current solution. The optimal assignment is:

M_1 is assigned to place A
M_2 is assigned to place B
M_3 is assigned to place E
M_4 is assigned to place D

There is no machine assigned to place C.

The assignment cost $= (4 + 4 + 2 + 2) \times 1000 = 12000$ \$.

1.3.4 Alternate Optimal Solutions

Sometimes, it is possible to have two or more ways to strike off all zero elements in the reduced matrix for a given problem. In such cases, there will be alternate optimal solutions with the same cost.Alternate optimal solutions offer a great flexibility to the management since it can select the one which is more suitable to its requirement.

Example (1.8):

Recall example (1.2), the optimal solution obtained is not unique. For example, we can make the following assignment:

	M_1	M_2	M_3	M_4
J_1	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$
J_2	5	$\cancel{0}$	$\boxed{0}$	2
J_3	$\boxed{0}$	1	$\cancel{0}$	2
J_4	8	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$

Without change in the optimal cost (23$).

Exercises 1 (In addition to the text book exercises)

Find the optimal assignment for the following:

	I	II	III	IV	V
1	11	17	8	16	20
2	9	7	12	6	15
3	13	16	15	12	16
4	21	24	17	28	26
5	14	10	12	11	15

	1	2	3	4
1	6	5	1	6
2	2	5	3	7
3	3	7	2	8
4	7	7	5	9
5	12	8	8	6
6	6	9	5	10

	1	2	3	4	5	6
A	19	15	--	16	13	22
B	13	--	15	--	21	14
C	15	17	19	20	12	18
D	20	22	16	18	17	--
E	--	16	14	19	18	15

Find the optimal assignment for the following assignment problem to maximize the profit.

	1	2	3	4	5
A	5	11	10	12	4
B	2	4	6	3	5
C	3	12	5	14	6
D	6	14	4	11	7
E	7	9	8	12	5