Ch.3: Network Models

Many operations research situations can be modeled and solved as networks such as determination of the shortest route between two cities, design of an offshore natural-gas pipeline.

3.1 Network Logic

Some of the terms commonly used in networks are defined below.

Definition (3.1):

An ***activity*** is physically identifiable part of a project which requires time and resources for its execution. An activity is represented by an ***arc*** or ***arrow***, the tail of which represents the start and the head, the finish of the activity.

Definition (3.2):

The beginning and end points of an activity are called ***events*** or ***nodes***. Event is a point in time and does not consume any resources. It is represented by a circle.

Definition (3.3):

An unbroken chain of activity arrows connecting the initial event to some other event is called a ***path***.

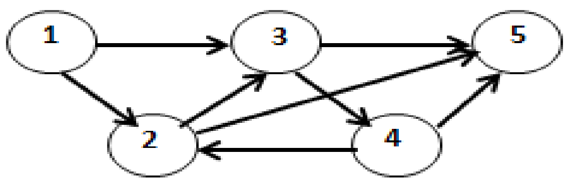
Definition (3.4):

A ***network*** is the graphical representation of logically and sequentially connected nodes and arcs (arrows) representing activities and events of a project. The notion for describing a network is (N, A), where N is the set of nodes, and A is the set of arcs.

Associated with each network is a ***flow***, e.g. oil products flow in pipeline and automobile flow in highway. The maximum flow in a network can be finite or infinite, depending on the capacity of its arcs.

Example (3.1):

The network in the following figure is described as:



N= {1, 2, 3, 4, 5}

A= { (1,2), (1,3), (2,3), (2,5), (3,4), (3,5), (4,2), (4,5)}

Each of: 1-2-5, 1-2-3-4-5 are paths between nodes 1 and 5.

Definition (3.5):

An arc is said to be ***directed*** or ***oriented*** if it allows positive flow in one direction only. A directed network has all directed arcs.

Definition (3.6):

An activity which only determines the dependency of one activity on the other, but does not consume any time is called ***dummy activity***. Dummies are usually represented by dotted line arrows.

Definition (3.7):

A path forms a ***cycle*** or a ***loop*** if it connects a node back to itself through other nodes.

Example (3.2):

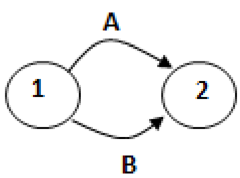
In example (3.1): 2-3-4-2 is a cycle

3.2 Remarks

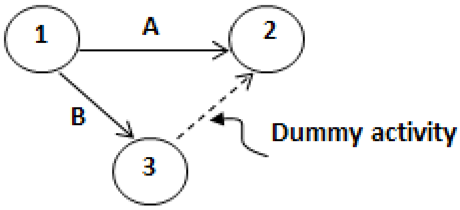
1- The length, shape and direction of the arrow have no relation to the size of the activity.
2- An arrow (activity) directed from node 1 to node 2 can be denoted either by (1, 2) or by 12 or by 1-2 or simply by a letter, e.g. A.
3- For each activity $(i, j), i < j$.
4- Each activity is represented by one and only one arc.
5- Each activity must have a tail and head event.
6- No two or more activities may have the same tail and head events. In this case dummy activities must be used.
7- In a network diagram there should be only one initial event and one end event.
8- An activity must end before its successor begins.
9- An activity occurs only once, that is loops are not allowed.

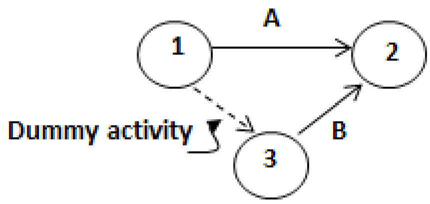
Example (3.3):

Consider the following:



Both activities A and B are joining nodes 1 and 2, this is not allowed, thus we insert a dummy activity as follows:

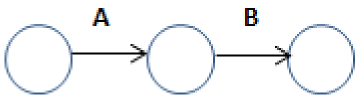




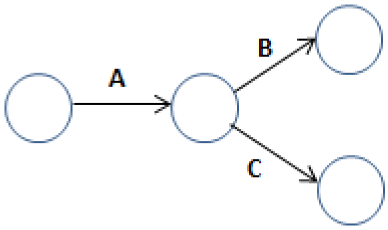
Example (3.4):

Consider the following:

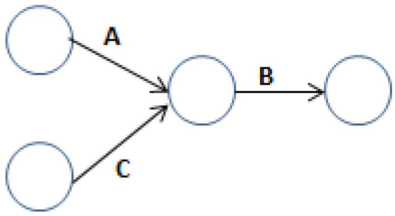
1- Activity A must end before activity B begins.



2- Activity A must end before activities B and C begins.



3- Activities A and C must ends before activity B begins.



3.3 The Critical Path Method (CPM)

The end result in CPM is a time schedule for the project. To achieve this goal, special computations are carried out to produce the following information:

1- Total duration needed to complete the project.
2- Classification of the activities of the project as critical and noncritical.

Definition (3.8):

An activity is ***critical*** if its start and finish times are predetermined (fixed).An activity is ***noncritical*** if it can be scheduled in a time span greater than its duration, permitting flexible start and finish times (within limits).

A delay in the start time of a critical activity definitely causes a delay in the completion of the entire project.

To carry out the necessary computations let:

ES_j = Earliest start time (Earliest occurrence time) of node (event) j (it will be denoted by □ in network)

LS_j = Latest finish time (Latest occurrence time) of node (event) j (it will be denoted by Δ in network)

D_{ij} = Duration of activity (i, j)

The critical path calculations involve two passes: the ***forward pass*** determines the earliest start time of events, and the ***backward pass*** determines the Latest finish time of events.

Forward pass (Earliest start times)

The computations start at node 1 and advance recursively to node n.

Initial step: Set $ES_1 = 0$.

General step j: Given that nodes $p_1, p_2, \dots,$ and p_m are linked directly to node j by incoming activities $(p_1, j), (p_2, j), \dots,$ and (p_m, j) and that the earliest occurrence times of events (nodes) $p_1, p_2, \dots,$ and p_m have already been computed, then the earliest occurrence time of event j is computed as:

$$ES_j = max\left\{ES_{p_1} + D_{p_1 j}, ES_{p_2} + D_{p_2 j}, \dots, ES_{p_m} + D_{p_m j}\right\}$$

The forward pass is complete when ES_n at node n has been computed. By definition, ES_j is the longest path (duration) to node j.

Backward pass (Latest start times)

The computations start at node n and ends at node 1.

Initial step: Set $LS_n = ES_n$ to indicate that latest occurrence of the last node equals the duration of the project.

General step j: Given that nodes $p_1, p_2, \dots,$ and p_m are linked directly to node j by outgoing activities $(j, p_1), (j, p_2), \dots,$ and (j, p_m) and that the latest occurrence times of events (nodes) $p_1, p_2, \dots,$ and p_m have already been computed, then the latest occurrence time of event j is computed as:

$$LS_j = min\left\{LS_{p_1} - D_{jp_1}, LS_{p_2} - D_{jp_2}, \dots, LS_{p_m} - D_{jp_m}\right\}$$

The backward pass is complete with $LS_1 = 0$ at node 1.

Based on the preceding computations, the activity (i, j) will be ***critical*** if it satisfies three conditions:

1- $LS_i = ES_i$
2- $LS_j = ES_j$
3- $LS_j - ES_i = D_{ij}$ (or equivalently: $LS_j - LS_i = ES_j - ES_i = D_{ij}$)

Example (3.5):

Determine the finishing time and the critical path for the following project network. All the durations are in days.

Solution:

Forward pass

Node 1: Let $ES_1 = 0$

Node 2: $ES_2 = ES_1 + D_{12} = 0 + 5 = 5$

Node 3: $ES_3 = max\ \{ES_1 + D_{13},\ ES_2 + D_{23}\} = max\{0 + 6, 5 + 3\} = 8$

Node 4: $ES_4 = ES_2 + D_{24} = 5 + 8 = 13$

Node 5: $ES_5 = max\ \{ES_3 + D_{35},\ ES_4 + D_{45}\} = max\{8 + 2, 13 + 0\} = 13$

Node 6: $ES_6 = max\ \{ES_3 + D_{36},\ ES_4 + D_{46}, ES_5 + D_{56}\} = max\{8 + 11,13 + 1, 13 + 12\} = 25$

The finishing time of the project is 25 days.

Backward pass

Node 6: Let $LS_6 = ES_6 = 25$

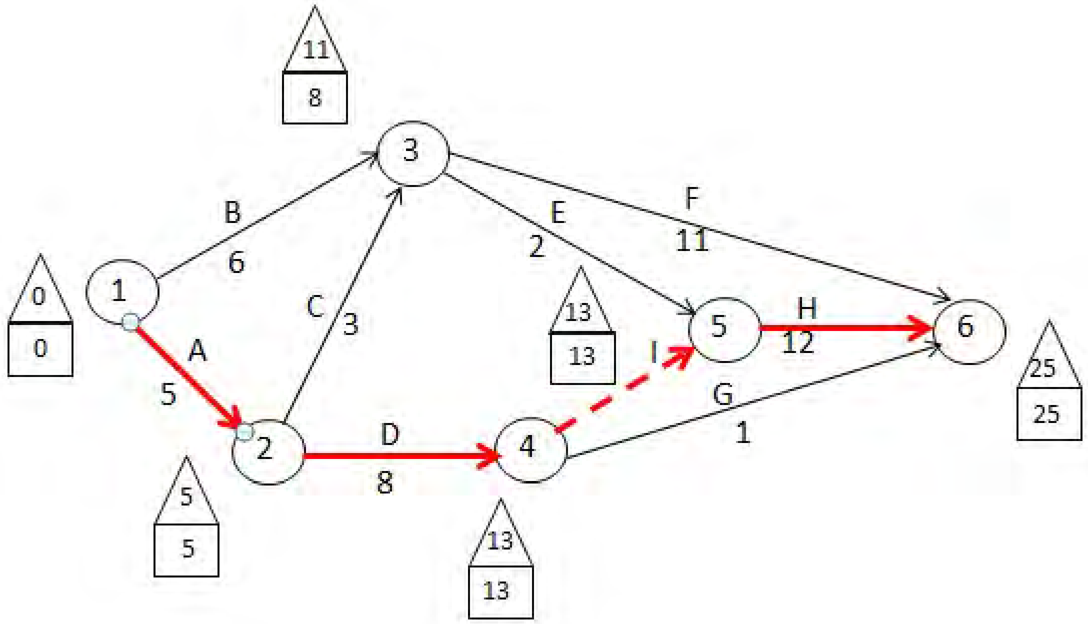
Node 5: $LS_5 = LS_6 - D_{56} = 25 - 12 = 13$

Node 4: $LS_4 = min\ \{LS_5 - D_{45},\ LS_6 - D_{46}\} = min\{13 - 0, 25 - 1\} = 13$

Node 3: $LS_3 = min\ \{LS_5 - D_{35},\ LS_6 - D_{36}\} = min\{13 - 2, 25 - 11\} = 11$

Node 2: $LS_2 = min\ \{LS_3 - D_{23},\ LS_4 - D_{24}\} = min\{11 - 3, 13 - 8\} = 5$

Node 1: $LS_1 = min\ \{LS_2 - D_{12},\ LS_3 - D_{13}\} = min\{5 - 5, 11 - 6\} = 0$



Then the critical activities are A, D, I, and H (or equivalently: (1,2), (2,4), (4,5), and (5,6)) and the critical path is: 1-2-4-5-6.

Example (3.6):

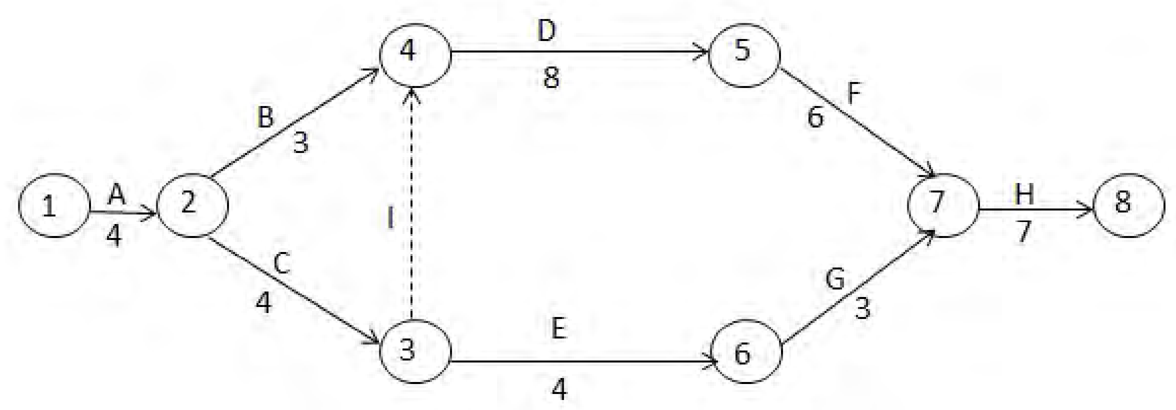
A project to produce radios requires the following activities according to times marked by each of them.

Activity	Description	Preceding activity	Duration/Days
A	Study the desired marketing specifications	--	4
B	Develop designs and geometric shapes	A	3
C	Provide the machinery and basic supplies for the production	A	4
D	Provide the manpower needed for the production	B, C	8
E	The organization of production lines within the plant	C	4
F	Training of workers on manufacturing processes	D	6
G	Provide the secondary supplies for the production	E	3
H	Production	G, F	7

Determine the finishing time and the critical path for the following project network. All the durations are in days.

Solution:

The project network is:



Forward pass

Node 1: Let $ES_1 = 0$

Node 2: $ES_2 = ES_1 + D_{12} = 0 + 4 = 4$

Node 3: $ES_3 = ES_2 + D_{23} = 4 + 4 = 8$

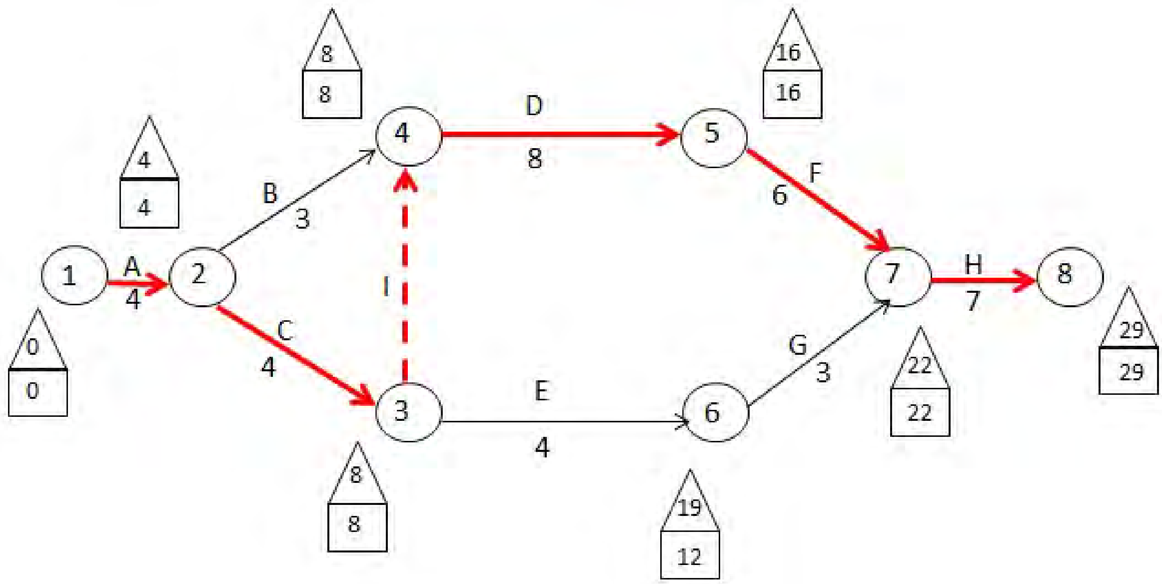
Node 4: $ES_4 = max\ \{ES_2 + D_{24},\ ES_3 + D_{34}\} = max\{4 + 3, 8 + 0\} = 8$

Node 5: $ES_5 = ES_4 + D_{45} = 8 + 8 = 16$

Node 6: $ES_6 = ES_3 + D_{36} = 8 + 4 = 12$
Node 7: $ES_7 = max\ \{ES_5 + D_{57},\ ES_6 + D_{67}\} = max\{16 + 6, 12 + 3\} = 22$
Node 8: $ES_8 = ES_7 + D_{78} = 22 + 7 = 29$
The finishing time of the project is 29 days.

Backward pass

Node 8: Let $LS_8 = ES_8 = 29$
Node 7: $LS_7 = LS_8 - D_{78} = 29 - 7 = 22$
Node 6: $LS_6 = LS_7 - D_{67} = 22 - 3 = 19$
Node 5: $LS_5 = LS_7 - D_{57} = 22 - 6 = 16$
Node 4: $LS_4 = LS_5 - D_{45} = 16 - 8 = 8$
Node 3: $LS_3 = min\ \{LS_4 - D_{34},\ LS_6 - D_{36}\} = min\{8 - 0, 19 - 4\} = 8$
Node 2: $LS_2 = min\ \{LS_3 - D_{23},\ LS_4 - D_{24}\} = min\{8 - 3, 8 - 4\} = 4$
Node 1: $LS_1 = LS_2 - D_{12} = 4 - 4 = 0$

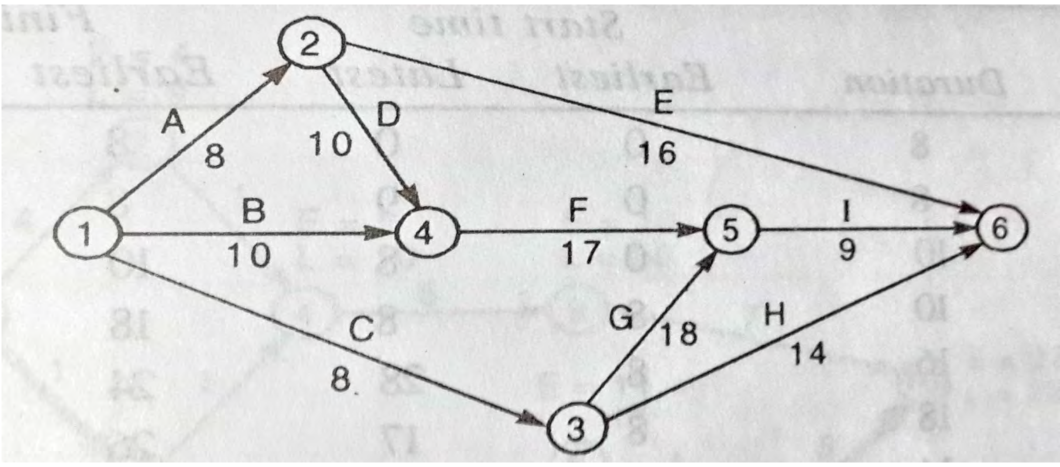


Then the critical activities are A, C, I, D, F and H (or equivalently: (1,2), (2,3), (3,4), (4,5),(5,7) and (7,8)) and the critical path is: 1-2-3-4-5-7-8.

Exercise 3.1 (in addition to text book exercises)

Determine the finishing time and the critical path for each of the following project networks.

1- Duration in days.



2- The R and D department is planning to bid on a large project for the development of a new communication system for commercial planes. The accompanying table shows the activities, times and sequence required.

Activity	Immediate predecessor	Time / weeks
A	---	3
B	A	2
C	A	4
D	A	4
E	B	6
F	C , D	6
G	D, F	2
H	D	3
I	E, G, H	3

Draw the network diagram. Determine the finishing time and the critical path for the network.

3.4 Program Evaluation and Review Technique (PERT)

PERT differs from CPM in that it assumes probabilistic duration times based on three estimates:

The optimistic time, ***a***, which occurs when execution goes extremely well.

The most likely time, ***m***, which occurs when execution is done under normal conditions.

The pessimistic time, ***b***, which occurs when execution goes extremely poorly.

The most likely time, m, falls in the range (a, b). Based on the estimates, the average duration time, $\bar{D}$, and variance, $v = \sigma^2$, are approximated as:

$$\bar{D} = \frac{a + 4m + b}{6}$$

$$\sigma^2 = v = \left(\frac{b-a}{6}\right)^2$$

CPM calculations can be applied directly, with $\overline{D}$, replacing the single estimate *D*. To find the probability of completing the project in time *S*, we calculate:

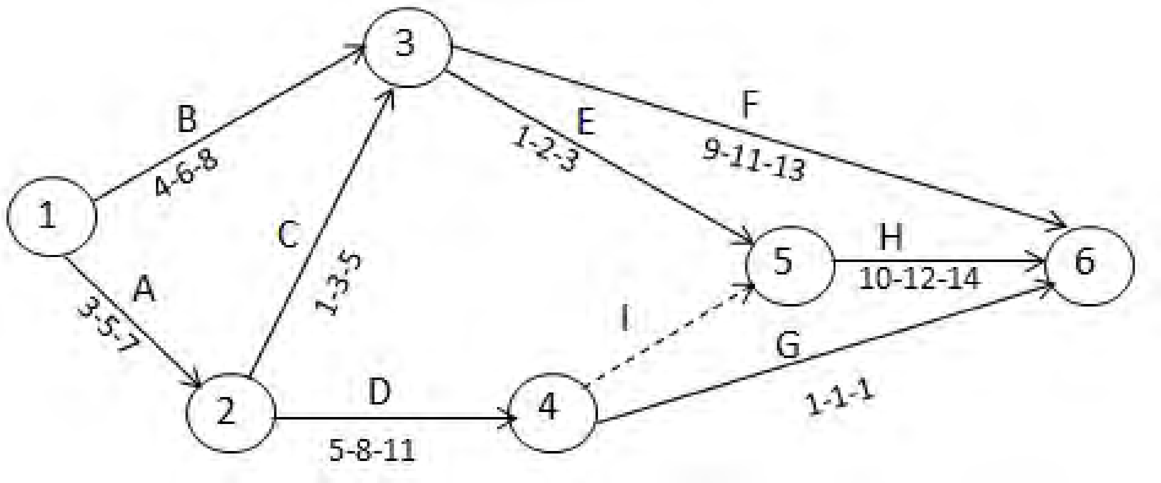
$$z = \frac{S - FT}{\sigma}$$

Where FT is the finishing time, $\sigma = \sqrt{\sum_{critical\ path} \sigma^2}$. The probability is then read from the standard normal probability distribution table for the value of z calculated above.

Example (3.7):

Determine for the following network:

1- The finishing time.
2- The critical path.
3- The probability that the project will be completed in a) $S_1 = 30, S_2 = 21,$ and $S_3 = 29$ days .



Solution:

We must calculate expected times as follows:

Activity	$\overline{D}_{ij}$
A or (1,2)	$\overline{D}_{12} = \frac{3+20+7}{6} = 5$
B or (1,3)	$\overline{D}_{13} = \frac{4+24+8}{6} = 6$
C or (2,3)	$\overline{D}_{23} = \frac{1+12+5}{6} = 3$
D or (2,4)	$\overline{D}_{24} = \frac{5+32+11}{6} = 8$

E or (3,5)	$\bar{D}_{35} = \frac{1+8+3}{6} = 2$
F or (3,6)	$\bar{D}_{36} = \frac{9+44+13}{6} = 11$
I or (4,5)	$\bar{D}_{45} = \frac{0+0+0}{6} = 0$
G or (4,6)	$\bar{D}_{46} = \frac{1+4+1}{6} = 1$
H or (5,6)	$\bar{D}_{56} = \frac{10+48+14}{6} = 12$

Forward pass

Node 1: Let $ES_1 = 0$

Node 2: $ES_2 = ES_1 + \bar{D}_{12} = 0 + 5 = 5$

Node 3: $ES_3 = max\ \{ES_1 + \bar{D}_{13},\ ES_2 + \bar{D}_{23}\} = max\{0 + 6, 5 + 3\} = 8$

Node 4: $ES_4 = ES_2 + \bar{D}_{24} = 5 + 8 = 13$

Node 5: $ES_5 = max\ \{ES_3 + \bar{D}_{35},\ ES_4 + \bar{D}_{45}\} = max\{8 + 2, 13 + 0\} = 13$

Node 6: $ES_6 = max\ \{ES_3 + \bar{D}_{36},\ ES_4 + \bar{D}_{46}, ES_5 + \bar{D}_{56}\} = max\{8 + 11, 13 + 1, 13 + 12\} = 25$

1- The finishing time of the project is 25 days.

Backward pass

Node 6: Let $LS_6 = ES_6 = 25$

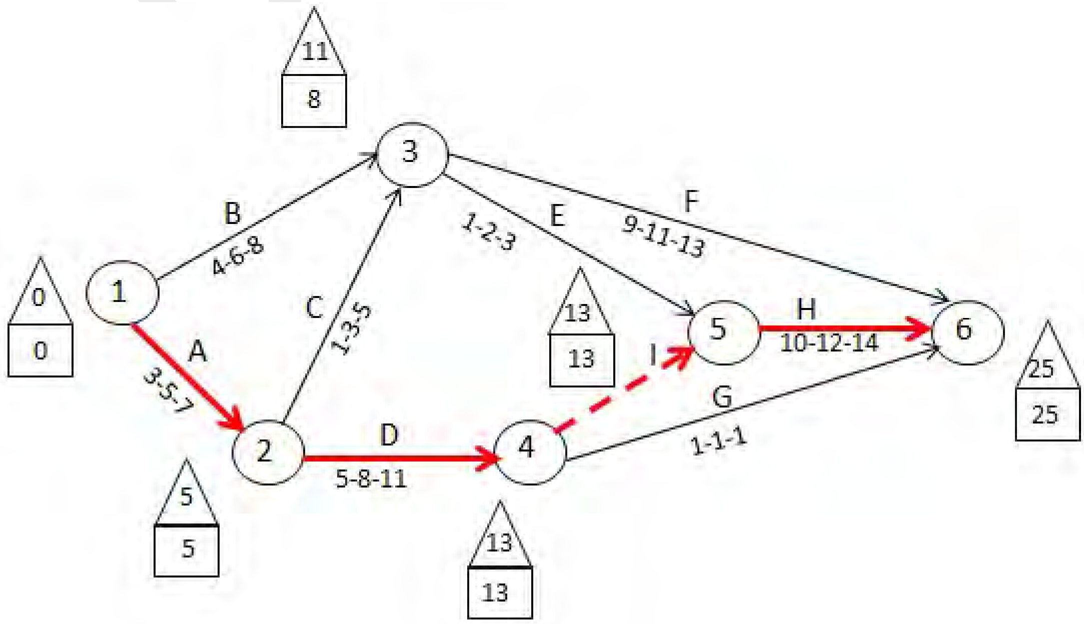
Node 5: $LS_5 = LS_6 - \bar{D}_{56} = 25 - 12 = 13$

Node 4: $LS_4 = min\ \{LS_5 - \bar{D}_{45},\ LS_6 - \bar{D}_{46}\} = min\{13 - 0, 25 - 1\} = 13$

Node 3: $LS_3 = min\ \{LS_5 - \bar{D}_{35},\ LS_6 - \bar{D}_{36}\} = min\{13 - 2, 25 - 11\} = 11$

Node 2: $LS_2 = min\ \{LS_3 - \bar{D}_{23},\ LS_4 - \bar{D}_{24}\} = min\{11 - 3, 13 - 8\} = 5$

Node 1: $LS_1 = min\ \{LS_2 - \bar{D}_{12},\ LS_3 - \bar{D}_{13}\} = min\{5 - 5, 11 - 6\} = 0$



2- Then the critical activities are A, D, I, and H (or equivalently: (1,2), (2,4), (4,5), and (5,6)) and the critical path is: 1-2-4-5-6.

3- To calculate the variance for critical activities:

Activity	$\sigma_{ij}^2 = v_{ij}$
A or (1,2)	$\sigma_{12}^2 = \left(\frac{7-3}{6}\right)^2 = 0.444$
D or (2,4)	$\sigma_{24}^2 = \left(\frac{11-5}{6}\right)^2 = 1$
I or (4,5)	$\sigma_{46}^2 = \left(\frac{0-0}{6}\right)^2 = 0$
H or (5,6)	$\sigma_{56}^2 = \left(\frac{14-10}{6}\right)^2 = 0.444$

$\sigma = \sqrt{\sigma_{12}^2 + \sigma_{24}^2 + \sigma_{45}^2 + \sigma_{56}^2} = \sqrt{0.444 + 1 + 0 + 0.444} = \sqrt{1.888} = 1.37$

$z_1 = \frac{S_1 - FT}{\sigma} = \frac{30-25}{1.37} = 3.65 \implies p(z_1 \leq 30) = 0.9999 = 99.99\%$

$z_2 = \frac{S_2 - FT}{\sigma} = \frac{21-25}{1.37} = -2.92 \implies p(z_2 \leq 21) = 1 - 0.9983 = 0.0017 = 0.17\%$

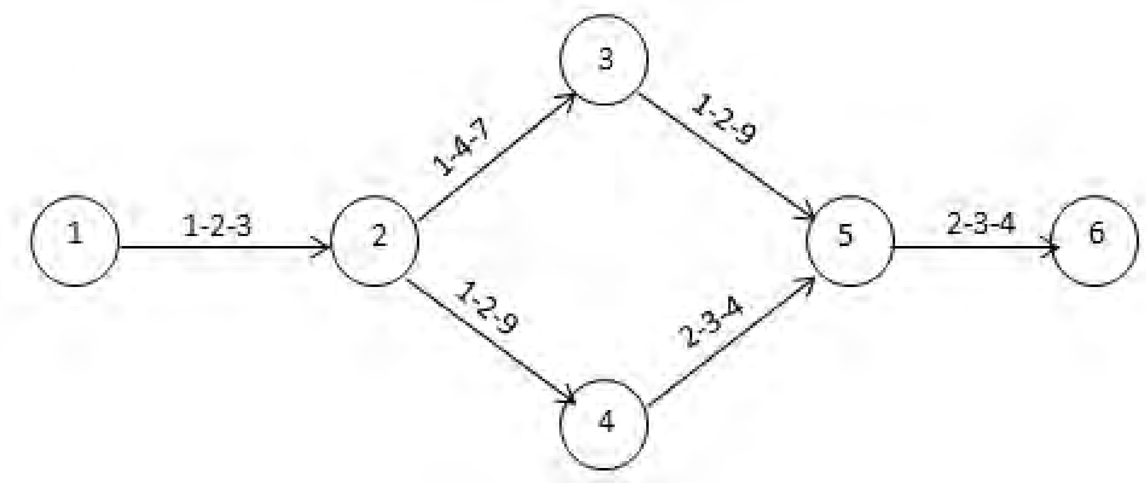
$z_3 = \frac{S_3 - FT}{\sigma} = \frac{29-25}{1.37} = 2.92 \implies p(z_3 \leq 29) = 0.9983 = 99.83\%$

$[p(z \leq -g) = p(z \geq g) = 1 - p(z \leq g)]$

Example (3.8):

Determine for the following network:

1- The finishing time.
2- The critical path.
3- The probability that the project will be completed in a) $S_1 = 12, S_2 = 14$, and $S_3 = 10$ days .



Solution:

We must calculate expected times as follows:

Activity	$\bar{D}_{ij}$
(1,2)	$\bar{D}_{12} = \frac{1+8+3}{6} = 2$
(2,3)	$\bar{D}_{23} = \frac{1+16+7}{6} = 4$
(2,4)	$\bar{D}_{24} = \frac{1+8+9}{6} = 3$
(3,5)	$\bar{D}_{35} = \frac{1+8+9}{6} = 3$
(4,5)	$\bar{D}_{45} = \frac{2+12+4}{6} = 3$
(5,6)	$\bar{D}_{56} = \frac{2+12+4}{6} = 3$

Forward pass

Node 1: Let $ES_1 = 0$

Node 2: $ES_2 = ES_1 + \bar{D}_{12} = 0 + 2 = 2$

Node 3: $ES_3 = ES_2 + \bar{D}_{23} = 2 + 4 = 6$

Node 4: $ES_4 = ES_2 + \bar{D}_{24} = 2 + 3 = 5$

Node 5: $ES_5 = max\ \{ES_3 + \bar{D}_{35},\ ES_4 + \bar{D}_{45}\} = max\{6 + 3, 5 + 3\} = 9$

Node 6: $ES_6 = ES_5 + \bar{D}_{56} = 9 + 3 = 12$

1- The finishing time of the project is 12 days.

Backward pass

Node 6: Let $LS_6 = ES_6 = 12$

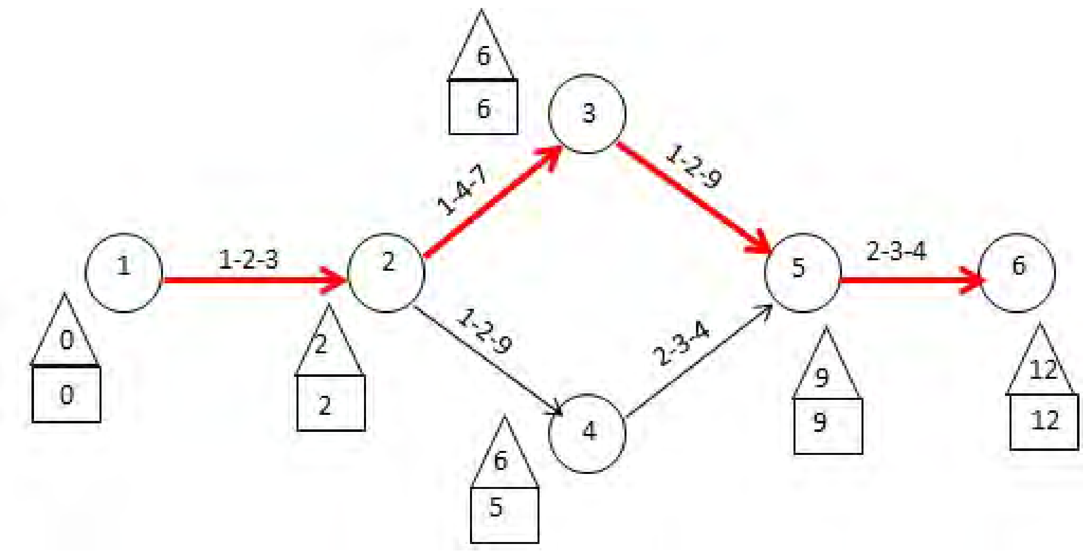
Node 5: $LS_5 = LS_6 - \bar{D}_{56} = 12 - 3 = 9$

Node 4: $LS_4 = LS_5 - \bar{D}_{45} = 9 - 3 = 6$

Node 3: $LS_3 = LS_5 - \bar{D}_{35} = 9 - 3 = 6$

Node 2: $LS_2 = min\ \{LS_3 - \bar{D}_{23},\ LS_4 - \bar{D}_{24}\} = min\{6 - 4, 6 - 3\} = 2$

Node 1: $LS_1 = LS_2 - \bar{D}_{12} = 2 - 2 = 0$



2- Then the critical activities are (1,2), (2,3), (3,5), and (5,6) and the critical path is: 1-2-3-5-6.

3- To calculate the variance for critical activities:

Activity	$\sigma_{ij}^2 = v_{ij}$
(1,2)	$\sigma_{12}^2 = \left(\frac{3-1}{6}\right)^2 = 1/9$
(2,3)	$\sigma_{23}^2 = \left(\frac{7-1}{6}\right)^2 = 1$
(3,5)	$\sigma_{35}^2 = \left(\frac{9-1}{6}\right)^2 = 16/9$
(5,6)	$\sigma_{56}^2 = \left(\frac{4-2}{6}\right)^2 = 1/9$

$$\sigma = \sqrt{\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{35}^2 + \sigma_{56}^2} = \sqrt{\frac{1}{9} + 1 + \frac{16}{9} + \frac{1}{9}} = 1.73$$

$$z_1 = \frac{S_1 - FT}{\sigma} = \frac{12-12}{1.73} = 0 \Longrightarrow p(z_1 \le 12) = 0.5000 = 50\%$$

$$z_2 = \frac{S_2 - FT}{\sigma} = \frac{14-12}{1.73} = 1.16 \Longrightarrow p(z_2 \le 14) = 0.877 = 87.7\%$$

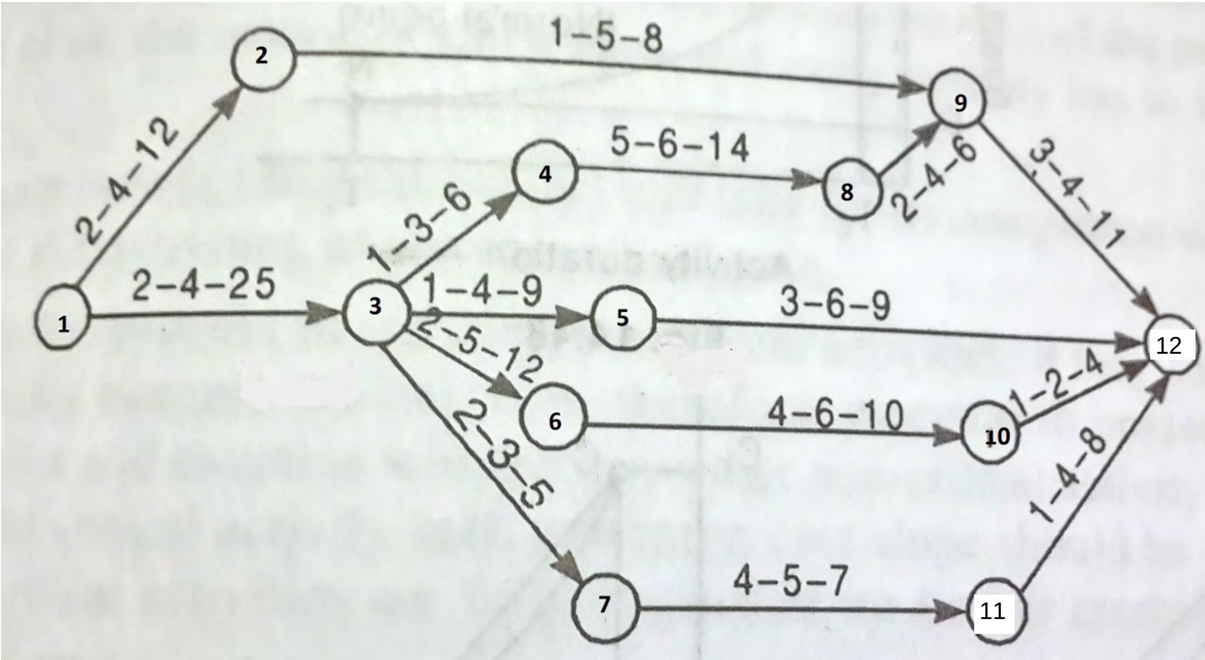
$$z_3 = \frac{S_3 - FT}{\sigma} = \frac{10-12}{1.73} = -1.16 \Longrightarrow p(z_3 \le 10) = 1 - 0.877 = 0.123 = 12.3\%$$

Exercise 3.2 (in addition to text book exercises)

Determine for the following network:

1- The finishing time.
2- The critical path.

3- The probability that the project will be completed in a) $S_1 = 32, S_2 = 27$, and $S_3 = 20$ days .



STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3.0	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.99900
3.1	.99903	.99906	.99910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.99940	.99942	.99944	.99946	.99948	.99950
3.3	.99952	.99953	.99955	.99957	.99958	.99960	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.99990	.99990	.99990	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997