Ch.3: Advanced Topics in Linear Programming

3.1 Special Cases in Linear Programming

There are some special cases that arise in the application.

3.1.1 Tie in the Choice of the Entering Variable

The non-basic variable that enters the basis is the one that gives the largest per unit improvement in the objective function. They are variable having minimum (maximum) negative (positive) value in a maximization (minimization) problem in Z-row is the entering variable. A tie in the choice of entering variable exists when more than one variable has the same largest negative (positive) value. To break this tie, select any one of them arbitrarily as the entering variable. There is no method to predict which of them is better. If there is a tie between a decision variable and a slack/surplus variable, select the decision variable.

3.1.2 Unbounded Solution

In some LP models, the values of the variables may be increased indefinitely without violating any of the constraint- meaning that the solution space is unbounded in at least one variable. As a result, the objective function value may increase (maximization case) or decrease (minimization case) indefinitely. In this case, both the solution space and the optimum objective value are unbounded. In simplex technique, this happens when all the constraint coefficients of the non-basic variable that is to enter the basis are negative or zero so that there is no minimum in the non-negative ratio. That it is not possible to determine the basic variable that should leave the basis.

Example (3.1):

Discuss the following LPP:

$$\begin{aligned} max \quad & Z = x_1 + 2x_2 \\ S.t \quad & x_1 - x_2 \leq 10 \\ & 2x_1 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution:

The standard form of the LPP (with modification of the objective function) is:

$$\begin{aligned} max \quad & Z - x_1 - 2x_2 = 0 \\ S.t \quad & x_1 - x_2 + s_1 = 10 \\ & 2x_1 + s_2 = 40 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$

Let $x_1 = x_2 = 0$, then $s_1 = 10$ and $s_2 = 40, Z = 0$. The simplex table is:

B.V.	x_1	x_2	s_1	s_2	Solution
s_1	1	-1	1	0	10
s_2	2	0	0	1	40
Z	-1	-2	0	0	0

The current solution is not optimal. In the starting table, the x_2-column is the pivot column. But, all the constraint coefficients under the x_2 are negative or zero. This means that there is no leaving variable and that x_2 can increase indefinitely without violating any of the constraints. Because each unit increase in x_2 will increase Z by 2, an infinite increase in x_2 leads to an infinite increase in Z. Thus, the problem has no bounded solution. We can see this graphically:

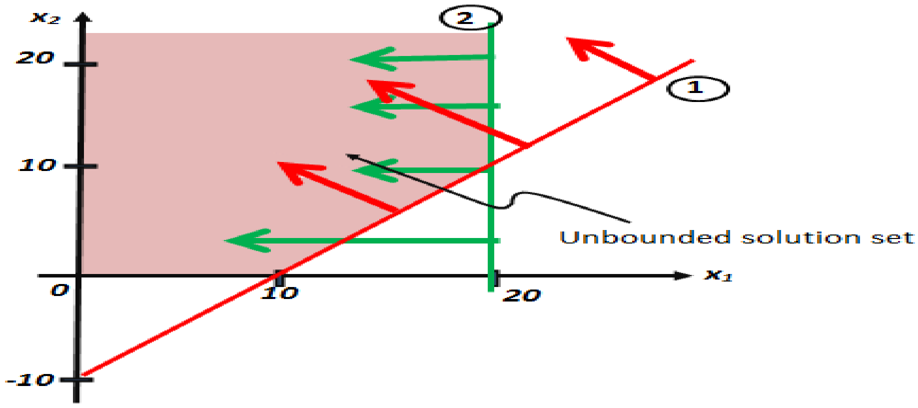

Figure (3.1)

3.1.3 Alternative Optima

This happens when there are multiple optimal solutions. Graphically, this happens when the objective function is parallel to a non-redundant constraint. In the optimal simplex table, if a non-basic variable has zero coefficients in the Z-row, there exists an alternate optimal solution. It is because that non-basic variable can enter the basis without changing the value of Z, but causing a change in the value of the basic variables. These variables may be a decision or slack or surplus variable.

Example (3.2) (Infinite number of solutions):

Discuss the following LPP:

$$max \quad Z = 2x_1 + 4x_2$$

$$S.t \quad x_1 + 2x_2 \le 5$$

$$x_1 + x_2 \le 4$$

$$x_1, x_2 \ge 0$$

Solution:

The standard form of the LPP (with modification of the objective function) is:

$max \quad Z - 2x_1 - 4x_2 = 0$

$S.t \quad x_1 + 2x_2 + s_1 = 5$

$\quad x_1 + x_2 + s_2 = 4$

$\quad x_1, x_2, s_1, s_2 \geq 0$

Let $x_1 = x_2 = 0$, then $s_1 = 5$ and $s_2 = 4, Z = 0$. The simplex table is:

B.V.	x_1	x_2	s_1	s_2	Solution	
s_1	1	2	1	0	5	5/2=2.5
s_2	1	1	0	1	4	4/1=4
Z	−2	−4	0	0	0	
x_2	½	1	½	0	5/2	
s_2	½	0	−1/2	1	3/2	
Z	0	0	2	0	10	
x_2	0	1	1	−1	1	
x_1	1	0	−1	2	3	
Z	0	0	2	0	10	

The first iteration gives the optimum solution $x_1 = 0, x_2 = 5/2$, and $Z_{max} = 10$, which coincides with point B in the graphical representation of the problem. The coefficient of the non-basic variable x_1 in the Z-equation is zero, indicating that x_1 can enter the basic solution without changing the value of Z, but causing a change in the values of variables. In second iteration: $x_1 = 3, x_2 = 1$, and $Z_{max} = 10$. This solution occurs at the corner point C (3, 1). Any point in the line segment BC represents an alternative optimum with $Z_{max} = 10$. The simplex method determines only the two corners B and C. Mathematically; we can determine all the points (x_1, x_2) on the line segment BC as a non-negative weighted average of the points B and C. Thus given

$B: x_1 = 0, x_2 = 5/2 \quad and \quad C: x_1 = 3, x_2 = 1$

Then all the points on the line segment BC are given by:

$$\left.\begin{aligned} \widehat{x_1} &= \alpha(0) + (1-\alpha)(3) = 3 - 3\alpha \\ \widehat{x_2} &= \alpha\left(\frac{5}{2}\right) + (1-\alpha)(1) = 1 + \frac{3}{2}\alpha \end{aligned}\right\} \quad 0 \leq \alpha \leq 1$$

When $\alpha = 0, (\widehat{x_1}, \widehat{x_2}) = (3,1)$ which is the point C. When $\alpha = 1, (\widehat{x_1}, \widehat{x_2}) = (0,5/2)$ which is the point B. For values of $\alpha (0 \leq \alpha \leq 1)$, $(\widehat{x_1}, \widehat{x_2})$ lies between B and C.

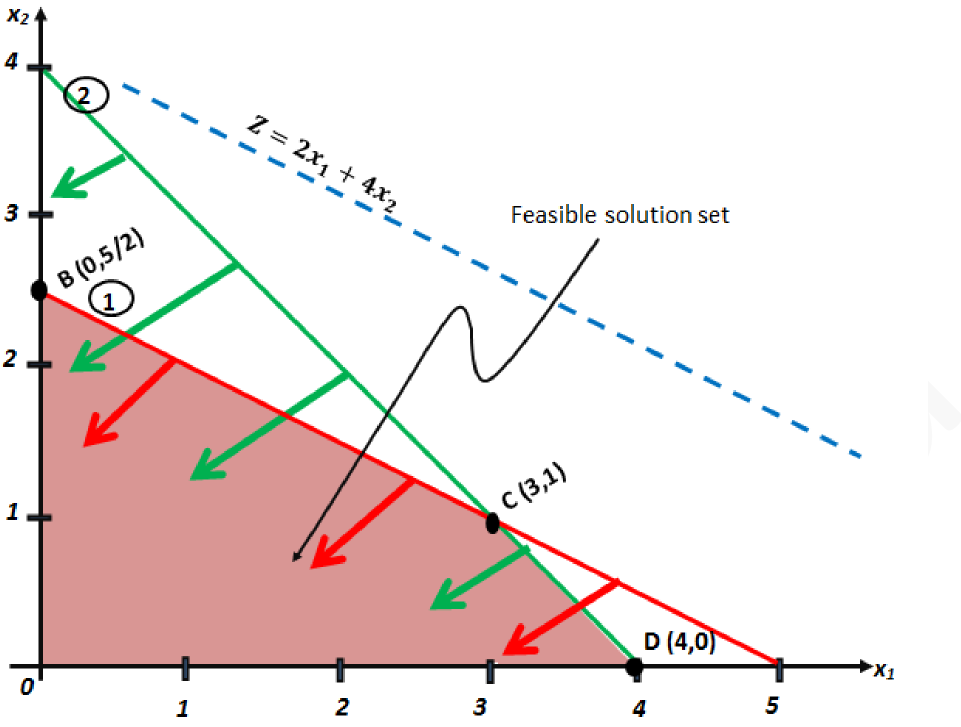

Figure (3.2)

3.1.4 No Feasible Solution (Infeasible Solution)

In this case, there is no feasible solution in LPP that satisfies all the constraints and non-negativity restrictions. It means that the constraints in the problem are conflicting and inconsistent. As an example, see examples (2.18) and (2.19).

Example (3.3):

Discuss the following LPP:

$$\begin{aligned} max \quad & Z = 3x_1 + 2x_2 \\ S.t. \quad & -2\,x_1 + 3x_2 \leq 9 \\ & 3\,x_1 - 2x_2 \leq -20 \\ & x_1\,, x_2 \geq 0 \end{aligned}$$

Solution:

The standard form of the LPP (with modification in Z-equation) is:

$$\begin{aligned} max \quad & Z + (-3 + 3M)x_1 + (-2 - 2M)x_2 + Ms_2 = -20M \\ S.t. \quad & -2\,x_1 + 3x_2 + s_1 = 9 \\ & -3\,x_1 + 2x_2 - s_2 + R_1 = 20 \\ & x_1\,, x_2, s_1,\ s_2,\ R_1 \geq 0 \end{aligned}$$

Let $x_1 = x_2 = s_2 = 0$, then $s_1 = 9, R_1 = 20$, and $Z = -20M$. The simplex iteration of the LP model is:

B.V.	x_1	x_2	s_1	s_2	R_1	Solution	
s_1	-2	$\boxed{3}$	1	0	0	9	9/3=3
R_1	-3	2	0	-1	1	20	20/2=10
Z	$-3+3M$	$-2-2M$	0	M	0	$-20M$	
x_2	$-2/3$	1	1/3	0	0	3	
R_1	$-5/3$	0	$-2/3$	-1	1	14	
Z	$-\frac{13}{3}+\frac{5}{3}M$	0	$\frac{2}{3}+\frac{5}{3}M$	M	0	6+4M	

Optimum iteration shows that the artificial variable R_1 is positive, which indicates that the problem is infeasible. The result is what we may call a ***pseudo-optimal solution***. The graphic representation of the problem shows clearly that the absence of feasible solution.

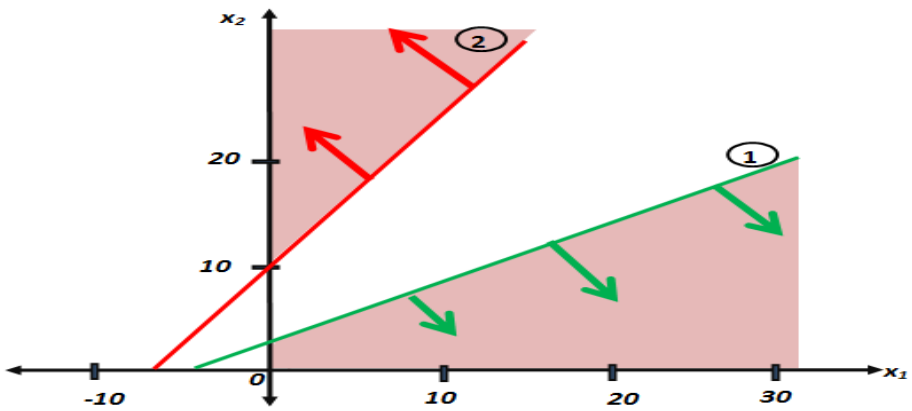

Figure (3.3)

3.1.5 Degeneracy (Tie in the Choice of the Leaving Variable)

Degeneracy in Linear Programming is said to occur when one or more basic variables have zero value. If the minimum ratio is zero for two or more basic variables, degeneracy may result and the simplex routine will cycle indefinitely. That is, the solution which we have obtained in one iteration may repeat after few iterations and therefore no optimum solution may be obtained. This concept is known as ***cycling*** or ***circling***.

To resolve degeneracy, we follow the following method which is called the ***perturbation method*** by A. Charnes:

1. Divide each element in the tied rows by positive coefficients of the pivot (key) column in that row.

2. Compare the resulting ratio, column by column, first in the identity and then in the body of the simplex table, from left to right.
3. The row which first contains the smallest algebraic ratio contains the outgoing variable. The simplex method is then continued to reach the optimal solution.

If any artificial variable is one of the tied variables, it should be immediately selected to leave the basis without following the above rules.

Example (3.4):

Discuss the following LPP:

$$\begin{aligned} max \quad & Z = 2x_1 + x_2 \\ S.t. \quad & 4x_1 + 3x_2 \leq 12 \\ & 4x_1 + x_2 \leq 8 \\ & 4x_1 - x_2 \leq 8 \\ & x_1 , x_2 \geq 0 \end{aligned}$$

Solution:

The standard form of the LPP (with modification in the Z-equation) is:

$$\begin{aligned} max \quad & Z - 2x_1 - x_2 = 0 \\ S.t. \quad & 4x_1 + 3x_2 + s_1 = 12 \\ & 4x_1 + x_2 + s_2 = 8 \\ & 4x_1 - x_2 + s_3 = 8 \\ & x_1 , x_2, s_1, s_2, s_3 \geq 0 \end{aligned}$$

Let $x_1 = x_2 = 0$, then: $s_1 = 12, s_2 = 8, s_3 = 8,$ and $Z = 0$.

B.V.	x_1	x_2	s_1	s_2	s_3	Solution	
s_1	4	3	1	0	0	12	12/4=3
s_2	4	1	0	1	0	8	8/4=2
s_3	$\boxed{4}$	−1	0	0	1	8	8/4=2
Z	−2	−1	0	0	0	0	

In the above table x_1 is the entering variable, as s_2 and s_3 are the tied rows, perturbation method is used to determine the outgoing variable. The first column of the identity has the elements 0 and 0 in the tied rows. Dividing them by the corresponding elements of the key column, the resulting ratios are 0 and 0. Hence first column of the identity fails to identify the outgoing variable. The second column of the identity has the elements 1 and 0 in the tied rows. Dividing them by the corresponding elements of the key column, the resulting ratios are 1/4 and 0. As s_3-row yields the smaller ratio, the s_3 is the leaving variable.

Performing iterations to get the optimal solution results in the following tables:

B.V.	x_1	x_2	s_1	s_2	s_3	Solution	
s_1	0	4	1	0	-1	4	4/4=1
s_2	0	2	0	1	-1	0	0/2=0
x_1	1	$-1/4$	0	0	1/4	2	
Z	0	$-3/2$	0	0	1/2	4	
s_1	0	0	1	-2	1	4	4/1=4
x_2	0	1	0	1/2	$-1/2$	0	
x_1	1	0	0	1/8	1/8	2	2/(1/8)=16
Z	0	0	0	3/4	$-1/4$	4	
s_3	0	0	1	-2	1	4	
x_2	0	1	1/2	$-1/2$	0	2	
x_1	1	0	$-1/8$	3/8	0	3/2	
Z	0	0	1/4	1/4	0	5	

Then the optimal solution is: $x_1 = \frac{3}{2}, x_2 = 2, and\ Z_{max} = 5$.

Exercises 3.1 (In addition to the text book exercises)

Discuss the following LPP's:

1. $min \quad Z = 6x_1 + 10x_2$

$S.t. \quad 6x_1 + 10x_2 \geq 30$

$x_1 \leq 4$

$x_2 \leq 2$

$x_1 , x_2 \geq 0$

2. $max \quad Z = 16x_1 + 2x_2$

$S.t. \quad 16x_1 + 2x_2 \leq 16$

$4x_1 + 2x_2 \leq 12$

$6x_1 + 2x_2 \leq 12$

$2\, x_1 + 12x_2 \leq 16$

$x_1 , x_2 \geq 0$

3. $max \quad Z = 12x_1 + 15x_2$

$S.t. \quad 3x_1 + 3x_2 \geq 3$

$-6x_1 + 3x_2 \leq 3$

$12x_1 - 3x_2 \leq 3$

$$x_1, x_2 \geq 0$$

4. $max \quad Z = 8x_1 + 105x_2 + 4x_3$

$S.t. \quad 4x_1 + 2x_2 + 2x_3 \leq 20$

$$2x_1 + 6x_2 + 2x_3 \leq 24$$

$$2x_1 + 2x_2 + 2x_3 = 12$$

$$x_1, x_2, x_3 \geq 0$$

5. $max \quad Z = 3x_1 - 6x_2 - 9x_3$

$S.t. \quad 6x_1 + 3x_2 + 9x_3 = 6$

$$6x_1 + 9x_2 + 12x_3 = 3$$

$$x_1, x_2, x_3 \geq 0$$

3.2 Sensitivity Analysis

In LPP, the parameters (input data) of the model can change within certain limits without causing the optimum solution to change. This is referred to as ***sensitivity analysis***. In LPP models, the parameters are usually not exact; we can ascertain the impact of this uncertainty on the quality of the optimum solution. The changes in (discrete) parameters of an LPP include changes in the values of few $b_i's$ or c_j or a_{ij} or addition/deletion of some constraints/ variables. Generally, these parameter changes result in one of the following three cases:

1- The optimal solution remains unchanged, i.e., the basic variables and their values remain unchanged.

2- The basic variables remain unchanged but their values change.

3- The basic variables as well as their values are changed.

3.2.1 Cost Changes

We will, first, consider the changing a cost value by Δ in the original problem. If we are given the original problem and an optimal tableau and If we had done exactly the same calculations beginning with the modified problem, we would have had the same final tableau except that the corresponding cost entry would be Δ lower (this is because we do nothing but to add or subtract scalar multiples of Rows 1 through m to other rows; we never add or subtract Z-row to other rows).

Example (3.5):

Consider the LPP:

$max \quad Z = 3x + 2y$

$S.t. \quad x + y \leq 4$

$$2x + y \leq 6$$
$$x, y \geq 0$$

Suppose that the cost for x is changed to $3 + \Delta$ in the original formulation.

a) What are the limits of Δ so as the solution remains optimal?

b) If the objective function is changed to $max \quad Z = 3.5x + 2y$, what is the optimal solution of the problem?

c) If the objective function is changed to $max \quad Z = x + 2y$, what is the optimal solution of the problem?

Solution:

a) The standard form of the LPP (If the cost of x is changed from 3 to $3 + \Delta$ with modification in the objective function) is:

$$max \quad Z - (3 + \Delta)x - 2y = 0$$
$$S.t. \quad x + y + s_1 = 4$$
$$2x + y + s_2 = 6$$
$$x, y, s_1, s_2 \geq 0$$

Let $x = y = 0$, then $s_1 = 4, s_2 = 6$, and $Z = 0$. The simplex table is:

B.V.	x	y	s_1	s_2	b
s_1	1	1	1	0	4
s_2	2	1	0	1	6
Z	$\boxed{-3-\Delta}$	-2	0	0	0
s_1	0	1/2	1	$-1/2$	1
x	1	1/2	0	1/2	3
Z	0	$\boxed{-1/2+\Delta/2}$	0	$3/2+\Delta/2$	$9+3\Delta$
y	0	1	2	-1	2
x	1	0	-1	1	2
Z	0	0	$1-\Delta$	$1+\Delta$	$10+2\Delta$

The solution is optimal if the elements of the Z-row are all non-negative. This is true if: $1 - \Delta \geq 0$ and $1 + \Delta \geq 0$ which holds if $-1 \leq \Delta \leq 1 (\Delta \in [-1,1])$. For any Δ in that range, our previous basis (and variable values) is optimal. The objective changes to $10 + 2\Delta$.

The optimal solution for the original problem is: $x = 2, y = 2$, and $Z_{max} = 10$ and the optimal tableau is:

B.V.	x	y	s_1	s_2	b
y	0	1	2	-1	2
x	1	0	-1	1	2
Z	0	0	1	1	10

Note that the table has the same basic variables and the same variable values (except for Z) that the previous solution had.

b) The value of Δ is obtained from subtracting cost coefficients of x in new and old objective functions, thus: $\Delta = 3.5 - 3 = 0.5. 1 - \Delta = 0.5 > 0$ and $1 + \Delta = 1.5 > 0$ then the solution remains optimal. That is: $x = 2, y = 2, Z_{max} = 10 + 2\Delta = 11$

c) $\Delta = 1 - 3 = -2$, then $1 - \Delta = 3 > 0$, but $1 + \Delta = -1 < 0$, then the solution is no longer optimal. To find the optimal solution we use the optimal table, that is:

B.V.	x	y	s_1	s_2	b
y	0	1	2	-1	2
x	1	0	-1	1	2
Z	0	0	3	-1	6
y	1	1	1	0	4
s_2	1	0	-1	1	2
Z	1	0	2	0	8

Then the optimal solution is: $x = 0, y = 4, Z\_max = 8$

In the previous example, we changed the cost of a basic variable. The next example will show what happens when the cost of a non-basic variable changes.

Example (3.6):

Consider the LPP:

$$max \quad Z = 3x + 2y + 2.5w$$
$$S.t. \quad x + y + 2w \leq 4$$
$$2x + y + 2w \leq 6$$
$$x, y, w \geq 0$$

Suppose that the cost for w is changed to $2.5 + \Delta$ in the original formulation. What are the limits of Δ so as the solution remains optimal?

Solution:

The standard form of the LPP (If the cost of w is changed from 2.5 to $2.5+\Delta$ with modification in the objective function) is:

$$max \quad Z - 3x - 2y - (2.5+\Delta)w = 0$$
$$S.t. \quad x + y + 2w + s_1 = 4$$
$$2x + y + 2w + s_2 = 6$$
$$x, y, w, s_1, s_2 \geq 0$$

Let $x = y = w = 0$, then $s_1 = 4, s_2 = 6,$ and $Z = 0$. The simplex table is:

B.V.	x	y	w	s_1	s_2	b
s_1	1	1	2	1	0	4
s_2	2	1	2	0	1	6
Z	$\boxed{-3}$	-2	$-2.5-\Delta$	0	0	0
s_1	0	1/2	1	1	$-1/2$	1
x	1	1/2	1	0	1/2	3
Z	0	$\boxed{-1/2}$	$0.5-\Delta$	0	3/2	9
y	0	1	2	2	-1	2
x	1	0	0	-1	1	2
Z	0	0	$1.5-\Delta$	1	1	10

In this case, we already have a valid tableau. This will represent an optimal solution if $1.5-\Delta \geq 0$, so $\Delta \leq 1.5$. Any change in the objective coefficient of the non-basic variable will affect only its index row coefficient and not others. Notice that, the optimal tableau is:

B.V.	x	y	w	s_1	s_2	b
y	0	1	2	2	-1	2
x	1	0	0	-1	1	2
Z	0	0	1.5	1	1	10

3.2.2 Right Hand Side Changes

Example (3.7):

Consider the LPP:

$$max \quad Z = 4x + 5y$$
$$S.t. \quad 2x + 3y \leq 12$$
$$x + y \leq 5$$
$$x, y \geq 0$$

a) Suppose that the value of the right-hand-side of the first constraint from 12 to $12 + \Delta$. What are the limits of Δ so as the solution remains feasible?
b) If the value of the right-hand-side of the first constraint is changed to 11, does the solution remains feasible, what is the optimal solution?
c) If the value of the right-hand-side of the first constraint is changed to 25, what is optimal solution?

Solution:

a) The standard form of the LPP (with modification in the objective function and changing the right-hand-side of the first constraint to $12 + \Delta$) is:

$$\begin{aligned} max \quad & Z - 4x - 5y = 0 \\ S.t. \quad & 2x + 3y + s_1 = 12 + \Delta \\ & x + y + s_2 = 5 \\ & x, y, s_1, s_2 \geq 0 \end{aligned}$$

Let $x = y = 0,$ then $s_1 = 12 + \Delta$ and $s_2 = 5$. The simplex table is:

B.V.	x	y	s_1	s_2	b
s_1	2	3	1	0	$12 + \Delta$
s_2	1	1	0	1	5
Z	-4	-5	0	0	0
y	$2/3$	1	$1/3$	0	$4+ \Delta/3$
s_2	$1/3$	0	$-1/3$	1	$1- \Delta/3$
Z	$-2/3$	0	$5/3$	0	$20+ 5\Delta/3$
y	0	1	1	-2	$2+\Delta$
x	1	0	-1	3	$3-\Delta$
Z	0	0	1	2	$22+\Delta$

This represents an optimal tableau as long as the right-hand-side is all non-negative. In other words, Δ must be between -2 and 3 in order for the basis not to change (remains feasible). The optimal tableau is:

B.V.	x	y	s_1	s_2	b
y	0	1	1	-2	2
x	1	0	-1	3	3
Z	0	0	1	2	22

The optimal solution is: $x = 3, y = 2, and\ Z_{max} = 22$.

b) $\Delta = 11 - 12 = -1$, then $2 + \Delta = 1 > 0$ and $3 - \Delta = 4 > 0$, thus, the solution remains feasible. The optimal solution is: $x = 4, y = 1, and\ Z_{max} = 21$

c) $\Delta = 25 - 12 = 13$, then $2 + \Delta = 15 > 0$ and $3 - \Delta = -10 < 0$. Thus, the solution is no longer feasible to manage this case we use the optimal table as follows:

B.V.	x	y	s_1	s_2	b
y	0	1	1	-2	15
x	1	0	-1	3	-10
Z	0	0	1	2	35
y	1	1	0	1	5
s_1	-1	0	1	-3	10
Z	1	0	0	5	25

The optimal solution is: $x = 0, y = 5, and\ Z_{max} = 25$.

Exercises 3.2 (In addition to the text book exercises)

1. Consider the following LPP:

$$\begin{aligned} max\quad & Z = 3x_1 + 7x_2 + 4x_3 + 9x_4 \\ S.t.\quad & x_1 + 4x_2 + 5x_3 + 8x_4 \le 9 \\ & x_1 + 2x_2 + 6x_3 + 4x_4 \le 7 \\ & x_i \ge 0 \qquad i = 1,2,3,4 \end{aligned}$$

a) Solve this linear program using the simplex method.

b) What are the values of the variables in the optimal solution?

c) What is the optimal objective function value?

d) What would you estimate the objective function would change to if:

- ❖ We change the right-hand side of the first constraint to 10.
- ❖ We change the right-hand side of the second constraint to 6.5.

2. Solve the problem :

$$\begin{aligned} max\quad & Z = 45x_1 + 100x_2 + 30x_3 + 50x_4 \\ S.t.\quad & 7x_1 + 10x_2 + 4x_3 + 9x_4 \le 1200 \\ & 3x_1 + 40x_2 + x_3 + x_4 \le 800 \\ & x_i \ge 0 \qquad i = 1,2,3,4 \end{aligned}$$

Find the effect of:

a) Changing the cost coefficients c_1 and c_4 from 45 and 50 to 40 and 60 respectively.

b) Changing c_1 to 30 and c_2 to 90.

c) Changing c_3 from 30 to 24.

3.3 Integer Programming

Integer linear programming problems (***ILPP***) are linear programming problems with some or all the variables restricted to integer (discrete) values. When all the variables are constrained to be integers, it is called an ***all*** (***pure***) ***integer programming problem***. In case only some of the variables are restricted to have integer values, the problem is said to be a ***mixed integer programming problem***. The ILPP algorithms are based on exploiting the tremendous computational success of LPP. The strategy of these algorithms involves three steps:

Step 1: Relax the solution space of the ILPP by deleting the integer restriction on all integer variables. The result of the relaxation is a regular LPP.

Step 2: Solve the LPP, and identify its optimum.

Step 3: Starting from the optimum point, add special constraints that iteratively modify the LPP solution space in a manner that will eventually render an optimum extreme point satisfying the integer requirements.

Two general methods have been developed for generating the special constraints in step 3:

1. Cutting - plane method.
2. Branch - and – bound (B & B) method.

3.3.1 Gomory's Cutting Plane Method

This systematic procedure for solving pure ILPP was first suggested by R.E. Gomory (1929-) in 1958. Later, he extends the procedure to cover mixed ILPP. The method consists in first solving the ILPP as ordinary continuous LPP and then introducing additional constraints one after the other to cut (eliminate) certain parts of the solution space until an integral solution is obtained.

Definition (3.1):

For all real number x, the ***greatest integer function*** (denoted by $[\,x\,]$) returns the largest integer less than or equal to *x*. In other words, the greatest integer function rounds down a real number to the nearest integer. The number *x* can be written in the form $x = [x] + e$, where $0 \leq e \leq 1$.We call e the fractional part of x.

Example (3.8):

1) $[0.41] = 0$ since integers less than 0.41 are: $\dots, -2, -1, 0$ and the greatest one of them is 0. The number 0.41 can be written in the form: $0.41 = 0 + 0.41 = [0.41] + 0.41$.
2) $[-0.41] = -1$ since integers less than -0.41 are: $\dots, -3, -2, -1$ and the greatest one of them is -1. The number -0.41 can be written in the form: $-0.41 = -1 + 0.59 = [-0.41] + 0.59$
3) $[9.73] = 9$
4) $[-7.26] = -8$
5) $[3] = 3$
6) $[-5] = -5$

According to definition (3.1), the structural coefficients and the stipulations can be written as:

$a_{ij} = [a_{ij}] + f_{ij},\ b_i = [b_i] + f_i, where\ 0 \le f_{ij} \le 1\ and\ 0 \le f_i \le 1$;
$i = 1, \dots, m; j = 1, \dots, n$

The steps of Gomory's cutting plane method for pure ILPP are:

***Step 1:* Integerise the constraints**: Transform the constraints so that all the coefficients are whole numbers. For example, the constraint equation:
$\frac{7}{4}x_1 + \frac{1}{5}x_2 + \frac{3}{4}x_3 = \frac{17}{5}$ can be expressed as: $35x_1 + 4x_2 + 15x_3 = 68$.

***Step 2:* Solve the problem**: Ignoring integrality restriction, find the optimal solution to the problem. If the solution is all integers, it is an optimal basic feasible integer solution. If not, proceed to step 3. Ignore non-integer values for slack variables since they represent unused resources only.

Step 3:* Develop a cutting plane:** From the final table select the constraint with the largest fractional cut. The selected row is called the ***source row. In case of a tie, choose the constraint having the highest contribution (maximization problem) or the lowest cost (minimization problem). Alternatively select the constraint with $max\ \frac{f_i}{\sum_{j=1}^{n} f_{ij}}$ $\dots\dots(1)$. Construct the Gomory's constraint:

$s_i = \sum_{j=1}^{n} f_{ij} y_j - f_i$ .....(2) (y_j may be decision or slack variable)

And add it to the final table. Add an additional column for s_i also.

***Step 4:* Solve using the dual simplex method:** Solve the augmented ILPP obtained above by the dual simplex method so that the outgoing variable is s_i . If the optimal solution thus obtained has all integral values, it is an optimal solution for the given ILPP. If not, repeat step 3 until an optimal feasible integer solution is obtained.

Remark (3.1):

In mixed ILPP, only constraints corresponding to integer variables are used to construct the cut.

Example (3.9):

Find the optimal solution of the following ILPP

$$
\begin{aligned}
max \quad & Z = 5x_1 + 6x_2 \\
S.t \quad & 2x_1 + 3x_2 \leq 18 \\
& 2x_1 + x_2 \leq 12 \\
& x_1 + x_2 \leq 8 \\
& x_1, x_2 \geq 0, x_1 \text{ and } x_2 \text{ are integers}
\end{aligned}
$$

Solution:

The standard form of the LPP (with modification in the objective function and ignoring integrality condition) is:

$$
\begin{aligned}
max \quad & Z - 5x_1 - 6x_2 = 0 \\
S.t \quad & 2x_1 + 3x_2 + s_1 = 18 \\
& 2x_1 + x_2 + s_2 = 12 \\
& x_1 + x_2 + s_3 = 8 \\
& x_1, x_2, s_1, s_2, s_3 \geq 0
\end{aligned}
$$

Let $x_1 = x_2 = 0$, then $s_1 = 18, s_2 = 12, s_3 = 8$, and $Z = 0$ and

B.V.	x_1	x_2	s_1	s_2	s_3	b	
s_1	2	3	1	0	0	18	
s_2	2	1	0	1	0	12	
s_3	1	1	0	0	1	8	
Z	-5	$\boxed{-6}$	0	0	0	0	
x_2	2/3	1	1/3	0	0	6	
s_2	4/3	0	$-1/3$	1	0	6	
s_3	1/3	0	$-1/3$	0	1	2	
Z	$\boxed{-1}$	0	2	0	0	36	
x_2	0	1	1/2	$-1/2$	0	3	3
x_1	1	0	$-1/4$	3/4	0	9/2	$4\frac{1}{2}$
s_3	0	0	$-1/4$	$-1/4$	1	1/2	$\frac{1}{2}$

Z	0	0	7/4	3/4	0	81/2

The non-integer optimal solution is $x_1 = \frac{9}{2}, x_2 = 3,$ and $Z_{max} = 40\frac{1}{2}$. To construct Gomory's constraint, select x_1-row which has the greatest fractional part ½ (s_3-row also has the fractional part ½ , but a decision variable is preferred than slack variable).

$1.x_1 + 0.x_2 - \frac{1}{4}.s_1 + \frac{3}{4}s_2 + 0.s_3 = \frac{9}{2}$

Or $(1+0)x_1 + \left(-1+\frac{3}{4}\right)s_1 + \left(0+\frac{3}{4}\right)s_2 = 4 + \frac{1}{2}$

By using equation (2), the Gomory's constraint (cut) is:

$s_4 = \frac{3}{4}s_1 + \frac{3}{4}s_2 - \frac{1}{2} \quad \Longrightarrow \quad -\frac{3}{4}s_1 - \frac{3}{4}s_2 + s_4 = -\frac{1}{2}$

The modified table after inserting this equation becomes:

B.V.	x_1	x_2	s_1	s_2	s_3	s_4	**b**	
x_2	0	1	1/2	−1/2	0	0	3	
x_1	1	0	−1/4	3/4	0	0	9/2	
s_3	0	0	−1/4	−1/4	1	0	1/2	
s_4	0	0	−3/4	$\boxed{-3/4}$	0	1	−1/2	
Z	0	0	7/4	3/4	0	0	81/2	
x_2	0	1	1	0	0	−2/3	10/3	$3\frac{1}{3}$
x_1	1	0	−1	0	0	1	4	4
s_3	0	0	0	0	1	−1/3	2/3	$\frac{2}{3}$
s_2	0	0	1	1	0	−4/3	2/3	$\frac{2}{3}$
Z	0	0	1	0	0	1	40	

($\left|\frac{7/2}{-3/4}\right| = 4.7, \left|\frac{3/4}{-3/4}\right| = 1$) . x_2 has a non-integer value (10/3). Since the fractional part of s_2 and s_3 are equal (=2/3), then from equation (1):

$\frac{f_i}{\sum_{j=1}^{n} f_{ij}}$ for s_2 − equation $= \frac{2/3}{2/3} = 1$

$\frac{f_i}{\sum_{j=1}^{n} f_{ij}}$ for s_3 − equation $= \frac{2/3}{2/3} = 1$

Since both ratios are equal, we choose s_2 arbitrarily to construct second Gomory's cut as follows:

$1.s_1 + 1.s_2 - \frac{4}{3}s_4 = \frac{2}{3}$

Or $(1+0)s_1 + (1+0)s_2 + \left(-2+\frac{2}{3}\right)s_4 = 0 + \frac{2}{3}$

Then from equation (2):

$s_5 = \frac{2}{3}s_4 - \frac{2}{3} \quad \Rightarrow \quad -\frac{2}{3}s_4 + s_5 = -\frac{2}{3}$

The modified table after inserting this equation becomes

B.V.	x_1	x_2	s_1	s_2	s_3	s_4	s_5	b
x_2	0	1	1	0	0	$-2/3$	0	10/3
x_1	1	0	-1	0	0	1	0	4
s_3	0	0	0	0	1	$-1/3$	0	2/3
s_2	0	0	1	1	0	$-4/3$	0	2/3
s_5	0	0	0	0	0	$\boxed{-2/3}$	1	$-2/3$
Z	0	0	1	0	0	1	0	40
x_2	0	1	1	0	0	0	-1	4
x_1	1	0	-1	0	0	0	1/2	3
s_3	0	0	0	0	1	0	$-1/2$	1
s_2	0	0	1	1	0	0	-2	2
s_4	0	0	0	0	0	1	$-3/2$	1
Z	0	0	1	0	0	0	3/2	39

The optimal solution is: $x_1 = 3, x_2 = 4$, and $Z_{max} = 39$.

Example (3.10):

Discuss, graphically, the effect of the cuts in example (3.9) on the feasible solutions space.

Solution:

For $2x_1 + 3x_2 = 18 \Rightarrow$ if $x_1 = 0$ then (0, 6) is the intersection point with the $x_2 -$ axis.

And if $x_2 = 0$ then (9, 0) is the intersection point with the $x_1 -$ axis.

For $2x_1 + x_2 = 12 \Rightarrow$ if $x_1 = 0$ then (0, 12) is the intersection point with the $x_2 -$ axis.

And if $x_2 = 0$ then (6, 0) is the intersection point with the $x_1 -$ axis.

For $x_1 + x_2 = 8 \Rightarrow$ if $x_1 = 0$ then (0, 8) is the intersection point with the $x_2 -$ axis.

And if $x_2 = 0$ then (8, 0) is the intersection point with the $x_1 -$ axis.

The point B is resulting from the intersection of the lines representing the first and the second constraints, so we use these constraints to find the coordinates of B.

$$\cancel{2x_1} + 3x_2 = 18$$
$$\cancel{\mp 2x_1} \mp x_2 = \mp 12$$

$$\Rightarrow 2x_2 = 6 \Rightarrow x_2 = 3 \xRightarrow{(constr.2)} x_1 = \frac{12-3}{2} = 4.5$$

The feasible solution region space is the shaded area OABC in figure (3.4) whose corners are the points $O(0,0)$, $A(0,6)$, $B(4.5,3)$, and $C(6,0)$.

Corner	Value of Z
O(0,0)	Z=0
$A(0,6)$	$Z = 5 \times 0 + 6 \times 6 = 36$
$B(4.5,3)$	$Z = 5 \times 4.5 + 6 \times 3 = 40.5$ *
$C(6,0)$	$Z = 5 \times 6 + 6 \times 0 = 30$

From the table, we see that the greatest value of Z occurs at corner B, then $x_1 = 4.5$, $x_2 = 3$, and $Z_{max} = 40.5$.

The first cut is: $0 \le s_4 = \frac{3}{4}s_1 + \frac{3}{4}s_2 - \frac{1}{2}$ ...(*), that is:

$$-\frac{3}{4}s_1 - \frac{3}{4}s_2 \le -\frac{1}{2} \qquad .... (**)$$

From the first and second constraints in the standard form:

$$s_1 = 18 - 2x_1 - 3x_2$$
$$s_2 = 12 - 2x_1 - x_2$$

Substitute in (**), then:

$$-\frac{3}{4}(18 - 2x_1 - 3x_2) - \frac{3}{4}(12 - 2x_1 - x_2) \le -\frac{1}{2}$$
$$\frac{-54-36}{4} + \frac{3}{2}x_1 + \frac{9}{4}x_2 + \frac{3}{2}x_1 + \frac{3}{4}x_2 \le -\frac{1}{2}$$
$$\Longrightarrow 3x_1 + 3x_2 \le 22$$

$3x_1 + 3x_2 = 22 \Longrightarrow$ if $x_1 = 0$ then (0, 7.3) is the intersection point with the x_2 – axis.

And if $x_2 = 0$ then (7.3, 0) is the intersection point with the x_1 – axis.

The first cut intersects the first constraint in the point (4,10/3), this solution is not optimal. The second constraint is: $0 \le s_5 = \frac{2}{3}s_4 - \frac{2}{3}$, that is

$$-\frac{2}{3}s_4 \le -\frac{2}{3} \qquad ...(***)$$

From equation (*):

$s_4 = \frac{3}{4}s_1 + \frac{3}{4}s_2 - \frac{1}{2} = \frac{3}{4}(18 - 2x_1 - 3x_2\,) + \frac{3}{4}(12 - 2x_1 - x_2\,) - \frac{1}{2}$
$s_4 = 22 - 3x_1 - 3x_2$
Substitute s_4 in (***) , the result is the second cut in terms of x_1 and x_2:
$2x_1 + 2x_2 \leq 14$
$2x_1 + 2x_2 = 14$ $\Rightarrow$ if $x_1 = 0$ then (0, 7) is the intersection point with the $x_2 -$ axis.
And if $x_2 = 0$ then (7, 0) is the intersection point with the $x_1 -$ axis.

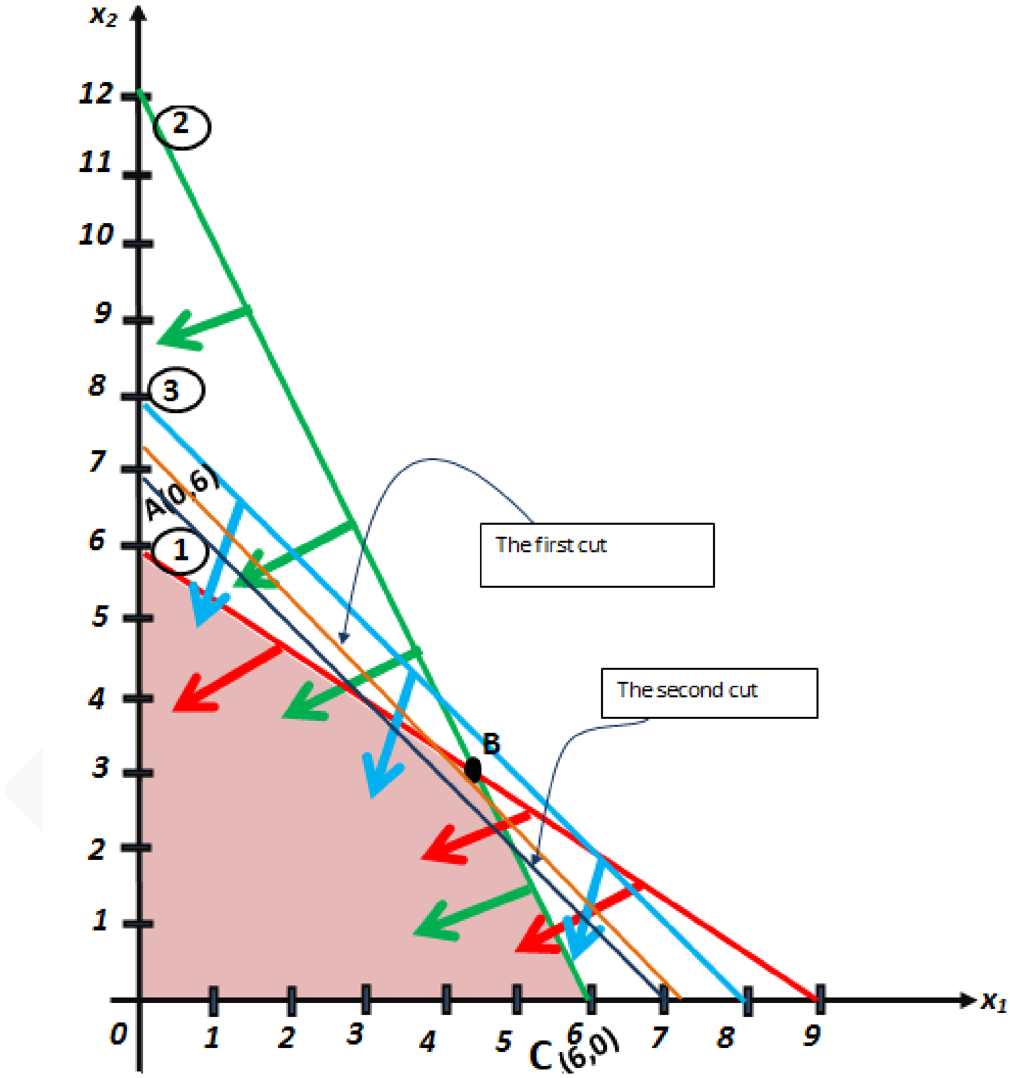

Figure (3.4)

The second cut passes through the point C(4,3) which is the optimal solution. Each cut neglecting a part of the feasible solutions set as we see in figures (3.4) and (3.5). Figure (3.5) shows the parts of the feasible solution set in which the extreme point exists.

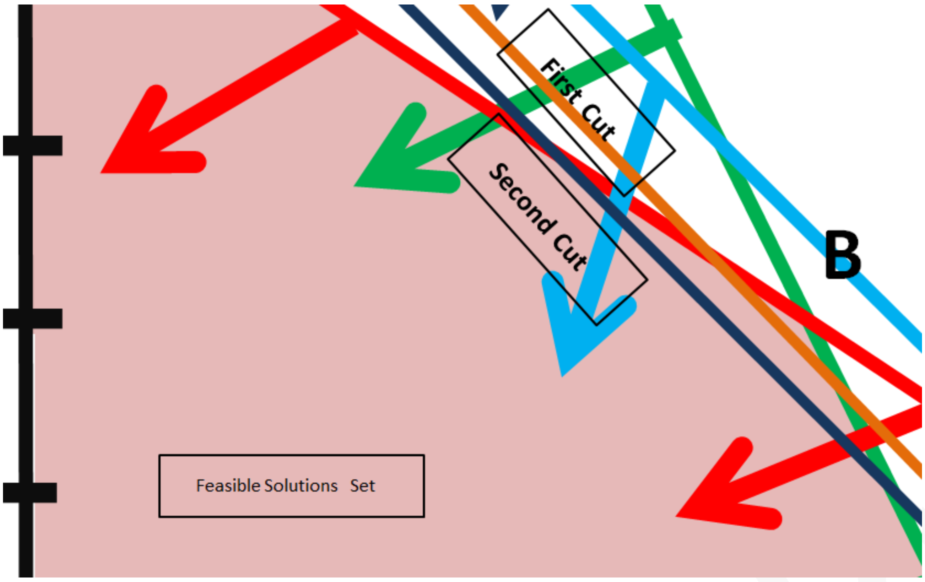

Figure (3.5)

Example (3.11):

Find the optimal solution of the following ILPP

$max \quad Z = -4x_1 + 5x_2$

$S.t \quad -\frac{3}{5}x_1 + \frac{3}{5}x_2 \leq \frac{6}{5}$

$2x_1 + 4x_2 \leq 12$

$x_1, x_2 \geq 0, x_1$ and x_2 are integers

Solution:

First of all, multiplying the first constraint by (5), so it will be:

$-3x_1 + 3x_2 \leq 6$

The standard form of the LPP (with modification in the objective function and ignoring integrality condition) is:

$max \quad Z + 4x_1 - 5x_2 = 0$

$S.t \quad -3x_1 + 3x_2 + s_1 = 6$

$2x_1 + 4x_2 + s_2 = 12$

$x_1, x_2, s_1, s_2 \geq 0$

Let $x_1 = x_2 = 0,$ then $s_1 = 6, s_2 = 12,$ and $Z = 0$ and

B.V.	x_1	x_2	s_1	s_2	b
s_1	-3	3	1	0	6
s_2	2	4	0	1	12
Z	4	$\boxed{-5}$	0	0	0
x_2	-1	1	1/3	0	2

s_2	6	0	$-4/3$	1	4
Z	$\boxed{-1}$	0	5/3	0	10
x_2	0	1	1/9	1/6	8/3
x_1	1	0	$-2/9$	1/6	2/3
Z	0	0	13/9	1/6	32/3

$2\frac{2}{3}$

$\frac{2}{3}$

The non-integer optimal solution is $x_1 = \frac{2}{3}, x_2 = \frac{8}{3}$, and $Z_{max} = \frac{32}{3}$. Since the fractional part of x_1 and x_2 are equal (=2/3), then from equation (1):

$$\frac{f_i}{\sum_{j=1}^{n} f_{ij}} \text{ for } x_1 - \text{equation} = \frac{2/3}{(7/9)+(1/6)} = 12/17$$

$$\frac{f_i}{\sum_{j=1}^{n} f_{ij}} \text{ for } x_2 - \text{equation} = \frac{2/3}{(1/9)+(1/6)} = 12/5$$

x_2 –equation is selected as the source row and Gomory's cut is:

$$1. x_2 + \frac{1}{9}s_1 + \frac{1}{6}s_2 = \frac{8}{3}$$

Or $(1+0)x_2 + (0+\frac{1}{9})s_1 + (0+\frac{1}{6})s_2 = 2 + \frac{2}{3}$

Then from equation (2):

$$s_3 = \frac{1}{9}s_1 + \frac{1}{6}s_2 - \frac{2}{3} \implies -\frac{1}{9}s_1 - \frac{1}{6}s_2 + s_3 = -\frac{2}{3}$$

The modified table after inserting this equation becomes

B.V.	x_1	x_2	s_1	s_2	s_3	b
x_2	0	1	1/9	1/6	0	8/3
x_1	1	0	$-2/9$	1/6	0	2/3
s_3	0	0	$-1/9$	$\boxed{-1/6}$	1	$-2/3$
Z	0	0	13/9	1/6	0	32/3
x_2	0	1	0	0	1	2
x_1	1	0	$-3/9$	0	1	0
s_2	0	0	6/9	1	-6	4
Z	0	0	4/3	0	1	10

The optimal solution is $x_1 = 0, x_2 = 2,$ and $Z_{max} = 10$.

Exercises 3.3 (In addition to the text book exercises)

Find the optimal solution of the following ILPP:

1. $max \quad Z = 14x_1 + 20x_2$

$S.t \quad -2x_1 + 6x_2 \leq 12$

$\quad 14x_1 + 2x_2 \leq 70$

$$x_1, x_2 \geq 0, x_1 \text{ and } x_2 \text{ are integers}$$

2. $max \quad Z = 2x_1 + 10x_2 + x_3$

$S.t \quad 5x_1 + 2x_2 + x_3 \leq 15$

$2x_1 + x_2 + 7x_3 \leq 20$

$x_1 + 3x_2 + 2x_3 \leq 25$

$x_1, x_2, x_3 \geq 0$ and are integers

3.3.2 Branch-and-Bound (B&B) Method

The first B&B algorithm was developed in 1960 by A.H.Land and A.G.Doig for the general mixed and pure ILPP. In this method also, the problem is first solved as a continuous LPP ignoring the integrality condition. Assume a maximization (minimization) problem, set an initial lower (upper) bound $Z = -\infty\ (\infty)$ on the optimum objective value of ILPP. Set $i = 0$.

***Step 1:* (Fathoming / bounding)**. Select Z_i , the next subproblem to be examined. Solve Z_i , and attempt to fathom it using one of three conditions:

a) The optimal Z-value of Z_i cannot yield a better objective value than the current lower bound.
b) Z_i yields a better feasible integer solution than the current lower bound.
c) Z_i has no feasible solution.

Two cases will arise:

a) If Z_i is fathomed and a better solution is found, update the lower bound. If all subproblems have been fathomed, stop; the optimum ILPP is associated with the current finite lower bound. If no finite lower bound exists, the problem has no feasible solution. Else, set $i = i + 1$, and repeat step 1.
b) If Z_i is not fathomed, go to step 2 for branching.

***Step 2:* (branching)**. Select one of the integer variables x_j, whose optimum value x_j^* is not integer. Eliminate the region:

$$[x_j^*] < x_j < [x_j^*] + 1$$

By creating two LP subproblems that correspond to:

$$x_j \leq [x_j^*] \text{ and } x_j \geq [x_j^*] + 1$$

x_j is called the ***branching variable***. These two conditions are mutually execlusive and when applied separately to the continuous LPP, form two different subproblems.Thus the original problem is branched (or partitioned) into two subproblems (also called ***nodes***). Geometrically, it means that the branching process eliminates that portion of the feasible region that contains no feasible integer solution. Set $i = i + 1$, and go to step 1.

Example (3.12):

Find the optimal solution of the following ILPP:

$max \quad Z = x_1 + x_2$

$S.t \quad 2x_1 + 5x_2 \leq 16$

$6x_1 + 5x_2 \leq 30$

$x_1, x_2 \geq 0, x_1$ and x_2 are integers

Solution:

For $2x_1 + 5x_2 = 16 \Rightarrow$ if $x_1 = 0$ then (0, 3.2) is the intersection point with the $x_2 -$ axis.

And if $x_2 = 0$ then (8, 0) is the intersection point with the $x_1 -$ axis.

For $6x_1 + 5x_2 = 30 \Rightarrow$ if $x_1 = 0$ then (0, 6) is the intersection point with the $x_2 -$ axis.

And if $x_2 = 0$ then (5, 0) is the intersection point with the $x_1 -$ axis.

The graphical representation is:

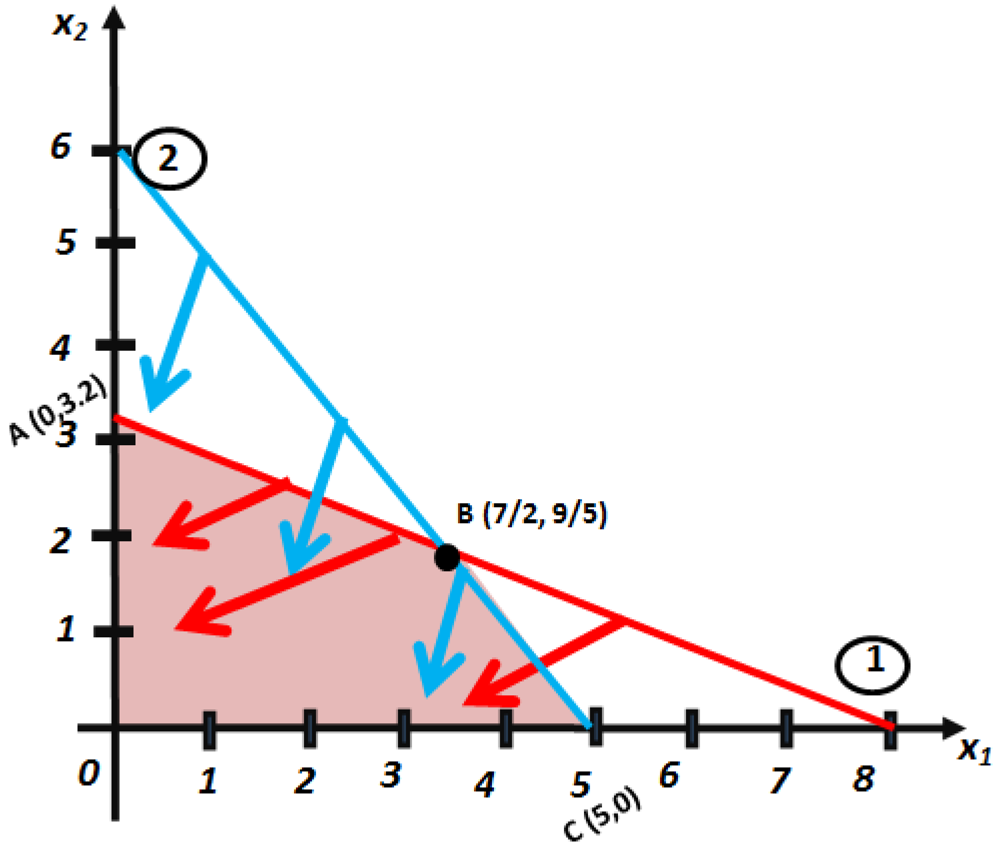

Figure (3.6)

The point B is resulting from the intersection of the lines representing the first and the second constraints, so we use these constraints to find the coordinates of B.

$2x_1 + \cancel{5x_2} = 16$
$\mp 6x_1 \mp \cancel{5x_2} = \mp 30$

$\Rightarrow -4x_1 = -14 \Rightarrow x_1 = 7/2 \xRightarrow{(constr.2)} x_2 = \frac{16-7}{5} = 9/5$

The feasible solution region space is the shaded area OABC in figure (3.6) whose corners are the points $O(0,0)$, $A(0,3.2)$, $B(7/2,9/5)$, and $C(5,0)$.

Corner	Value of Z
O(0,0)	Z=0
$A(0,3.2)$	$Z = 0 + 3.2 = 3.2$
$B(7/2,9/5)$	$Z = \frac{7}{2} + \frac{9}{5} = \frac{53}{10} = 5.3$ *
$C(5,0)$	$Z = 5 + 0 = 5$

From the table, we see that the greatest value of Z occurs at corner B, then $x_1 = 7/2$, $x_2 = 9/5$, and $Z_{max} = 5.3$. The solution is not optimal, then choose $x_1 = 3.5$ as a branching variable, $3 \leq x_1 \leq 4$. Construct two new problems by adding the constructs: $x_1 \leq 3, x_1 \geq 4$.

Subproblem Z_1:

$max \quad Z = x_1 + x_2$

$S.t \quad 2x_1 + 5x_2 \leq 16$

$6x_1 + 5x_2 \leq 30$

$x_1 \leq 3$

$x_1, x_2 \geq 0$

x_1 and x_2 are integers

The solution is: $x_1 = 3$, $x_2 = 2, Z_{max} = 5$

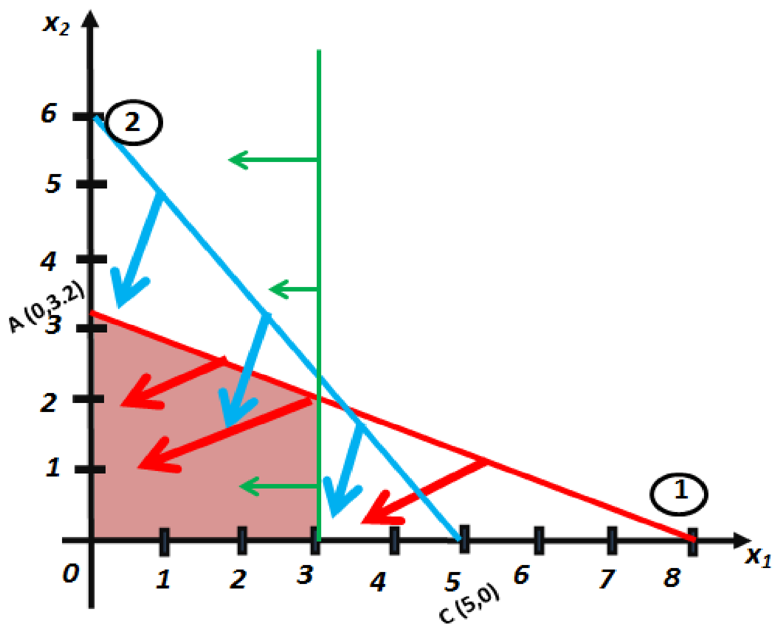

Figure (3.7)

Subproblem Z_2:

$max \quad Z = x_1 + x_2$

$S.t \quad 2x_1 + 5x_2 \leq 16$

$6x_1 + 5x_2 \leq 30$

$x_1 \geq 4$

$x_1, x_2 \geq 0$, x_1 and x_2 are integers

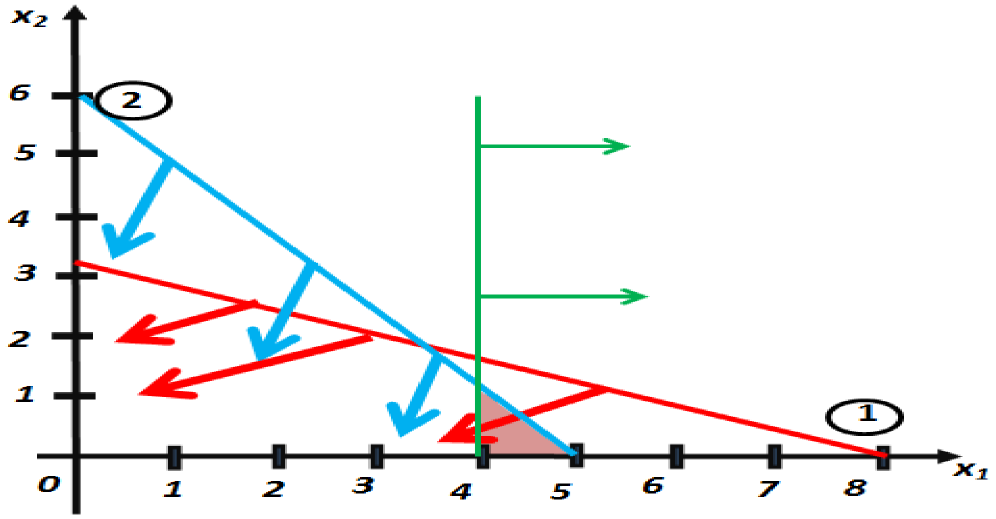

Figure (3.8)

The solution is $x_1 = 4, x_2 = \frac{6}{5} = 1.2$, and $Z_{max} = \frac{26}{5} = 5.2$.

Since the solution of subproblem Z_1 are integers, there is no need to branch subproblem Z_1(subproblem Z_1 is fathomed). The lower bound is now $Z_{max} = 5$. We branch from subproblem Z_2. Since $1 \leq x_2 \leq 2$, then choose x_2 as a branching variable. Construct two new problems by adding the constructs: $x_2 \leq 1, x_2 \geq 2$.

Subproblem Z_3 :

$max \quad Z = x_1 + x_2$

$S.t \quad 2x_1 + 5x_2 \leq 16$

$6x_1 + 5x_2 \leq 30$

$x_1 \geq 4$

$x_2 \leq 1$

$x_1, x_2 \geq 0$, x_1 and x_2 are integers

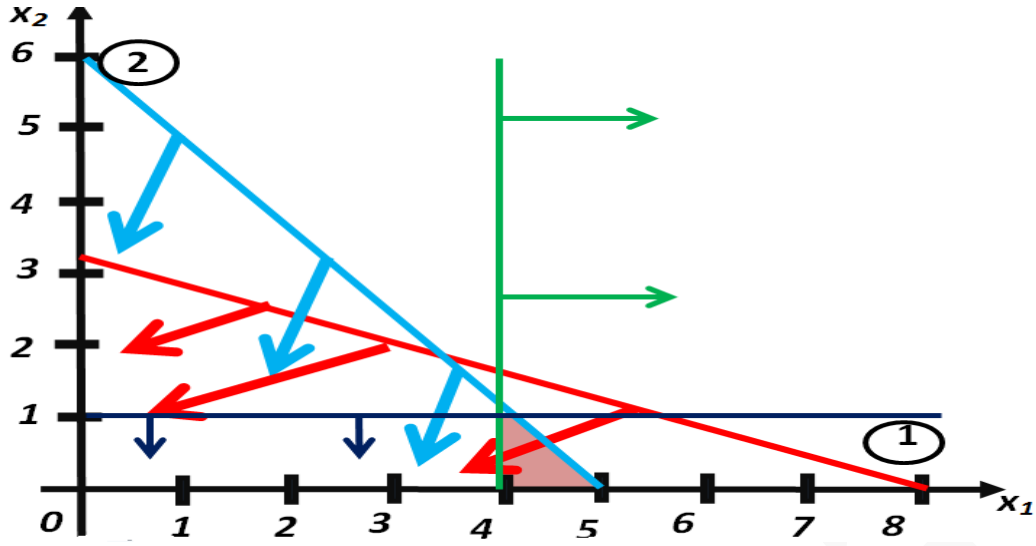

Figure (3.9)

The solution is: $x_1 = 25/6, x_2 = 1,$ and $Z_{max} = \frac{31}{6} = 5.17.$

Subproblem Z_4:

$$\begin{aligned} &max \quad Z = x_1 + x_2 \\ &S.t \quad 2x_1 + 5x_2 \le 16 \\ &\qquad 6x_1 + 5x_2 \le 30 \\ &\qquad x_1 \ge 4 \\ &\qquad x_2 \ge 2 \\ &\qquad x_1, x_2 \ge 0 \text{ , } x_1 \text{ and } x_2 \text{ are integers} \end{aligned}$$

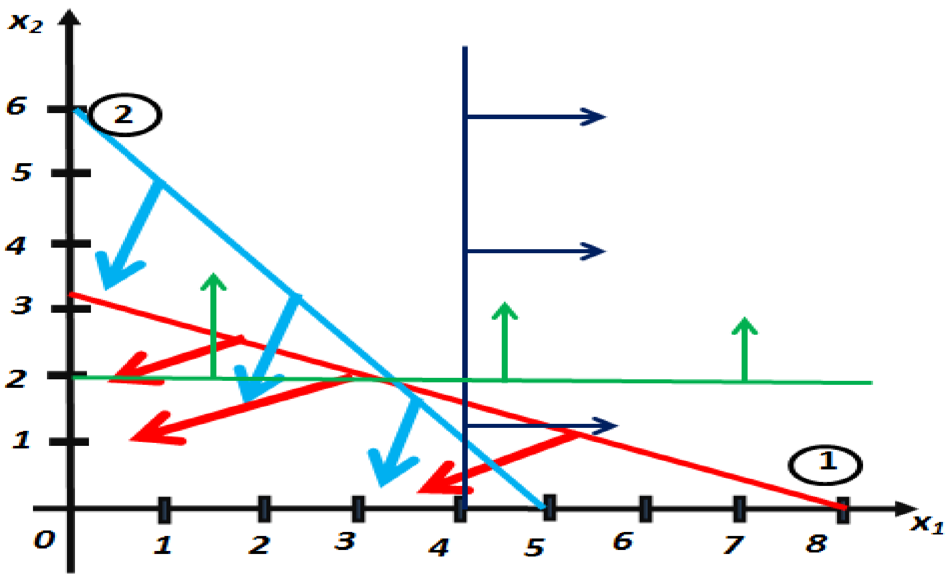

Figure (3.10)

There is no feasible solution.

Subproblem Z_4 is fathomed. Since the solution of subproblem Z_3 is $Z_{max} = \frac{31}{6} = 5.17$, which is not inferior to the lower bound. Therefore it can be branched from subproblem Z_3 into further subproblems. Since x_1 is the only fractional valued variable. Since $4 \le x_1 \le 5$ construct two new problems by adding the constructs: $x_1 \le 4, x_1 \ge 5$.

Subproblem Z_5:

$$
\begin{aligned}
max \quad & Z = x_1 + x_2 \\
S.t \quad & 2x_1 + 5x_2 \le 16 \\
& 6x_1 + 5x_2 \le 30 \\
& x_1 = 4 \\
& x_2 \le 1 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

x_1 and x_2 are integers

The solution is: $x_1 = 4$, $x_2 = 1$, and $Z_{max} = 5$.

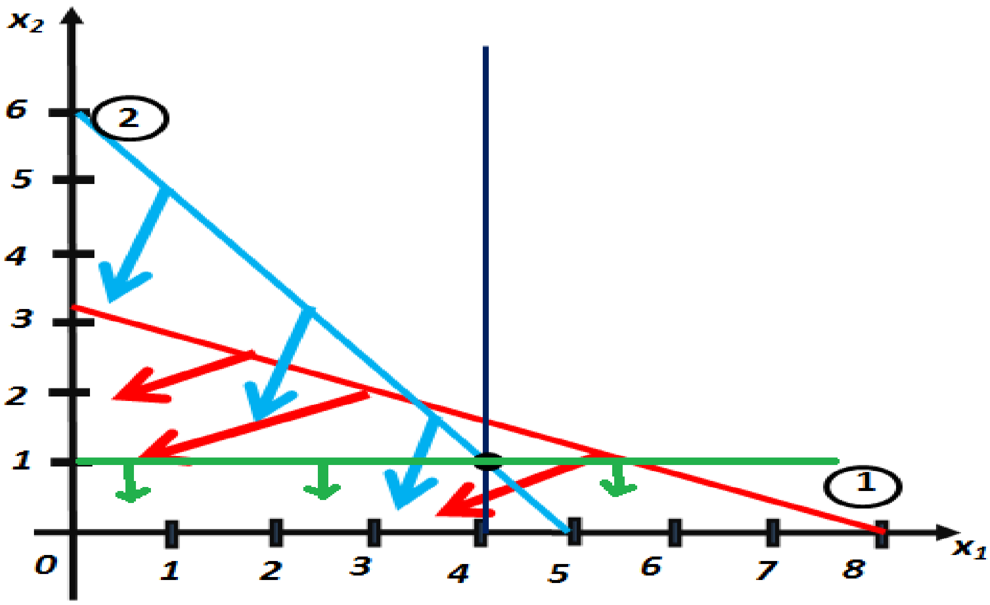

Figure (3.11)

Subproblem Z_6:

$$
\begin{aligned}
max \quad & Z = x_1 + x_2 \\
S.t \quad & 2x_1 + 5x_2 \le 16 \\
& 6x_1 + 5x_2 \le 30 \\
& x_1 \ge 5
\end{aligned}
$$

$$x_2 \le 1$$
$$x_1, x_2 \ge 0$$

x_1 and x_2 are integers

The solution is: $x_1 = 5$, $x_2 = 0$, and $Z_{max} = 5$.

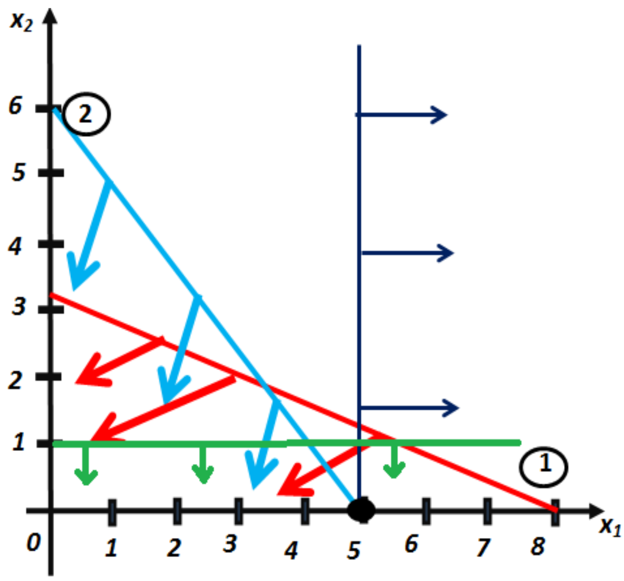

Figure (3.12)

There is more than one solution to this problem, they are:

$x_1 = 3, x_2 = 2$, and $Z_{max} = 5$

$x_1 = 4, x_2 = 1$, and $Z_{max} = 5$

$x_1 = 5, x_2 = 0$, and $Z_{max} = 5$

Figure (3.13) summarize the generated subproblems in the form of a tree.

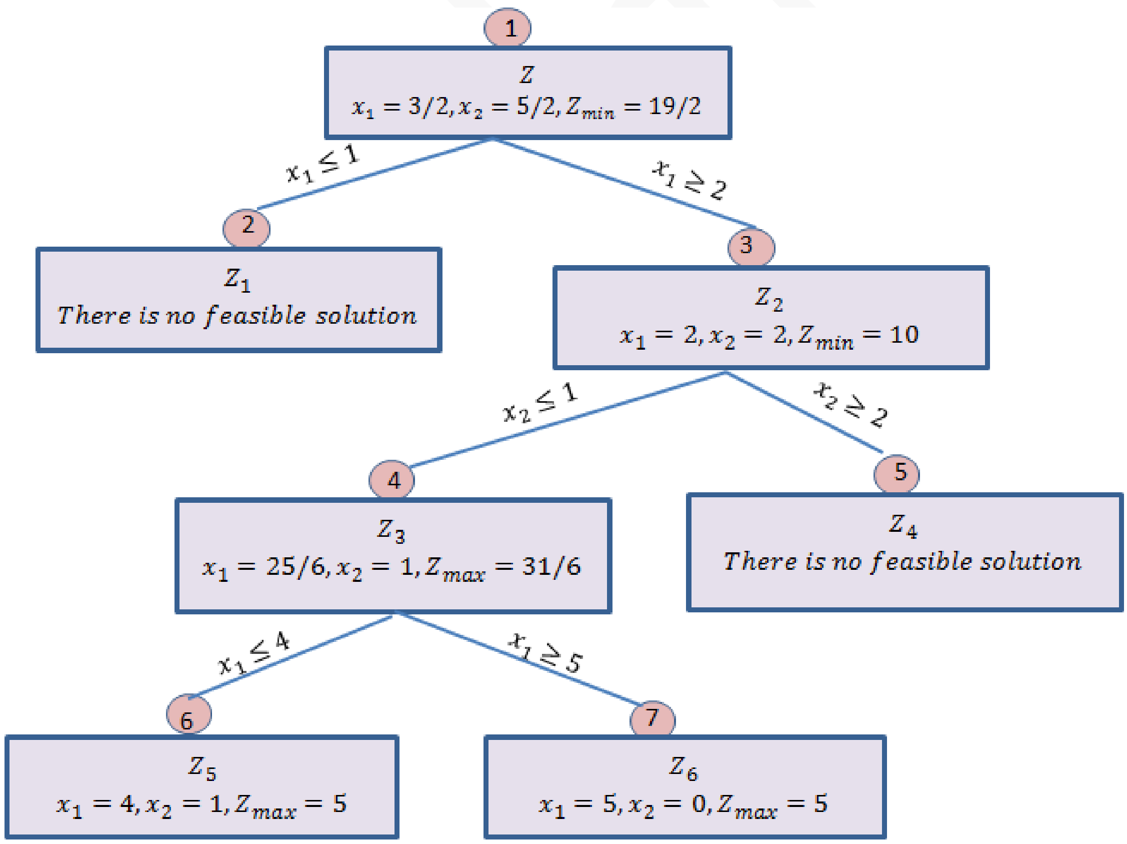

Figure (3.13)

Example (3.13):

Find the optimal solution of the following ILPP:

$$\min \quad Z = 3x_1 + 2x_2$$

$$S.t \quad x_1 + x_2 = 4$$

$$x_1 + 3x_2 \geq 6$$

$$5x_1 + 3x_2 \geq 15$$

$$x_1, x_2 \geq 0, x_1 \text{ and } x_2 \text{ are integers}$$

Solution:

The standard form of the ILPP is:

$$\min \quad Z = 3x_1 + 2x_2 + MR_1 + MR_2 + MR_3$$

$$S.t \quad x_1 + x_2 + R_1 = 4$$

$$x_1 + 3x_2 - s_1 + R_2 = 6$$

$$5x_1 + 3x_2 - s_2 + R_3 = 15$$

$$x_1, x_2, s_1, s_2, R_1, R_2, R_3 \geq 0, x_1 \text{ and } x_2 \text{ are integers}$$

From the constraints:

$$R_1 = 4 - x_1 - x_2$$

$$R_2 = 6 - x_1 - 3x_2 + s_1$$

$$R_3 = 15 - 5x_1 - 3x_2 + s_2$$

Substitute $R_1, R_2,$ and R_3 in the Z=equation and rearrange Z-equation, the standard form will be:

$$\min \quad Z + (-3+7M)x_1 + (-2 + 7M)x_2 - Ms_1 + Ms_2 = 25M$$

$$S.t \quad x_1 + x_2 + R_1 = 4$$

$$x_1 + 3x_2 - s_1 + R_2 = 6$$

$$5x_1 + 3x_2 - s_2 + R_3 = 15$$

$$x_1, x_2, s_1, s_2, R_1, R_2, R_3 \geq 0, x_1 \text{ and } x_2 \text{ are integers}$$

Let $x_1 = x_2 = s_1 = s_2 = 0, then\ R_1 = 4, R_3 = 6, R_3 = 15, Z = 25M.$

B.V.	x_1	x_2	s_1	s_2	R_1	R_2	R_3	b
R_1	1	1	0	0	1	0	0	4
R_2	1	3	−1	0	0	1	0	6
R_3	$\boxed{5}$	3	0	−1	0	0	1	15
Z	−3+7M	−2+7M	−M	−M	0	0	0	25M

B.V.	x_1	x_2	s_1	s_2	R_1	R_2	b
R_1	0	2/5	0	1/5	1	0	1
R_2	0	$\boxed{12/5}$	−1	1/5	0	1	3

x_1	1	3/5	0	−1/5	0	0	3
Z	0	−1/5+14/5M	−M	$-\frac{3}{5}+2/5M$	0	0	4M+9

B.V.	x_1	x_2	s_1	s_2	R_1	b
R_1	0	0	1/6	1/6	1	1/2
x_2	0	1	−5/12	1/12	0	5/4
x_1	1	0	1/4	−1/4	0	9/4
Z	0	0	$-\frac{1}{12}+1/6M$	$-\frac{7}{12}+1/6M$	0	1/2M+37/4

B.V.	x_1	x_2	s_1	s_2	b
s_1	0	0	1	1	3
x_2	0	1	0	1/2	5/2
x_1	1	0	0	−1/2	3/2
Z	0	0	0	−1/2	19/2

$\therefore\ x_1 = 3/2, x_2 = 5/2, and\ Z_{min} = 19/2$. The solution is not optimal, we need two constraints $x_1 \le 1\ and\ x_1 \ge 2$.

Subproblem Z_1:

$min \quad Z = 3x_1 + 2x_2$

$S.t \qquad x_1 + x_2 = 4$

$\qquad x_1 + 3x_2 \ge 6$

$\qquad 5x_1 + 3x_2 \ge 15$

$\qquad x_1 \le 1$

$\qquad x_1, x_2 \ge 0, x_1$ and x_2 are integers

Either we solve the above problem in the usual way from the beginning or by using the last table. The additional constraint is written as: $x_1 + s_3 = 1$

x_1 is a basic variable, so we substitute x_1 from the table:

$x_1 + 0.x_2 + 0.s_1 - \frac{1}{2}s_2 = \frac{3}{2}$, then: $x_1 = \frac{3}{2} + \frac{1}{2}s_2$

$\Longrightarrow\ \frac{3}{2} + \frac{1}{2}s_2 + s_3 = 1 \quad \Longrightarrow\ \frac{1}{2}s_2 + s_3 = \frac{-1}{2}$

B.V.	x_1	x_2	s_1	s_2	s_3	b
s_1	0	0	1	1	0	3
x_2	0	1	0	1/2	0	5/2
x_1	1	0	0	−1/2	0	3/2

s_3	0	0	0	$\boxed{1/2}$	1	$-1/2$
Z	0	0	0	$-1/2$	0	19/2

There is no feasible solution

Subproblem Z_2:

$min \quad Z = 3x_1 + 2x_2$

$S.t \quad x_1 + x_2 = 4$

$x_1 + 3x_2 \geq 6$

$5x_1 + 3x_2 \geq 15$

$x_1 \geq 2$

$x_1, x_2 \geq 0, x_1$ and x_2 are integers

Either we solve the above problem in the usual way from the beginning or by using the last table. The additional constraint is written as: $-x_1 \leq -2$
$\Rightarrow -x_1 + s_3 = -2$

x_1 is a basic variable, so we substitute x_1 from the table:

$x_1 - \frac{1}{2}s_2 = \frac{3}{2}$, then: $\quad x_1 = \frac{3}{2} + \frac{1}{2}s_2$

$\Longrightarrow \quad -\frac{3}{2} - \frac{1}{2}s_2 + s_3 = -2 \quad \Longrightarrow -\frac{1}{2}s_2 + s_3 = \frac{-1}{2}$

B.V.	x_1	x_2	s_1	s_2	s_3	b
s_1	0	0	1	1	0	3
x_2	0	1	0	1/2	0	5/2
x_1	1	0	0	$-1/2$	0	3/2
s_3	0	0	0	$\boxed{-1/2}$	1	$-1/2$
Z	0	0	0	$-1/2$	0	19/2
s_1	0	0	1	0	2	3
x_2	0	1	0	0	1	2
x_1	1	0	0	0	-1	2
s_2	0	0	0	1	-2	1
Z	0	0	0	0	-1	10

$\therefore$ the optimal solution is $x_1 = 2, x_2 = 2, and \ Z_{min} = 10.$

Figure (3.14) summarize the generated subproblems in the form of a tree.

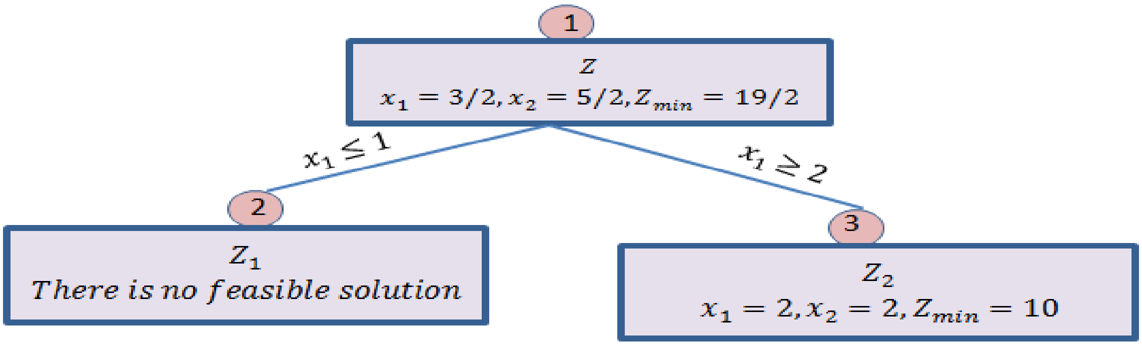

Figure (3.14)

Exercises 3.4 (In addition to the text book exercises)

Use B & B method to solve the following ILPP:

1. $$\max \quad Z = 9x_1 + 3x_2 + 9x_3$$

$$\begin{aligned} S.t \qquad -3x_1 + 6x_2 + 3x_3 &\le 12 \\ 12\,x_1 - 9x_3 &\le 18 \\ 3\,x_1 - 9x_2 + 6x_3 &\le 9 \end{aligned}$$

$x_1, x_2, x_3 \ge 0, x_1$ and x_3 are integers

2. $$\min \quad Z = 5x_1 + 4x_2$$

$$\begin{aligned} S.t \qquad x_1 + x_2 &\le 5 \\ 10x_1 + 6x_2 &\le 45 \end{aligned}$$

$x_1, x_2 \ge 0, x_1$ and x_2 are integers