Ch. 2: Game Theory

The theory of games (or game theory or competitive strategies) is a mathematical theory that deals with the general feature of competitive situations. This theory is helpful when two or more opponents (individuals, companies,… etc.) with conflicting objectives try to make decision. In such situations, a decision made by one decision-maker affects the decision made by one or more of the remaining decision-makers and the final outcome depend on the decision of all parties.

The game theory is based on the ***minimax principle*** put forward by J. von Neuman (1903-1957) which implies that each competitor will act so as to minimize his maximum loss (or maximize his minimum gain) or achieve the best of the worst. The theory does not describe how a game should be played; it describes only the procedure and principles by which plays should be selected.

2.1 Characteristics of the Game

A competitive game has the following characteristics:

a) There is finite number of participants or competitors. If the number of participants is 2, the game is called ***two- person game***; for number greater than two, it is called ***n-person game***.

b) Each participant has a list of finite number of possible courses of actions available to him. The list may not be the same for each participant.

c) Each participant knows all the possible choices available to others but does not know which of them is going to be chosen by them.

d) A ***play*** is said to occur when each of the participants chooses one of the courses of actions available to him. The choices are assumed to be made simultaneously so that no participant knows the choices made by others until he has decided his own.

e) Every combination of courses of actions determines an outcome which results in gains of the participants. The gain (***payoff***) may be positive, negative or zero. Negative gain is called ***loss***.

f) The gain of a participant depends not only on his own actions but also on those of others.

g) The gains of each and every play are fixed and specified in advance and are known to each player.

h) The players make individual decisions without direct communication.

2.2 Definitions

Definition (2.1):

A ***game*** is an activity between two or more persons, involving action by each one of them according to a set of rules which results in some gain (+ve, −ve or zero) for each.

Definition (2.2):

Each participant or competitor playing a game is called a ***player***.

Definition (2.3):

A ***strategy*** is a predetermined rule by which a player decides his course of action from his list of courses of actions during the game. To decide a particular strategy the player needs to know the other's strategy.

Definition (2.4):

A ***pure strategy*** is the decision rule to always select a particular course of action.

Definition (2.5):

Mixed strategy is the decision, in advance of all plays, to choose a course of action for each play in accordance with some probability distribution. Thus, a mixed strategy is a selection among pure strategies with some fixed probabilities.

Definition (2.6):

The strategy that puts the player in the most preferred position irrespective of the strategy of his opponents is called an ***optimal strategy***. Any deviation from this strategy would reduce his payoff.

Definition (2.7):

Zero-sum game is a game in which the sum of payments to all the players, after the play of the game, is zero. In such a game, the gain of players that win is exactly equal the loss of players that lose.

Definition (2.8):

Two-person zero-sum game is a game involving only two players in which the gain of one player equals the loss of the other. It is also called a ***rectangular game*** or ***matrix game*** because the payoff matrix is rectangular in form.

Definition (2.9):

A ***nonzero- game*** is a game in which a third party receives or makes some payment.

Definition (2.10):

Payoff (***gain*** or ***game***) ***matrix*** is the table showing the amounts received by the player named at the left-hand-side after all possible plays of the game. The payment is made by the player named at the top of the table.

In a two-person zero-sum game, the cell entries in B's payoff matrix will be the negative of the corresponding cell entries in A's payoff matrix. A is called ***maximizing player*** as he would try to maximize his gains, while B is called ***minimizing player*** as he would try to minimize his losses.

Player B

Player A

	1	**2**	**…**	***j***	**…**	***n***
1	a_{11}	a_{12}	…	a_{1j}	…	a_{1n}
2	a_{21}	a_{22}	…	a_{2j}	…	a_{2n}
⋮	⋮	⋮		⋮		⋮
i	a_{i1}	a_{i2}	…	a_{ij}	…	a_{in}
⋮	⋮	⋮		⋮		⋮
m	a_{m1}	a_{m2}	…	a_{mj}	…	a_{mn}

A's payoff matrix

Player B

Player A

	1	2	…	j	…	n
1	$-a_{11}$	$-a_{12}$	…	$-a_{1j}$	…	$-a_{1n}$
2	$-a_{21}$	$-a_{22}$	…	$-a_{2j}$	…	$-a_{2n}$
⋮	⋮	⋮		⋮		⋮
i	$-a_{i1}$	$-a_{i2}$	…	$-a_{ij}$	…	$-a_{in}$
⋮	⋮	⋮		⋮		⋮
m	$-a_{m1}$	$-a_{m2}$	…	$-a_{mj}$	…	$-a_{mn}$

B's payoff matrix

Thus the sum of payoff matrices for A and B is a null matrix. Then, we shall usually omit B's payoff matrix; keeping in mind that it is just the negative of A's payoff matrix. That is if $a_{ij} > 0$, it is a gain for player A, $a_{ij} < 0$, it is a gain for player B, $a_{ij} = 0$, players gain nothing.

2.3 Rule 1: Look for a Pure Strategy (Saddle Point)

The steps required to detect a saddle point:

1) At the right of each row, write the row minimum and ring the largest of them (***maximin***).
2) At the bottom of each column, write the column maximum and ring the smallest of them (***minimax***).
3) If minimax = maximin, the cell where the corresponding row and column meet is a ***saddle point*** (***equilibrium point***) and the element in that cell is the value of the game, the game is called ***stable game***.
4) If minimax ≠ maximin , there is no saddle point and the value of the game lies between these two values.
5) If there are more than one saddle points then there will be more than one solution, each solution corresponding to each saddle point.

Example (2.1):

In a game of matching coins, the payoff matrix is given in the following table. Determine the best strategies for each player and the value of the game>

		B	
		H	**T**
A	**H**	0	5
	T	−2	0

Solution:

First, we search for a saddle point:

		B		
		H	**T**	min
A	**H**	0	5	0
	T	−2	0	-2
	max	0	5	

Minimax=0, maximin=0. Since minimax=maximin, then there is a saddle point (1,1)[means first strategy of A and first strategy of B].

Optimal strategy for player A :(1, 0)

Optimal strategy for player B :(1, 0)

The value of the game V=0

Example (2.2):

Does the following game have a saddle point?

		B		
		B_1	**B_2**	**B_3**
A	**A_1**	0	7	6
	A_2	3	12	1

Solution:

		B			
		B_1	**B_2**	**B_3**	min
A	**A_1**	0	7	6	0
	A_2	3	12	1	1
	max	3	12	6	

Minimax=3, maximin=1. Since minimax≠maximin, then there is no saddle point.

2.4 Rule 2: Reduce the Game

If no pure strategy exists, the next step is to eliminate certain strategies (rows and/or columns) by dominance. The resulting game can be solved for some mixed strategy. The ***dominance rules*** are:

For rows: The row i dominating row k if : $a_{ij} \geq a_{kj}\, , j = 1, \dots , n.$

For columns: The column j dominating column k if : $a_{ij} \leq a_{ik}\, , i = 1, \dots , m.$

Example (2.3):

Two players P and Q play a game. Each of them has to choose one of the three colors, white (W), black (B), and red (R) independently of the other. Thereafter the colors are compared. If both P and Q have chosen white (W,W), neither wins anything. The payoff matrix is shown below. Does the game have a saddle point? If not reduce the game.

		P		
		W	B	R
Q	W	0	−2	7
	B	2	5	6
	R	3	−3	8

Solution:

		P			
		W	B	R	min
Q	W	0	−2	7	−2
	B	2	5	6	②
	R	3	−3	8	−3
	max	③	5	8	

Minimax=3, maximin=2. Since minimax≠maximin, then there is no saddle point. $2 \le V \le 3$. To reduce the matrix: the first column dominating the third column $(0 < 7,2 < 6,3 < 8)$. The resulting matrix is:

		P	
		W	B
Q	W	0	−2
	B	2	5
	R	3	−3

The second row dominating the first row $(2 > 0, 5 > -2)$. The resulting matrix is:

		P	
		W	B
Q	B	2	5
	R	3	−3

Remark (2.1)

Sometimes the previous dominance rules are not useful; in this case we can use the ***average rule***:

For rows: The rows i and k dominating row h if every element in the average of rows i and k is greater than or equal the corresponding element in row h.
For columns: The columns j and k dominating column h if every element in the average of columns j and k is smaller than or equal the corresponding element in column h.

Example (2.4):

Consider the following game:

			B	
		1	**2**	**3**
	1	6	1	3
A	**2**	0	9	7
	3	2	3	4

This game has no saddle point, since:

			B		
		1	**2**	**3**	min
	1	6	1	3	1
A	**2**	0	9	7	0
	3	2	3	4	②
	max	⑥	9	7	

Minimax=6, maximin=2, minimax≠maximin. $2 \leq V \leq 6$. The game cannot be reduced by dominance rules. The average of A's first and second strategy is: $\left(\frac{6+0}{2}, \frac{1+9}{2}, \frac{3+7}{2}\right) = (3,5,5)$. By comparing each element in the average with the corresponding element in the third row: $3 > 2,\ 5 > 3, 5 > 4$. The resulting matrix will be:

			B	
		1	**2**	**3**
A	**1**	6	1	3
	2	0	9	7

2.5 Rule 3: Solve for a Mixed Strategy

In case where there is no saddle point and dominance has been used to reduce the game matrix, players will use mixed strategies. Such games are called ***unstable games***.

2.6 Mixed Strategies for 2 x 2 Games

2.6.1 Arithmetic method (Odds Method)

It provides an easy method for finding the optimum strategies for each player in a 2 x 2 game without a saddle point. It consists of the following steps:

i) Subtract the two digits in column 1 and write the difference under column 2, ignoring sign.

ii) Subtract the two digits in column 2 and write the difference under column 1, ignoring sign.

iii) Similarly proceed for the two rows, write the results to the right of each row.

These values are called oddments.

iv) To find the frequency (probability) in which the players must use their courses of action in their optimum strategy, divide the oddment of each player on the sum of his oddments.

v) The value of the game result by multiplying the elements of a row or column by the probabilities corresponding to these elements.

Example (2.5):

Consider the game in example (2.3), find the optimal strategy for each player and the value of the game.

Solution:

The game is reduced to a 2 x 2 game which we must check the existence of a saddle point:

		P		
		W	**B**	min
Q	**B**	2	5	2
	R	3	−3	−3
	max	3	5	

Minimax=3, maximin=2. Since minimax≠maximin, then there is no saddle point and $2 \leq V \leq 3$. Then:

		P			
		W	**B**		
Q	**B**	2	5	6	6/9
	R	3	−3	3	3/9
		8	1		
		8/9	1/9		

Optimal strategy for player P is : (8/9, 1/9, 0)

Optimal strategy for player Q is : (0, 6/9, 3/9)

To obtain the value of the game:

By using Q's oddments:

$V=\frac{2\times6+3\times3}{9}=21/9$ when Q plays B

V= $\frac{5\times6-3\times3}{9} = 21/9$ when Q plays R

By using P’s oddments:

V= $\frac{2\times8+5\times1}{9} = 21/9$ when P plays W

V= $\frac{3\times8-3\times1}{9} = 21/9$ when P plays B

Remark (2.2)

The above values of V are equal only if sum of the oddments vertically and horizontally are equal.

Example (2.6):

In a game of matching coins, the payoff matrix is given in the following table. Determine the best strategies for each player and the value of the game>

		B	
		H	T
A	H	2	−1
	T	−1	0

Solution:

First, we search for a saddle point:

		B		
		H	T	min
A	H	2	−1	−1
	T	−1	0	−1
	max	2	0	

Minimax=0, maximin= -1. Since minimax≠ maximin, then there is no saddle point and $-1 \leq V \leq 0$.

		B			
		H	T		
A	H	2	−1	1	1/4
	T	−1	0	3	3/4
		1	3		
		1/4	3/4		

Optimal strategy for player A is : (1/4, 3/4)

Optimal strategy for player B is : (1/4, 3/4)

V=$\frac{2\times1-1\times3}{4} = -1/4$, that is B is the winner.

Example (2.7):

Find the optimal strategy of each player and the value of the following game:

$$\begin{array}{cc|c|c|c|c|}
 & & & \mathbf{B} & & \\
 & & \mathbf{I} & \mathbf{II} & \mathbf{III} & \mathbf{IV} \\ \hline
 & \mathbf{1} & 3 & 2 & 4 & 0 \\ \hline
\mathbf{A} & \mathbf{2} & 3 & 4 & 2 & 4 \\ \hline
 & \mathbf{3} & 4 & 2 & 4 & 0 \\ \hline
 & \mathbf{4} & 0 & 4 & 0 & 8 \\ \hline
\end{array}$$

Solution:

$$\begin{array}{cc|c|c|c|c|c}
 & & & \mathbf{B} & & & \\
 & & \mathbf{I} & \mathbf{II} & \mathbf{III} & \mathbf{IV} & \text{min} \\ \cline{3-6}
 & \mathbf{1} & 3 & 2 & 4 & 0 & 0 \\ \cline{3-6}
\mathbf{A} & \mathbf{2} & 3 & 4 & 2 & 4 & \enclose{circle}{2} \\ \cline{3-6}
 & \mathbf{3} & 4 & 2 & 4 & 0 & 0 \\ \cline{3-6}
 & \mathbf{4} & 0 & 4 & 0 & 8 & 0 \\ \cline{3-6}
 & \text{max} & \enclose{circle}{4} & 4 & 4 & 8 &
\end{array}$$

Minimax=4, maximin=2. Since minimax≠ maximin, then there is no saddle point and $2 \leq V \leq 4$. Then we try to reduce the matrix:

$$\begin{array}{cc|c|c|c|c|}
 & & & \mathbf{B} & & \\
 & & \mathbf{I} & \mathbf{II} & \mathbf{III} & \mathbf{IV} \\ \hline
 & \mathbf{1} & 3 & 2 & 4 & 0 \\ \hline
\mathbf{A} & \mathbf{2} & 3 & 4 & 2 & 4 \\ \hline
 & \mathbf{3} & 4 & 2 & 4 & 0 \\ \hline
 & \mathbf{4} & 0 & 4 & 0 & 8 \\ \hline
\end{array}
\xRightarrow{R1\ vs\ R3}
\begin{array}{cc|c|c|c|c|}
 & & & \mathbf{B} & & \\
 & & \mathbf{I} & \mathbf{II} & \mathbf{III} & \mathbf{IV} \\ \hline
 & \mathbf{2} & 3 & 4 & 2 & 4 \\ \hline
\mathbf{A} & \mathbf{3} & 4 & 2 & 4 & 0 \\ \hline
 & \mathbf{4} & 0 & 4 & 0 & 8 \\ \hline
\end{array}$$

No saddle point

$$\xRightarrow{CI\ vs\ CIII}
\begin{array}{cc|c|c|c|}
 & & & \mathbf{B} & \\
 & & \mathbf{II} & \mathbf{III} & \mathbf{IV} \\ \hline
 & \mathbf{2} & 4 & 2 & 4 \\ \hline
\mathbf{A} & \mathbf{3} & 2 & 4 & 0 \\ \hline
 & \mathbf{4} & 4 & 0 & 8 \\ \hline
\end{array}
\xRightarrow{CII\ vs\ (CIII+CIV)/2}
\begin{array}{cc|c|c|}
 & & \mathbf{B} & \\
 & & \mathbf{III} & \mathbf{IV} \\ \hline
 & \mathbf{2} & 2 & 4 \\ \hline
\mathbf{A} & \mathbf{3} & 4 & 0 \\ \hline
 & \mathbf{4} & 0 & 8 \\ \hline
\end{array}$$

No saddle point

No saddle point

$$\xRightarrow{R2\ vs\ (R3+R4)/2}
\begin{array}{cc|c|c|}
 & & \mathbf{B} & \\
 & & \mathbf{III} & \mathbf{IV} \\ \hline
\mathbf{A} & \mathbf{3} & 4 & 0 \\ \hline
 & \mathbf{4} & 0 & 8 \\ \hline
\end{array}$$

The last matrix has no saddle point (maximin=0, minimax=4), then:

B

		III	IV		
A	3	4	0	8	2/3
	4	0	8	4	1/3
		8	4		
		2/3	1/3		

Optimal strategy for player A is: (0, 0, 2/3,1/3)

Optimal strategy for player B is: (0, 0, 2/3,1/3)

$V=\frac{4\times2+0\times1}{3}=8/3$

Example (2.8):

Reduce the following game and find the optimal strategy of each player and the value of the following game:

B

		1	2	3	4	5
A	I	1	3	2	7	4
	II	3	4	1	5	6
	III	6	5	7	6	5
	IV	2	0	6	3	1

Solution:

$\xrightarrow{RIV\ vs\ RIII}$

B

		1	2	3	4	5
A	I	1	3	2	7	4
	II	3	4	1	5	6
	III	6	5	7	6	5

$\xrightarrow[C5\ vs\ C2]{C4\ vs\ C2}$

B

		1	2	3
A	I	1	3	2
	II	3	4	1
	III	6	5	7

$\xrightarrow[RII\ vs\ RIII]{RI\ vs\ RIII}$

B

		1	2	3
A	III	6	5	7

Optimal strategy for player A is: (0, 0, 1,0)

Optimal strategy for player B is: (0, 1, 0,0,0) [B must play strategy 2 in order to minimize his losses]

V=5

Example (2.9):

A company is currently involved in negotiations with its union on the upcoming wage contract. Positive signs in the following table represent wages increase while negative sign represents wage reduction. What are the optimal strategies for the company as well as the union and what is the value of the game?

		Union strategies			
		U₁	**U₂**	**U₃**	**U₄**
Company strategies	**C₁**	+0.25	+0.27	+0.35	−0.02
	C₂	+0.20	+0.16	+0.08	+0.08
	C₃	+0.14	+0.12	+0.15	+0.13
	C₄	+0.30	+0.14	+0.19	+0.00

Solution:

Since in a game matrix, player to its left is a maximizing player and the one at the top is a minimizing player, the above table is transposed and rewritten as the following table since company's interest is to minimize the wage increase while union's interest is to get the maximum wage increase.

		Company strategies			
		C₁	**C₂**	**C₃**	**C₄**
Union strategies	**U₁**	0.25	0.2	0.14	0.3
	U₂	0.27	0.16	0.12	0.14
	U₃	0.35	0.08	0.15	0.19
	U₄	−0.02	0.08	0.13	0.00

First, we must look for a saddle point:

		Company strategies				
		C₁	**C₂**	**C₃**	**C₄**	**min**
Union strategies	**U₁**	0.25	0.2	0.14	0.3	0.14
	U₂	0.27	0.16	0.12	0.14	0.12
	U₃	0.35	0.08	0.15	0.19	0.08
	U₄	−0.02	0.08	0.13	0.00	−0.02
	max	0.35	0.2	0.15	0.3	

Maximin=0.14, minimax=0.15, since maximin $\neq$ minimax, then there is no saddle point and $0.14 \leq V \leq 0.15$

$\overset{U_3 vs. U_4}{\Longrightarrow}$

		Company strategies			
		C₁	**C₂**	**C₃**	**C₄**
Union strategies	**U₁**	0.25	0.2	0.14	0.3
	U₂	0.27	0.16	0.12	0.14
	U₃	0.35	0.08	0.15	0.19

There is no saddle point

$\overset{\substack{C_1 vs. C_2 \\ C_4 vs. C_3}}{\Longrightarrow}$

		Company strategies	
		C₂	**C₃**
Union strategies	**U₁**	0.2	0.14
	U₂	0.16	0.12
	U₃	0.08	0.15

There is no saddle point

$\overset{U_2 vs. U_1}{\Longrightarrow}$

		Company strategies		
		C₂	**C₃**	min
Union strategies	**U₁**	0.2	0.14	0.14
	U₃	0.08	0.15	0.08
	max	0.2	0.15	

There is no saddle point.

		Company strategies			
		C₂	**C₃**		
Union strategies	**U₁**	0.2	0.14	0.07	7/13
	U₃	0.08	0.15	0.06	6/13
		0.01	0.12		
		1/13	12/13		

Optimal strategy for the company: (0, 1/13, 12/13,0)

Optimal strategy for the union: (7/13, 0, 6/13, 0)

The value of the game is V= $\frac{0.2*7+0.08*6}{13} = \frac{1.88}{13} = 0.145$

2.6.2 Algebraic Method for Finding Optimum Strategies and Game Value

Consider the following 2 x 2 game:

		B		
		B₁	**B₂**	
A	**A₁**	a	b	x
	A₂	c	d	$1-x$
		y	$1-y$	

While applying this method it is assumed that x represents the fraction of time (frequency) for which player A uses strategy 1 and $(1-x)$ represents the fraction of time (frequency) for which player A uses strategy 2. Then the value of the game:

$$V = a*x + c*(1-x) = b*x + d*(1-x)$$

Solve these equations to find the value of x. Similarly y and $(1-y)$ represents the fraction of time (frequency) for which player B uses strategies 1 and 2 respectively. Then the value of the game:

$$V = a*y + b*(1-y) = c*y + d*(1-y)$$

Solve these equations to find the value of y.

Example (2.10):

Two armies are at war. Army A has two airbases, one of which is thrice as valuable as the other. Army B can destroy an undefended airbase, but it can destroy only one of them. Army A can also defend only one of them. Find the best strategy for A to minimize his losses and find the optimal strategy for B.

Solution:

Since both armies have only two possible courses of action, the gain matrix for the game is:

			Army A	
			1 **Defend the smaller airbase**	**2** **Defend the larger airbase**
Army B	**1**	**Attack the smaller airbase**	0	1
	2	**Attack the larger airbase**	3	0

First, we check for the existence of a saddle point:

		Army A		
		1	**2**	**min**
Army B	**1**	0	1	0
	2	3	0	0
max		3	1	

Maximin=0, minimax=1. Since minimax $\neq$ maximin, then there is no saddle point and $0 \leq V \leq 1$.

Let x and $(1-x)$ represents the fraction of time (frequency) for which player B uses strategies 1 and 2 respectively. Then the value of the game:

$V = 0 \times x + 3 \times (1-x) = 1 \times x + 0 \times (1-x)$

$\Rightarrow 3 - 3x = x \Rightarrow 3 = 4x \;\Rightarrow x = \frac{3}{4} \Rightarrow 1 - x = \frac{1}{4}$

Similarly let y and $(1-y)$ represents the fraction of time (frequency) for which player A uses strategies 1 and 2 respectively. Then the value of the game:

$V = 0 \times y + 1 \times (1-y) = 3 \times y + 0 \times (1-y)$

$\Rightarrow 1 - y = 3y \Rightarrow 1 = 4y \;\Rightarrow y = \frac{1}{4} \Rightarrow 1 - y = \frac{3}{4}$

The optimal strategy for player A : (1/4, 3/4)

The optimal strategy for player B : (3/4, 1/4)

The value of the game $V = 0 \times \frac{1}{4} + 1 \times \frac{3}{4} = \frac{3}{4}$

Exercise 2.1 (in addition to text book exercises)

Find the optimum strategies for each player and the value of the games:

1-

		B		
		1	**2**	**3**
	1	−1	−2	8
A	**2**	7	5	−1
	3	6	0	12

2- Two breakfast food manufacturers, A and B are competing for an increased market share. The payoff matrix, represented in the following table, shows the increase in market share for A and decrease in in market share for B:

		B			
		Give gifts	**Decrease price**	**Maintain present strategy**	**Increase advertising**
A	**Give gifts**	2	−2	4	1
	Decrease price	6	1	12	3
	Maintain present strategy	−3	2	0	6
	Increase advertising	2	−3	7	1

Find the optimal strategies for both manufacturers and the value of the game.

2.7 Mixed Strategies for 2 x n or m x 2 Games

These are games in which one player has only two courses of action open to him while his opponent may have any number. If the game has no saddle point and cannot be reduced to a 2 x 2 game, it can be still solved by method of subgames or graphical method.

2.7.1 Method of Subgames for 2 x n or m x 2 Games

This method subdivides the given 2 x n or m x 2 game into a number of 2 x 2 games, each of which is then solved and then the optimal strategies are determined. If $k = n$ (for 2 x n games) or $k = m$ (for m x 2 games), then the number of subgames is: $\frac{k!}{2!(k-2)!}$.

Example (2.11):

Find the optimal strategy for each player and the value of the following game:

$$\begin{array}{cc|c|c|c|}
 & & & B & \\
 & & 1 & 2 & 3 \\ \hline
A & 1 & 275 & -50 & -75 \\ \hline
 & 2 & 125 & 130 & 150 \\ \hline
\end{array}$$

Solution:

First we search for a saddle point:

$$\begin{array}{cc|c|c|c|c}
 & & & B & & \\
 & & 1 & 2 & 3 & \textbf{min} \\ \cline{3-5}
A & 1 & 275 & -50 & -75 & -75 \\ \cline{3-5}
 & 2 & 125 & 130 & 150 & \enclose{circle}{125} \\ \cline{3-5}
 & \textbf{max} & 275 & \enclose{circle}{130} & 150 &
\end{array}$$

There is no saddle point and $125 \leq V \leq 130$. The game cannot be reduced. This game can be thought as three 2 x 2 games.

Subgame 1:

$$\begin{array}{cc|c|c|c} & & & B & \\ & & 1 & 2 & \textbf{min} \\ \hline A & 1 & 275 & -50 & -50 \\ \hline & 2 & 125 & 130 & \enclose{circle}{125} \\ \hline & \textbf{max} & 275 & \enclose{circle}{130} & \end{array}$$

There is no saddle point, then:

$$\begin{array}{cc|c|c|cc} & & & B & & \\ & & 1 & 2 & & \\ \hline A & 1 & 275 & -50 & 5 & 1/66 \\ \hline & 2 & 125 & 130 & 325 & 65/66 \\ \hline & & 180 & 150 & & \\ & & 36/66 & 30/66 & & \end{array}$$

The strategy for A: (1/66, 65/66)

The strategy for B: (36/66, 30/66, 0)

The value of the game: $V = \frac{275\times1+125\times65}{66} = 127.3$

Subgame 2:

$$\begin{array}{cc|c|c|c} & & & B & \\ & & 1 & 3 & \textbf{min} \\ \hline A & 1 & 275 & -75 & -75 \\ \hline & 2 & 125 & 150 & \enclose{circle}{125} \\ \hline & \textbf{max} & 275 & \enclose{circle}{150} & \end{array}$$

There is no saddle point, then:

$$\begin{array}{cc|c|c|cc} & & & B & & \\ & & 1 & 3 & & \\ \hline A & 1 & 275 & -75 & 25 & 1/15 \\ \hline & 2 & 125 & 150 & 350 & 14/15 \\ \hline & & 225 & 150 & & \\ & & 9/15 & 6/15 & & \end{array}$$

The strategy for A: (1/15, 14/15)

The strategy for B: (9/15, 0, 6/15)

The value of the game: $V = \frac{275\times1+125\times14}{15} = 135$

Subgame 3:

		B		
		2	**3**	**min**
A	**1**	−50	−75	−75
	2	130	150	130
	max	130	150	

There is a saddle point (2, 2), thus:

The strategy for A: (0, 1)

The strategy for B: (0, 1, 0)

The value of the game: $V = 130$

Since player B has the flexibility to play any two of the courses of action available to him, he will play those strategies for which his loss is minimum. AS the value of all subgames are positive, player A is the winner. Hence Player B will play subgame 1 for which the loss is minimum, i.e. 127.3. The complete solution of the problem is:

The optimal strategy for A: (1/66, 65/66)

The optimal strategy for B: (36/66, 30/66, 0)

The value of the game: $V = 127.3$

2.7.2 Graphical Method for 2 x n or m x 2 Games

Graphical method is applicable to only those games in which one of the players has two strategies only. The advantage of this method is that it is relatively fast. It reduces the 2 x n or m x 2 game to 2 x 2 game and the game can then be solved by the methods discussed earlier. The resulting solution is also the solution of the original problem.

Example (2.12):

Solve the game given in the following table:

		B			
		B₁	**B₂**	**B₃**	**B₄**
A	**A₁**	19	6	7	5
	A₂	7	3	14	6
	A₃	12	8	18	4
	A₄	8	7	13	−1

Solution:

First, we must search for a saddle point:

$$\begin{array}{cc|c|c|c|c|c}
 & & & & \mathbf{B} & & \\
 & & \mathbf{B_1} & \mathbf{B_2} & \mathbf{B_3} & \mathbf{B_4} & \mathbf{min} \\
\hline
 & \mathbf{A_1} & 19 & 6 & 7 & 5 & \boxed{5} \\
\hline
\mathbf{A} & \mathbf{A_2} & 7 & 3 & 14 & 6 & 3 \\
\hline
 & \mathbf{A_3} & 12 & 8 & 18 & 4 & 4 \\
\hline
 & \mathbf{A_4} & 8 & 7 & 13 & -1 & -1 \\
\hline
 & \text{max} & 19 & 8 & 18 & \boxed{6} &
\end{array}$$

There is no saddle point and $5 \leq V \leq 6$. Columns B$_1$ and B$_3$ are dominated by column B$_2$, then the reduced matrix will be:

$$\begin{array}{cc|c|c|}
 & & \mathbf{B} & \\
 & & \mathbf{B_2} & \mathbf{B_4} \\
\hline
 & \mathbf{A_1} & 6 & 5 \\
\hline
\mathbf{A} & \mathbf{A_2} & 3 & 6 \\
\hline
 & \mathbf{A_3} & 8 & 4 \\
\hline
 & \mathbf{A_4} & 7 & -1 \\
\hline
\end{array}$$

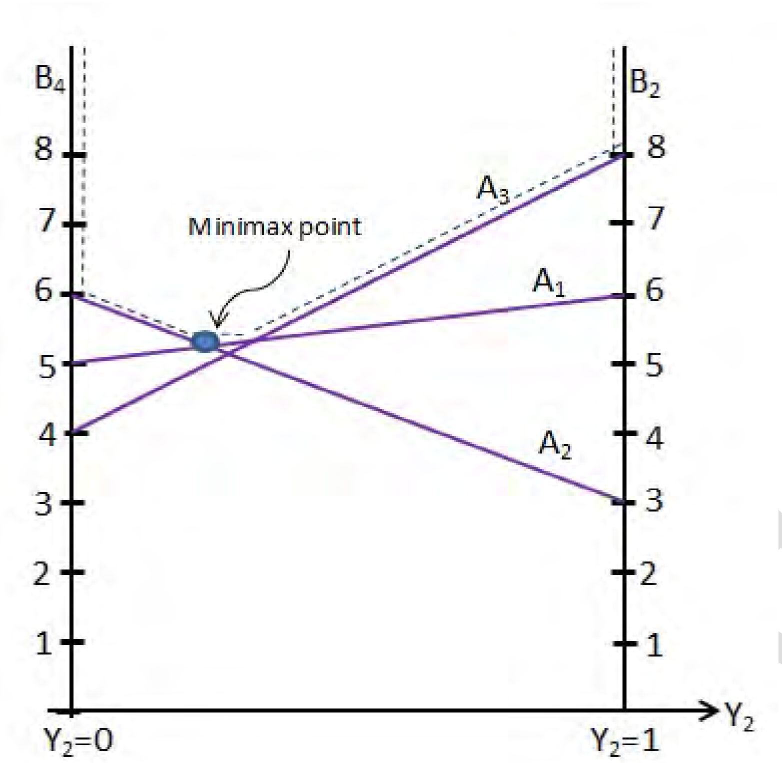
Row A$_3$ dominates row A$_4$ and the reduced matrix will be:

$$\begin{array}{cc|c|c|}
 & & \mathbf{B} & \\
 & & \mathbf{B_2} & \mathbf{B_4} \\
\hline
 & \mathbf{A_1} & 6 & 5 \\
\hline
\mathbf{A} & \mathbf{A_2} & 3 & 6 \\
\hline
 & \mathbf{A_3} & 8 & 4 \\
\hline
 & & y_2 & y_4 = 1 - y_2
\end{array}$$

Let A$_1$, A$_2$, and A$_3$ be the strategies which A mixes with probabilities x_1, x_2, and x_3 respectively and B$_2$, B$_4$ be the strategies which B mixes with probabilities y_2 and $y_4 = 1 - y_2$.When B adopts strategy B$_2$, $y_2 = 1$ and the probability with which he will adopt strategy B$_4$, i.e. $y_4 = 0$. B's expected Payoffs corresponding to A's pure strategies are given below:

A's pure strategies	**B's expected Payoffs**
A₁	$6y_2 + 5y_4 = 6y_2 + 5(1 - y_2) = y_2 + 5$
A₂	$3y_2 + 6y_4 = 3y_2 + 6(1 - y_2) = -3y_2 + 6$
A₃	$8y_2 + 4y_4 = 8y_2 + 4(1 - y_2) = 4y_2 + 4$

These three lines can be plotted as functions of y_2 as follows: draw two lines B$_2$ and B$_4$ parallel to each other one unit apart and mark a scale on each of them. To represent A's first strategy, A$_1$, join mark 5 on B$_4$ (when $y_2 = 0$) to 6 on B$_2$ (when $y_2 = 1$). Similarly for other A's strategies, A$_2$ and A$_3$, and bound the figure from above as shown since B is a minimization player.



Since player B wishes to minimize his maximum expected losses, the two lines which intersect at the lowest point of the upper bound show the two courses of action A should choose in his best strategy, i.e. A_1 and A_2. Thus, we can reduce the 3 x 2 game to the following 2 x 2 game which has no saddle point:

		B			
		B_2	**B_4**		
A	**A_1**	6	5	3	3/4
	A_2	3	6	1	1/4
		1	3		
		1/4	3/4		

The optimal strategies are: A (3/4, 1/4, 0, 0), B (0, 1/4, 0, 3/4)

The value of the game is: $V = \frac{6\times1+5\times3}{4} = \frac{21}{4}$

Example (2.13):

Solve the following 2 x 5 game:

				B		
		B_1	**B_2**	**B_3**	**B_4**	**B_5**
A	**A_1**	−5	5	0	−1	8
	A_2	8	−4	−1	6	−5

Solution:

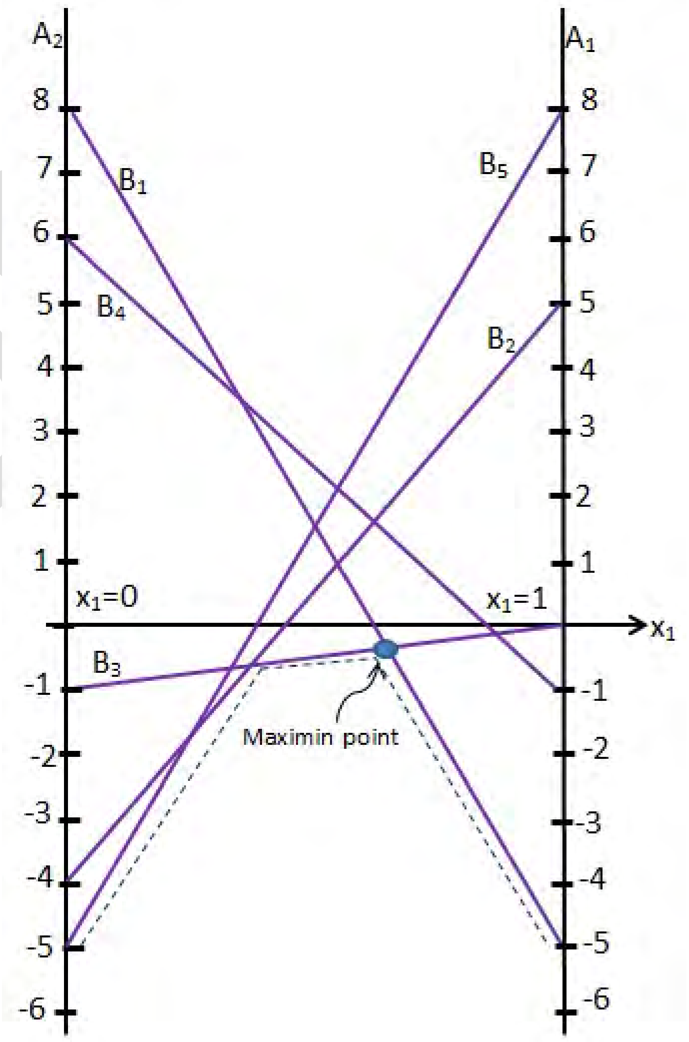
First, we must look for a saddle point; it does not exist in this problem.

$$\begin{array}{cc|c|c|c|c|c|cc}
 & & & & \mathbf{B} & & & & \\
 & & \mathbf{B_1} & \mathbf{B_2} & \mathbf{B_3} & \mathbf{B_4} & \mathbf{B_5} & \min & \\
\hline
\mathbf{A} & \mathbf{A_1} & -5 & 5 & 0 & -1 & 8 & -5 & x_1 \\
\hline
 & \mathbf{A_2} & 8 & -4 & -1 & 6 & -5 & -5 & x_2 = 1 - x_1 \\
\hline
 & \mathbf{max} & 8 & 5 & 0 & 6 & 8 & &
\end{array}$$

In this problem, the matrix cannot be reduced to a smaller matrix. The A's expected payoffs corresponding to B's pure strategies are:

B's pure strategies	A's expected payoffs
1	$-5x_1 + 8x_2 = -5x_1 + 8(1 - x_1) = -13x_1 + 8$
2	$5x_1 - 4x_2 = 5x_1 - 4(1 - x_1) = 9x_1 - 4$
3	$0x_1 - 1x_2 = -(1 - x_1) = x_1 - 1$
4	$-1x_1 + 6x_2 = -1x_1 + 6(1 - x_1) = -7x_1 + 6$
5	$8x_1 - 5x_2 = 8x_1 - 5(1 - x_1) = 13x_1 - 5$

The five lines can be plotted as a function of x_1 as follows: draw two lines A₁ and A₂ parallel to each other one unit apart and mark a scale on each of them. To represent B's first strategy, B₁, join mark 8 on A₂ (when $x_1 = 0$) to −5 on A₁ (when $x_1 = 1$) and so on. Bound the figure from below as shown since A is a maximization player.



Since player A wishes to maximize his minimum expected payoff, the two lines which intersect at the highest point of the lower bound show the two courses of action B should choose in his best strategy, i.e. B_1 and B_3. Thus, we can reduce the 2 x 5 game to the following 2 x 2 game which has no saddle point:

		B			
		B_1	**B_3**		
A	**A_1**	−5	0	9	9/14
	A_2	8	−1	5	5/14
		1	13		
		1/14	13/14		

The optimal strategies are: A (9/14, 5/14), B (1/14, 0, 13/14, 0, 0)

The value of the game is: $V = \frac{-5\times1+0\times13}{14} = \frac{-5}{14}$

Exercise 2.2 (in addition to text book exercises)

Solve the following game in two ways:

		B	
		1	2
	1	3	−1
	2	0	5
A	3	7	−2
	4	−3	4
	5	6	2

2.8 Mixed strategies for 3 x 3 Game – Method of Matrices

If the game has no saddle point and it reduced to a 3 x 3 matrix, the game can be solved by the method of matrices. The steps of this method are as follows:

Step 1: subtract 2nd row from the 1st and 3rd row from the 2nd and write down the values below the matrix.

Step 2: similarly, subtract each column from the column to its left (i.e. subtract 2nd column from the 1st and 3rd column from the 2nd) and write down the values to the right of the matrix.

Step 3: Calculate the oddments for A_1, A_2, A_3, B_1, B_2, and B_3. The oddment of each strategy is the determinant of the numbers calculated in steps 1 and 2 , after neglecting the strategy numbers. Write down these elements to the right and down the table, neglecting their signs.

Step 4: If the sum of the oddments of the players are equal, then there is a solution to the game; if not, this method fails.

Step 5: For each player calculate the probability in which he uses his strategies by dividing his oddments on the sum of oddments.

Example (2.14):

Solve the following game:

		B		
		1	**2**	**3**
	1	7	1	7
A	**2**	9	−1	1
	3	5	7	6

Solution:

		B			
		1	**2**	**3**	min
	1	7	1	7	1
A	**2**	9	−1	1	−1
	3	5	7	6	⑤
	max	9	⑦	7	

There is no saddle point and $5 \leq V \leq 7$. The matrix cannot be reduced, then:

		B				
		1	**2**	**3**		
	1	7	1	7	6	−6
A	**2**	9	−1	1	10	−2
	3	5	7	6	−2	1
		−2	2	6		
		4	−8	−5		

The oddments are:

$$\text{Oddment for } A_1 = \left|\begin{vmatrix} 10 & -2 \\ -2 & 1 \end{vmatrix}\right| = 6$$

$$\text{Oddment for } A_2 = \left|\begin{vmatrix} 6 & -6 \\ -2 & 1 \end{vmatrix}\right| = 6$$

$$\text{Oddment for } A_3 = \left|\begin{vmatrix} 6 & -6 \\ 10 & -2 \end{vmatrix}\right| = 48$$

$$\text{Oddment for } B_1 = \left|\begin{vmatrix} 2 & 6 \\ -8 & -5 \end{vmatrix}\right| = 38$$

$$\text{Oddment for } B_2 = \left|\begin{vmatrix} -2 & 6 \\ 4 & -5 \end{vmatrix}\right| = 14$$

$$\text{Oddment for } B_3 = \left|\begin{vmatrix} -2 & 2 \\ 4 & -8 \end{vmatrix}\right| = 8$$

sum of oddments for $A = 6 + 6 + 48 = 60$, sum of oddments for $B = 38 + 14 + 8 = 60$. Then:

			B			
		1	2	3		
	1	7	1	7	6	3/30
A	2	9	-1	1	6	3/30
	3	5	7	6	48	24/30
		38	14	8		
		19/30	7/30	4/30		

The optimal strategies are:

A (3/30, 3/30, 24/30), B (19/30, 7/30, 4/30)

The value of the game: $V = \frac{7\times3+9\times3+5\times24}{30} = \frac{168}{30} = \frac{28}{5}$

Exercise 2.3 (in addition to text book exercises)

Solve the following game:

			B	
		1	2	3
	1	1	-1	-1
A	2	-1	-1	3
	3	-1	2	-1

2.9 Method of Linear Programming

Game theory bears a strong relationship to linear programming, since every finite two-person zero-sum game can be expressed as a linear program and vice versa. Linear programming is the most general method of solving any two-person zero-sum game. Consider the following game:

				Player B			
		1	2	…	j	…	n
Player A	1	a_{11}	a_{12}	…	a_{1j}	…	a_{1n}
	2	a_{21}	a_{22}	…	a_{2j}	…	a_{2n}
	⋮	⋮	⋮		⋮		⋮
	i	a_{i1}	a_{i2}	…	a_{ij}	…	a_{in}
	⋮	⋮	⋮		⋮		⋮
	m	a_{m1}	a_{m2}	…	a_{mj}	…	a_{mn}

Let $p_1, p_2, \dots, p_m$ and $q_1, q_2, \dots, q_n$ be the probabilities by which A and B respectively select their strategies and let V be the value of the game. Consider the game from A's point of view, A is trying to maximize V, that is:

$$a_{11}p_1 + a_{21}p_2 + \cdots + a_{m1}p_m \geq V$$

$$a_{12}p_1 + a_{22}p_2 + \cdots + a_{m2}p_m \geq V$$

$$\vdots$$

$$a_{1n}p_1 + a_{2n}p_2 + \cdots + a_{mn}p_m \geq V$$
$$p_1 + p_2 + \cdots + p_m = 1$$
$$p_i \geq 0 \qquad i = 1,2,\dots,m$$

Since $V > 0$, then divide by V, the above system will be:

$$a_{11}\frac{p_1}{V} + a_{21}\frac{p_2}{V} + \cdots + a_{m1}\frac{p_m}{V} \geq 1$$
$$a_{12}\frac{p_1}{V} + a_{22}\frac{p_2}{V} + \cdots + a_{m2}\frac{p_m}{V} \geq 1$$
$$\vdots$$
$$a_{1n}\frac{p_1}{V} + a_{2n}\frac{p_2}{V} + \cdots + a_{mn}\frac{p_m}{V} \geq 1$$
$$\frac{p_1}{V} + \frac{p_2}{V} + \cdots + \frac{p_m}{V} = \frac{1}{V}$$
$$\frac{p_i}{V} \geq 0 \qquad i = 1,2,\dots,m$$

Let $x_i = \frac{p_i}{V}, i = 1,2,\dots,m$. Since A is trying to maximize *V*, i.e. minimize *1/V*, then let $Z = \frac{1}{V} = x_1 + x_2 + \cdots + x_m$, the LPP will be:

$$\begin{aligned}
&min \qquad Z = x_1 + x_2 + \cdots + x_m \\
&S.t. \qquad a_{11}x_1 + a_{21}x_2 + \cdots + a_{m1}x_m \geq 1 \\
&\qquad\qquad a_{12}x_1 + a_{22}x_2 + \cdots + a_{m2}x_m \geq 1 \\
&\qquad\qquad \vdots \\
&\qquad\qquad a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{mn}x_m \geq 1 \\
&\qquad\qquad x_i \geq 0 \qquad i = 1,2,\dots,m
\end{aligned}$$

In a similar way, we can write the LP model for the player B, which is, in fact, the dual of the LP model for player A. That is:

$$\begin{aligned}
&max \qquad W = y_1 + y_2 + \cdots + y_n \\
&S.t. \qquad a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \leq 1 \\
&\qquad\qquad a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \leq 1 \\
&\qquad\qquad \vdots \\
&\qquad\qquad a_{m1}y_1 + a_{m2}y_2 + \cdots + a_{mn}y_n \leq 1 \\
&\qquad\qquad y_j \geq 0 \qquad j = 1,2,\dots,n \qquad (\text{ where } y_j = \frac{q_j}{V}, j = 1,2,\dots,n)
\end{aligned}$$

By the duality principal, the optimal solution of one problem automatically yields the optimal solution of the other.

Example (2.15):

Use linear programming to solve the following game:

			B	
		1	**2**	**3**
	1	−1	1	1
A	**2**	2	−2	2
	3	3	3	−3

Solution:

			B		
		1	**2**	**3**	**min**
	1	−1	1	1	−1
A	**2**	2	−2	2	−2
	3	3	3	−3	−3
	max	3	3	2	

There is no saddle point , $-1 \leq V \leq 2$, and the game cannot be reduced to a smaller game. Player A's linear program:

$$\begin{aligned} &min && Z = x_1 + x_2 + x_3 \\ &S.t. && -x_1 + 2x_2 + 3x_3 \geq 1 \\ & && x_1 - 2x_2 + 3x_3 \geq 1 \\ & && x_1 + 2x_2 - 3x_3 \geq 1 \\ & && x_1, x_2, x_3 \geq 0 \end{aligned}$$

The dual of A's LP; which is B's linear program is:

$$\begin{aligned} &max && W = y_1 + y_2 + y_3 \\ &S.t. && -y_1 + y_2 + y_3 \leq 1 \\ & && 2y_1 - 2y_2 + 2y_3 \leq 1 \\ & && 3y_1 + 3y_2 - 3y_3 \leq 1 \\ & && y_1, y_2, y_3 \geq 0 \end{aligned}$$

The standard form of the last LPP (with modification in the objective function) is:

$$\begin{aligned} &max && W - y_1 - y_2 - y_3 = 0 \\ &S.t. && -y_1 + y_2 + y_3 + s_1 = 1 \\ & && 2y_1 - 2y_2 + 2y_3 + s_2 = 1 \\ & && 3y_1 + 3y_2 - 3y_3 + s_3 = 1 \\ & && y_1, y_2, y_3, s_1, s_2, s_3 \geq 0 \end{aligned}$$

Let $y_1 = y_2 = y_3 = 0$, then $s_1 = s_2 = s_3 = 1$

Basic Var.'s	y_1 ↓	y_2	y_3	s_1	s_2	s_3	**Solution**	
s_1	−1	1	1	1	0	0	1	
s_2	2	−2	2	0	1	0	1	1/2
← s_3	$\boxed{3}$	3	−3	0	0	1	1	1/3
W	−1	−1	−1	0	0	0	0	

Basic Var.'s	y_1	y_2	y_3 ↓	s_1	s_2	s_3	Solution
s_1	0	2	0	1	0	1/3	4/3
← s_2	0	−4	$\boxed{4}$	0	1	−2/3	1/3
y_1	1	1	−1	0	0	1/3	1/3
W	0	0 ↓	−2	0	0	1/3	1/3
← s_1	0	$\boxed{2}$	0	1	0	1/3	4/3
y_3	0	−1	1	0	1/4	−1/6	1/12
y_1	1	0	0	0	1/4	1/6	5/12
W	0	−2	0	0	1/2	0	1/2
y_2	0	1	0	1/2	0	1/6	2/3
y_3	0	0	1	1/2	1/4	0	3/4
y_1	1	0	0	0	1/4	1/6	5/12
W	0	0	0	1	1/2	1/3	11/6

$\Rightarrow W_{max} = Z_{min} = \frac{11}{6} \Rightarrow V = \frac{6}{11}. x_1 = 1, x_2 = \frac{1}{2}, x_3 = \frac{1}{3}$. Since $p_i = x_i V, i = 1,2,3, then: p_1 = x_1 V = 1 * \frac{6}{11} = \frac{6}{11}, p_2 = x_2 V = \frac{1}{2} * \frac{6}{11} = \frac{3}{11}, p_3 = x_3 V = \frac{1}{3} * \frac{6}{11} = \frac{2}{11}$.

$y_1 = \frac{5}{12}, y_2 = \frac{2}{3}, y_3 = \frac{3}{4}$. Since $q_j = y_j V, j = 1,2,3, then: q_1 = y_1 V = \frac{5}{12} * \frac{6}{11} = \frac{5}{22}, q_2 = y_2 V = \frac{2}{3} * \frac{6}{11} = \frac{8}{22}, q_3 = y_3 V = \frac{3}{4} * \frac{6}{11} = \frac{9}{22}$.

∴ The optimal strategy for player A: (6/11, 3/11, 2/11)

The optimal strategy for player B: (5/22, 8/22, 9/22)

The value of the game: V=6/11

Exercise 2.4 (in addition to text book exercises)

Solve the following games by linear programming:

		B		
		1	**2**	**3**
	1	0	2	2
A	**2**	3	−1	3
	3	4	4	−2

		B			
		1	**2**	**3**	**4**
	1	3	−2	1	4
A	**2**	2	3	−5	0
	3	−1	2	−2	2
	4	−3	−5	4	1