Ch. 2: Linear Programming Models Solution

2.1 History

The problem of solving a system of linear inequalities dates back at least as far as Fourier (1768–1830). The linear programming method was first developed by Leonid Kantorovich (1912-1986) in 1939. Leonid Kantorovich developed the earliest linear programming problems in 1939 for use during World War II. The method was kept secret until 1947 when George B. Dantzig (1914–2005) published the simplex method and John von Neumann (1903–1957) developed the theory of duality as a linear optimization solution, and applied it in the field of game theory. A larger theoretical and practical breakthrough in the field came in 1984 when Narendra Karmarkar(1957-) introduced a new interior-point method for solving linear-programming problems.

2.2 Linear Programming

Let us start by considering optimization problem

Definition (2.1):

An ***optimization problem*** (or ***mathematical programming problem***) is that branch of mathematics dealing with techniques for maximizing or minimizing an objective function subject to linear, nonlinear, and integer constraints. In other words, it is the problem of minimizing or maximizing the objective function $z = f(x_1, x_2, \dots, x_n)$ subject to the constraints:

$$g_1\ (x_1, x_2, \dots, x_n) \le l_1$$
$$\vdots$$
$$g_m\ (x_1, x_2, \dots, x_n) \le l_2$$

Definition (2.2):

Linear programming (***LP*** or ***linear optimization***) is a mathematical technique for the optimization (maximization or minimization) of a linear objective function subject to a set of linear constraints.

The objective function may be profit, cost, production capacity or any other measure of effectiveness.

2.3 Conditions for a Linear Programming Problem

1- There must be a well-defined objective function which is to be either maximized or minimized and which can be expressed as a linear function of decision variables.

2- There must be constraints or limitations on the available resources. These constraints must be capable of being expressed as linear equations or inequalities in terms of decision variables.
3- There must be alternative course of action. For example, many grades of row material may be available.
4- Since the negative values of (any) physical quantity has no meaning, therefore all the variables must assume non-negative values.
5- Linear programming assumes the presence of a finite number of activities and constraints without which it is not possible to obtain the best or optimal solution.

2.4 The General Linear Programming Problem

The general linear programming problem can be expressed as follows:
Find the values of variables $x_1, x_2, \dots, x_n$ which maximize (or minimize) an objective function Z, i.e.

$$\left.\begin{array}{ll} Optimize \quad Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n & \dots(1) \\ Subject\ to: & \\ \left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n\ (\le, =, \ge)\ b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n\ (\le, =, \ge)\ b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n\ (\le, =, \ge)\ b_m \end{array}\right\} & \dots(2) \\ x_1, x_2, \dots, x_n \ge 0 \quad \dots\ (3) & \end{array}\right\} \dots LPP$$

The above formulation may be put in the following compact form by using the summation sign:

$$\left.\begin{array}{llll} & max.(\,or\ min.) \quad Z = \sum_{j=1}^{n} c_jx_j & & \dots(1\,a) \\ S.t & \sum_{j=1}^{n} a_{ij}x_j\ (\le, =, \ge)\ b_i & i = 1,2,\dots,m & \dots(2\,a) \\ & x_j \ge 0 \quad j = 1,2,\dots,n & & \dots.(3\,a) \end{array}\right\} \dots LPP$$

The variables $x_j\ (j = 1, \dots, n)$ are called ***decision variables***, c_j, a_{ij}, and $b_i\ (i = 1, \dots, m; j = 1, \dots, n)$ are constants determined from the statement of the problem. The constants $c_j\ (j = 1, \dots, n)$ are called ***cost coefficients***, constants $b_i\ (i = 1, \dots, m)$ are called ***stipulations***, and constants $a_{ij}\ (i = 1, \dots, m; j = 1, \dots, n)$ are called ***structural coefficients***. The system consisting of the ***objective function*** (1), the ***constraints*** (2), and the ***non-negativity condition*** (3) is called ***linear programming problem*** (***LPP***).

2.5 Standard Form of LPP

After formulating LPP, the next step is to obtain its solution. But before any analytic method is used to obtain the solution, the problem must be available in a form. One of these forms is the standard form.

The standard form is used to develop the general procedure for solving any LPP. The main characteristics of the standard from are:

1- All variables are non-negative.
2- The right-hand side of each constraint is non-negative.
3- Objective function may be of maximization or minimization type.
4- All constraints are expressed as equations. The inequality constraint can be changed to equality by adding or subtracting a non-negative variable from the left-hand side of such constraint. These new variables are called ***slack variables*** or simple ***slack***. They are added if the constraint is ($\leq$) and subtracted if the constraint is ($\geq$). Since in the case of ($\geq$) constraints the subtracted variables represent the surplus of left-hand side over right-hand side, it is commonly known as ***surplus variables***. Both decision variables as well as the slack and surplus variables are called the ***admissible variables***.

Remark (2.1)

1- The minimization of a function, $f(x)$, is equivalent to the maximization of $-f(x)$.
2- An inequality constraint of ($\geq$) type can be changed to an inequality constraint of ($\leq$) type by multiplying both sides of the inequality by -1.
3- An equation may be replaced by two weak inequalities in opposite directions.
4- It is possible, in actual practice, that a variable may be unconstrained (unrestricted) in sign, i.e. it may be positive, zero or negative. If a variable is unconstrained, it is expressed as the difference between two non-negative variables.

Example (2.1):

$min\ Z = 3x_1 - 4x_2 + 9x_3$ is equivalent to: $max\ G = -Z = -3x_1 + 4x_2 - 9x_3$

Example (2.2):

The equation $x_1 + 3x_2 = 5$ is equivalent to the two simultaneous constraints: $x_1 + 3x_2 \leq 5$ and $x_1 + 3x_2 \geq 5$

Or $x_1 + 3x_2 \leq 5$ and $-x_1 - 3x_2 \leq -5$.

Example (2.3):

If x is unconstrained variable, then it can be expressed as: $x = x' - x''$ where $x', x'' \geq 0$.

Example (2.4):

Express the following LPP in the standard form

$$\begin{aligned} &min && Z = 7x_1 + 5x_2 \\ &S.t. && 3x_1 + 4x_2 \leq 17 \\ &&& x_1 + x_2 \geq 20 \\ &&& x_1 , x_2 \geq 0 \end{aligned}$$

Solution:

The standard form of the above LPP is:

$$\begin{aligned} &min && Z = 7x_1 + 5x_2 \\ &S.t. && 3x_1 + 4x_2 + s_1 = 17 \\ &&& x_1 + x_2 - s_2 = 20 \\ &&& x_1 , x_2 , s_1 , s_2 \geq 0 \end{aligned}$$

Example (2.5):

Express the following LPP in the standard form

$$\begin{aligned} &max && Z = 2x_1 + 5x_2 + 3x_3 \\ &S.t. && 2x_1 + 7x_2 \leq 9 \\ &&& x_1 + 3x_2 + 4x_3 \geq 7 \\ &&& 5x_1 - x_3 \leq 2 \\ &&& x_1 , x_2 \geq 0 \end{aligned}$$

Solution:

Here x_3 is unrestricted, so let $x_3 = x_4 - x_5$, where $x_4 , x_5 \geq 0$ .Thus the above problem will be:

$$\begin{aligned} &max && Z = 2x_1 + 5x_2 + 3x_4 - 3x_5 \\ &S.t. && 2x_1 + 7x_2 \leq 9 \\ &&& x_1 + 3x_2 + 4x_4 - 4x_5 \geq 7 \\ &&& 5x_1 - x_4 + x_5 \leq 2 \\ &&& x_1 , x_2 , x_4 , x_5 \geq 0 \end{aligned}$$

Introducing the slack variables, the standard form is:

$$\begin{aligned} &max && Z = 2x_1 + 5x_2 + 3x_4 - 3x_5 \\ &S.t. && 2x_1 + 7x_2 + s_1 = 9 \\ &&& x_1 + 3x_2 + 4x_4 - 4x_5 - s_2 = 7 \\ &&& 5x_1 - x_4 + x_5 + s_3 = 2 \end{aligned}$$

$$x_1, x_2, x_4, x_5, s_1, s_2, s_3 \geq 0$$

Remark (2.2):

The slack and surplus variables yielding zero profit (or incurring zero cost).

Remark (2.3):

Linear programs are problems that can be expressed in matrix form:

$$\begin{array}{ll} max.(\,or\ min.) & Z = cX \\ S.t. & AX\ (\leq, =, \geq) b \\ And & X \geq \mathbf{0} \end{array}$$

Where $X_{n\times 1}$ represents the vector of variables (to be determined), $c_{1\times n}$ and $b_{m\times 1}$ are vectors of known coefficients, $A_{m\times n}$ is a (known) matrix of coefficients, $\mathbf{0}_{n\times 1}$ is the zero vector.

Example (2.6):

Express the following LPP in the standard matrix form

$$\begin{array}{ll} min & Z = 3x_1 + 4x_2 - 2x_3 \\ S.t. & 4x_1 + 5x_2 - x_3 \geq 5 \\ & 2x_1 - 7x_2 + x_3 \leq 9 \\ & x_1 + 5x_3 = 8 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Solution:

The standard form of the LPP is:

$$\begin{array}{ll} min & Z = 3x_1 + 4x_2 - 2x_3 + 0.x_4 + 0.x_5 \\ S.t. & 4x_1 + 5x_2 - x_3 - x_4 = 5 \\ & 2x_1 - 7x_2 + x_3 + x_5 = 9 \\ & x_1 + 5x_3 = 8 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \quad (x_4, x_5 \text{ are slack variables}) \end{array}$$

Thus, the given problem in the matrix form is:

$$\begin{array}{ll} \min & Z = cX \\ S.t. & AX = b \\ & X \geq \mathbf{0} \end{array}$$

Where: $X = [x_1, x_2, x_3, x_4, x_5]^T$, $c = [3, 4, -2, 0, 0]$, $b = [5,9,8]^T$,and

$$A = \begin{bmatrix} 4 & 5 & -1 & -1 & 0 \\ 2 & -7 & 1 & 0 & 1 \\ 1 & 0 & 5 & 0 & 0 \end{bmatrix}.$$

2.6 Some Important Definitions

Consider the general LPP defined in (2.4):

Definition (2.3):

$x_j \ (j = 1, 2, \dots, n)$ is a ***solution*** of the general linear programming problem if it satisfies the constraints (2).

Definition (2.4):

$x_j \ (j = 1, 2, \dots, n)$ is a ***feasible solution*** of the general linear programming problem if it satisfies the conditions (2) and (3) .

Definition (2.5):

The solution obtained by setting $n - m$ variables equal to zero and solving for the values of the remaining m variables is called a ***basic solution***. These m variables (some of them may be zero) are called ***basic variables*** (***BV***) and each of the remaining $n - m$ that have been put equal to zero is called ***non-basic variable*** (***NBV***).

Definition (2.6):

A ***basic feasible solution*** (***BFS***) is a basic solution that satisfies the non-negativity restriction (3).

Definition (2.7):

A basic feasible solution is said to be ***non-degenerate*** if it has exactly m positive (non-zero) x_j . The solution, on the other hand, is ***degenerate*** if one or more of the m basic variables are equal to zero.

Definition (2.8):

A basic feasible solution is said to be ***optimal*** (or ***optimum***) if it is also optimize the objective function (1).

Definition (2.9):

If the value of the objective function can be increased or decreased indefinitely, the solution is called ***unbounded solution***.

Definition (2.10):

A set (of points) $\boldsymbol{S}$ is said to be a ***convex set*** if for any two points in the set, the line segment joining these points lies entirely in the set.

Example (2.7):

Figure (2.1) represents convex and non-convex sets.

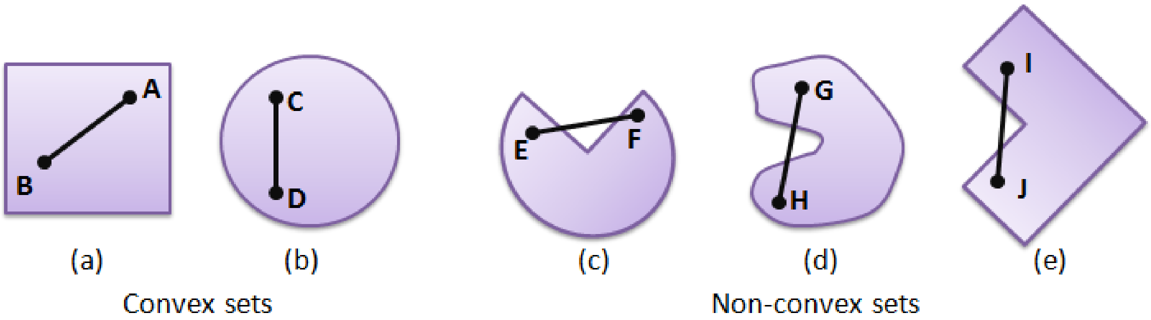

Figure (2.1)

2.7 Formation of LPP

First, the given problem must be presented in LP form. This requires defining the variables of the problem, establishing inter-relationships between them, and formulating the objective function and constraints. A model which approximates as closely as possible to the given problem is then developed. If some constraints happen to be non-linear, they are approximated the appropriate linear functions to fit the LP format. To formulate an LP model:

Step 1: From the study of the situation find the key-decision to be made.

Step 2: Assume symbols for variable quantities noticed in step 1.

Step 3: Express the objective function as a linear function of variables in step 2.

Step 4: List down all the constraints.

Step 5: Presenting the problem.

Example (2.8):

A furniture manufacturer produces tables and chairs. Both the products must be processed through two machines M1 and M2. The total machine hours available are: 200 hours of M1 and 400 hours of M2 respectively. Time in hours required for producing a chair and a table on both machines is as follows:

Time in hours		
Machine	**Table**	**Chair**
M1	7	4
M2	5	9

Profit from the sale of table is 20 \$ and that from a chair is 15 \$. Formulate LP model for the problem to maximize the total profit.

Solution:

For this example:

Step 1: The key-decision is to decide the number of tables and chairs produced to maximize the total profit.

Step 2: Let x_1=no. of tables produced

x_2=no. of chairs produced

Step 3: The objective function is to maximizing the profit. Since profit per unit from a table is 20 $ and a chair is 15 $, then the objective function is:

$$max \quad Z = 20x_1 + 15x_2$$

Step 4: The constraints are:

i) Total time on machine M1 cannot exceed 200 hours. Since it takes 7 hours to produce a table and 4 hours to produce a chair on machine M1, then:

$$7x_1 + 4x_2 \le 200$$

ii) Total time on machine M2 cannot exceed 400 hours. Since it takes 5 hours to produce a table and 9 hours to produce a chair on machine M1, then:

$$5x_1 + 9x_2 \le 400$$

Step 5: The LP model is:

$$\begin{aligned} max \quad & Z = 20x_1 + 15x_2 \\ \text{S.t} \quad & 7x_1 + 4x_2 \le 200 \\ & 5x_1 + 9x_2 \le 400 \\ & x_1 , x_2 \ge 0 \end{aligned}$$

Since if $x_1 \le 0$ and $x_2 \le 0$ it means that negative quantities of products are being manufactured which has no meaning.

Example (2.9) (Diet problem):

A person wants to decide the components of a diet which will fulfill his daily requirements of proteins, fats, and carbohydrates at the minimum cost. The choice is to be made from four different types of foods. The yields per unit of these foods are given in the following table:

Food type	Yield per unit			Cost per unit (ID)
	Proteins	Fats	Carbohydrates	
1	3	2	6	45
2	4	2	4	40
3	8	7	7	85
4	6	5	4	65
Minimum requirement	800	200	700	

Formulate linear programming model for the problem.

Solution:

Let $x_1, x_2, x_3,$ and x_4 denote the number of units of food of type 1, 2, 3, and 4 respectively.

The LP model is:

$$\begin{aligned} min \quad & Z = 45x_1 + 40x_2 + 85x_3 + 65x_4 \\ S.t. \quad & 3x_1 + 4x_2 + 8x_3 + 6x_4 \geq 800 \\ & 2x_1 + 2x_2 + 7x_3 + 5x_4 \geq 200 \\ & 6x_1 + 4x_2 + 7x_3 + 4x_4 \geq 700 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Example (2.10):

An advertising company wishes to plan its advertising strategy in three different media-television, radio and magazines. The purpose of advertising is to reach as large as potential customers as possible. Following data have been obtained from market survey (cost in ID):

	Television	Radio	Magazine I	Magazine II
Cost of an advertising unit	30000	20000	15000	10000
No. of potential customers reached per unit	200000	600000	150000	100000
No. of female customers reached per unit	150000	400000	70000	50000

The company wants to spend no more than 450000 ID on advertising. Following are the further requirements that must be met:

i) At least 1 million exposures take place among female costumers.

ii) Advertising on magazines be limited to 150000 ID.

iii) At least 3 advertising units are bought on magazine I and 2 units on magazine II.

iv) The number of advertising units on television and radio should each be between 5 and 10.

Formulate an LP model for the problem.

Solution:

Step 1: The company wants to maximize the number of potential customers reached.

Step 2: Let $x_1, x_2, x_3,$ and x_4 denote the number of advertising units to be bought on television, radio, magazine I, and magazine II.

Step 3: The objective function is:

$max \quad Z = 10^5(2x_1 + 6x_2 + 1.5x_3 + x_4)$

Step 4: The constraints are:

On advertising:

Budget: $30000x_1 + 20000x_2 + 15000x_3 + 10000x_4 \leq 450000$

Or $30x_1 + 20x_2 + 15x_3 + 10x_4 \leq 450$

On number of females:

Customers reached by the advertising

Campaign: $150000x_1 + 400000x_2 + 70000x_3 + 50000x_4 \geq 1000000$

Or $15x_1 + 40x_2 + 7x_3 + 5x_4 \geq 100$

On expenses:

Magazine advertising: $15000x_3 + 10000x_4 \leq 150000$

Or $15x_3 + 10x_4 \leq 150$

On no. of units on magazine: $x_3 \geq 3$

$x_4 \geq 2$

On no. of units on television: $5 \leq x_1 \leq 10$ or $x_1 \geq 5$, $x_1 \leq 10$

On no. of units on radio: $5 \leq x_2 \leq 10$ or $x_2 \geq 5$, $x_2 \leq 10$

Where $x_1 , x_2, \ x_3, \ x_4 \geq 0$

Step 5: The LP model is:

$$
\begin{aligned}
max \quad & Z = 10^5(2x_1 + 6x_2 + 1.5x_3 + x_4) \\
S.t. \quad & 30x_1 + 20x_2 + 15x_3 + 10x_4 \leq 450 \\
& 15x_1 + 40x_2 + 7x_3 + 5x_4 \geq 100 \\
& 15x_3 + 10x_4 \leq 150 \\
& x_1 \geq 5 \\
& x_1 \leq 10 \\
& x_2 \geq 5 \\
& x_2 \leq 10 \\
& x_3 \geq 3 \\
& x_4 \geq 2 \\
& x_1 , x_2, \ x_3, \ x_4 \geq 0
\end{aligned}
$$

Example (2.11):

A company has two grades of inspectors, I and II to undertake quality control inspection. At least 1500 pieces must be inspected in an 8-hour day. Grade I inspector can check 20 pieces in an hour with an accuracy of 96%. Grade II inspector can check 14 pieces in an hour with an accuracy of 92%. Wages of grade I inspector is 5 \$ per hour while those of grade II inspector is 4 \$ per hour. Any error made by an inspector cost 3 \$ to the company. If there are, in all, 10 grade I inspectors and 15 grade II inspectors in the company, formulate an LP model to minimize the daily inspection cost.

Solution:

Let x_1=no. of grade I inspectors

x_2=no. of grade II inspectors

Objective is to minimize the daily inspection cost. The company has to incur two types of costs: wages paid to the inspectors and the cost of their inspection error.

The cost of grade I inspector= 5+3x0.04x20 =7.40 \$

The cost of grade II inspector= 4+3x0.08x14 =7.36 \$

∴ The objective function is:

$$min \quad Z = 8(7.40x_1 + 7.36x_2) = 59.20x_1 + 58.88x_2$$

Constraints are:

On the number of grade I inspector: $x_1 \le 10$

On the number of grade II inspector: $x_2 \le 15$

On the number of pieces to be inspected daily: $20 \times 8x_1 + 14 \times 8x_2 \ge 1500$

Or $160x_1 + 112x_2 \ge 1500$

Where $x_1, x_2 \ge 0$

The LP model is:

$$\begin{aligned} min \quad & Z = 59.20x_1 + 58.88x_2 \\ S.t. \quad & 160x_1 + 112x_2 \ge 1500 \\ & x_1 \le 10 \\ & x_2 \le 15 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Example (2.12) (Blending problem):

A firm produces an alloy having the following specifications:

i) Specific gravity ≤ 0.98 ii) chromium ≥ 8% iii) melting point ≥ 450° C.

Raw materials A, B, and C having the properties shown in the following table can be used to make the alloy:

property	Properties of raw material		
	A	B	C
Specific gravity	0.92	0.97	1.04
Chromium	7%	13%	16%

Melting point	440°C	490°C	480°C

Costs of the various raw materials per ton are: 90 $ for A, 280 $ for B, and 40 $ for C. Formulate the LP model to find the properties in which A, B, and C be used to obtain an alloy of desired properties while the cost of raw materials is minimum.

Solution:

Let $x_1, x_2,$ and x_3 denote the percentage contents of raw materials A, B, and C to be used for making the alloy.

The LP model is:

$$min \quad Z = 90x_1 + 280x_2 + 40x_3$$

$$S.t \quad 0.92x_1 + 0.97x_2 + 1.04x_3 \leq 0.98$$

$$7x_1 + 13x_2 + 16x_3 \geq 8$$

$$440x_1 + 490x_2 + 480x_3 \geq 450$$

$$x_1 + x_2 + x_3 = 100$$

$$x_1, x_2, x_3 \geq 0$$

Example (2.13):

An investment company wants to invest up to 10 million ID into various bonds. The management is currently considering four bonds, the detail on return and maturity of which are as follows:

Bond	Type	Return	Maturity time
A	Government	22%	15 years
B	Government	18%	5 years
C	Industrial	28%	20 years
D	Industrial	16%	3 years

The company has decided to put less than half of its investment in the government bonds and that the average age of the portfolio should not be more than 6 years. The investment should maximize the return on investment, subject to the above restrictions. Formulate the LP model.

Solution:

Let $x_1, x_2, x_3,$ and x_4 denote the amount to be invested in bonds A, B, C, and D respectively.

The objective function is: $max \quad Z = 0.22x_1 + 0.18x_2 + 0.28x_3 + 0.16x_4$

Subject to the constraints:

$$x_1 + x_2 + x_3 + x_4 \leq 10^7$$

$$x_1 + x_2 \leq 5 \times 10^6$$

$\frac{15x_1+5x_2+20x_3+3x_4}{x_1+x_2+x_3+x_4} \le 6 \quad \Longrightarrow \quad 15x_1 + 5x_2 + 20x_3 + 3x_4 \le 6x_1 + 6x_2 + 6x_3 + 6x_4 \Longrightarrow 15x_1 - 6x_1 + 5x_2 - 6x_2 + 20x_3 - 6x_3 + 3x_4 - 6x_4 \le 0 \Longrightarrow 9x_1 - x_2 + 14x_3 - 3x_4 \le 0$

Then the LP model will be:

$$\begin{aligned} max \quad & Z = 0.22x_1 + 0.18x_2 + 0.28x_3 + 0.16x_4 \\ \text{S.t.} \quad & x_1 + x_2 + x_3 + x_4 \le 10^7 \\ & x_1 + x_2 \le 5 \times 10^6 \\ & 9x_1 - x_2 + 14x_3 - 3x_4 \le 0 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

Exercises 2.1 (In addition to the text book exercises)

1. A paper mill produces rolls of papers used in making cash registers. Each roll of paper is 100 m length and can be used in width of 3,4,6, and 10 cm. The company's production process results in rolls that are 24 cm in width. Thus, the company must cut its 24 cm roil to the desired width. It has six basic cutting alternatives as follows:

Cutting alternatives	Width of rolls (cm)				Waste (cm)
	3	4	6	10	
1	4	3	---	---	---
2	---	3	2	---	---
3	1	1	1	1	1
4	---	---	2	1	2
5	---	4	1	---	2
6	3	2	1	---	1

The minimum demand for the four rolls is as follows:

Roll width (cm)	Demand
3	2000
4	3600
6	1600
10	500

The paper mill wishes to minimize the waste resulting from trimming to size. Formulate the LP model.

2. A manufacturer of biscuits is considering four types of gift-packs containing three types of biscuits: orange cream (o.c.), chocolate cream (c.c.) and wafers (w.). Market research conducted to assess the preferences of the consumers shows the following types of assortments to be in good demand:

Assortment	Contents	Selling price/kg(thousands of ID)
A	Not less than 40% of o.c., not more than 20% of c.c.	20
B	Not less than 20% of o.c., not more than 40% of c.c	25
C	Not less than 50% of o.c., not more than 10% of c.c	22
D	No restriction	12

For the biscuits, the manufacturing capacity and costs are given below:

Biscuit variety	Plan200t capacity200(kg/day)	Manufacturing cost(thousands of ID)
o.c.	200	8
c.c.	200	9
w	150	7

Formulate the LP model to find the production schedule which maximizes the profit assuming that there are no market restrictions.

3. A farmer has 1000 acres (1 acre≈4047 m^2) of land on which he can grow corn, wheat or soya bean. Each acre of corn costs 100 $ for preparation, requires 7 man-days of work and yields a profit of 30 $. An acre of wheat costs 120 $ to prepare, requires 10 man-days of work and yields a profit of 40 $. An acre of soya bean costs 70 $ to prepare, requires 8 man-days of work and yields a profit of 20 $. If the farmer has 100000 $ for preparation, and can count 8000 man-days work, formulate the LP model to allocate the number of acres to each crop to maximize the total profit.

Methods of Solutions for LPP:

2.8 Graphical Solution of Linear Programming Models

The solution of a LPP with only two variables can be derived using a graphical method. The graphical procedure consists of the following steps:

Step 1: represent the given problem in mathematical form.

Step 2: Draw the x_1 and the x_2 –axes. The non-negativity restrictions imply that the values of the variables x_1 and x_2 can lie only in the first quadrant.

Step 3: Plot each of the constraints on the graph. First replace each inequality with an equation and then graph the resulting straight line by locating two distinct points on it. Simply we can take the points of intersection with the x_1 and the x_2 –axes.

Step 4: Each inequality (constraint) will divide the $(x_1,\ x_2)$- plane into two half-spaces, one on each side of the graphed plane. Only one of these two halves satisfy the inequality. To determine the correct side, choose (0, 0) as a ***reference point*** (you can choose any other point). If it satisfies the inequality, then the side in which it lies is the feasible half-space, otherwise the other side is. Find the feasible region of each constraint, the ***feasible solution space*** of the problem represents the area in the first quadrant in which all the constraints are satisfied simultaneously.

Step 5: Determine the optimal solution. For this, plot the objective function by assuming $Z = 0$. This will be a line passing through the origin (drawn as a dotted line). As the value of Z is increased from zero, the dotted line moves to the right, parallel to itself until it passes through a corner of the feasible solution space in which the value of the objective function is optimized.

Alternatively use the ***extreme point enumeration approach***. For this, find the coordinates of each extreme point (or corner point or vortex) of the feasible region. Find the value of the objective function at each extreme point. The point at which objective function is maximum/ minimum is the optimal point and its coordinates give the optimal solution.

Example (2.14):

A firm has two bottling plants, one located at Baghdad and, other at Erbil. Each plant produces three drinks, Coca-Cola, Fanta, and Seven-up, named A, B, and C respectively. The number of bottles produced per day is, as follows:

	Plant at	
	Baghdad (Bg)	**Erbil (E)**
Coca-Cola	15000	15000
Fanta	30000	10000
Seven-up	20000	50000

A market survey indicates that, during the month of April, there will be a demand of 200000 bottles of Coca-Cola, 400000 bottles of Fanta, and 440000 bottles of Seven-up. The operating cost per day for plants at Baghdad and Erbil is 600 and 400 monetary units respectively. For how many days each plant is run in April to minimize the production cost, while still meeting the market demand.

Solution:

Let x_1= the number of running days of the planet at Baghdad.

x_2= the number of running days of the planet at Erbil.

The LP model is:

$$\left.\begin{aligned} &min \quad Z = 600x_1 + 400x_2 \\ &S.t. \quad 15000x_1 + 15000x_2 \geq 200000 \\ &\qquad\;\; 30000x_1 + 10000x_2 \geq 400000 \\ &\qquad\;\; 20000x_1 + 50000x_2 \geq 440000 \\ &\qquad\;\; x_1, x_2 \geq 0 \end{aligned}\right\} \quad \text{Or} \quad \begin{aligned} &min \quad Z = 600x_1 + 400x_2 \\ &S.t. \quad 15x_1 + 15x_2 \geq 200 \\ &\qquad\;\; 3x_1 + x_2 \geq 40 \\ &\qquad\;\; 2x_1 + 5x_2 \geq 44 \\ &\qquad\;\; x_1, x_2 \geq 0 \end{aligned}$$

Then:

For $15x_1 + 15x_2 = 200 \Rightarrow$ if $x_1 = 0$ then $x_2 = \frac{200}{15} = 13.3 \Rightarrow$ (0, 13.3) is the intersection point with the $x_2 -$ axis.

And if $x_2 = 0$ then $x_1 = \frac{200}{15} = 13.3 \Rightarrow$ (13.3, 0) is the intersection point with the $x_1 -$ axis.

For $3x_1 + x_2 = 40 \Rightarrow$ if $x_1 = 0$ then $x_2 = \frac{40}{1} = 40 \Rightarrow$ (0, 40) is the intersection point with the $x_2 -$ axis.

And if $x_2 = 0$ then $x_1 = \frac{40}{3} = 13.3 \Rightarrow$ (13.3, 0) is the intersection point with the $x_1 -$ axis.

For $2x_1 + 5x_2 = 44 \Rightarrow$ if $x_1 = 0$ then $x_2 = \frac{44}{5} = 8.8 \Rightarrow$ (0, 8.8) is the intersection point with the $x_2 -$ axis.

And if $x_2 = 0$ then $x_1 = \frac{44}{2} = 22 \Rightarrow$ (22, 0) is the intersection point with the $x_1 -$ axis.

The graphical representation is:

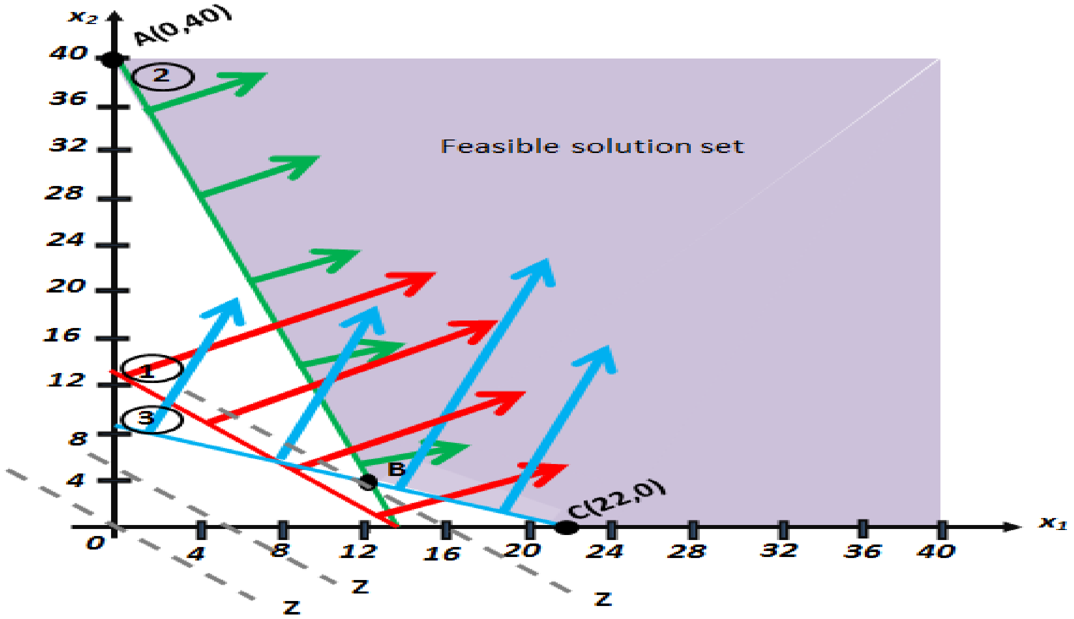

Figure (2.2)

The point B is resulting from the intersection of the lines representing the second and the third constraints, so we use these constraints to find the coordinates of B.

$$\left.\begin{array}{l}3x_1 + x_2 = 40\\ 2x_1 + 5x_2 = 44\end{array}\right\} \xrightarrow{5\times(2)} - \begin{array}{l}15x_1 + \cancel{5x_2} = 200\\ \mp 2x_1 \mp \cancel{5x_2} = \mp 44\end{array}$$

$$\Rightarrow 13x_1 = 156 \Rightarrow x_1 = 12 \overset{(2)}{\Rightarrow} x_2 = \frac{400-360}{1} = 4$$

The feasible solution region space is the shaded area in figure (2.2) whose corners are the points $A(0,40)$, $B(12,4)$, and $C(22,0)$ (i.e. any point in or at the boundary of the shaded region).

Corner	Value of Z
$A(0,40)$	$Z = 600 \times 0 + 400 \times 40 = 16000$
$B(12,4)$	$Z = 600 \times 12 + 400 \times 4 = 8800$ *
$C(22,0)$	$Z = 600 \times 22 + 400 \times 0 = 13200$

From the table, we see that the optimum occurs at corner B, then $x_1 = 12$ days, $x_2 = 4$ days, and $Z_{min} = 8800$ monetary units.

Note that the first constraint: $15000x_1 + 15000x_2 \geq 200000$ does not affect the solution space, such a constraint is called a ***redundant constraint***.

Example (2.15):

In one of the stages of production, a carpets company cuts lengths of carpet after its production in another department of the company. After cutting lengths by special machines to certain lengths, the lengths are folded in the form of rolls and then wrapped by certain substances for the purpose of selling in the markets. The following table shows the data for the two types of carpets A and B:

Production department	Product A	Product B	Available time
Cutting	8	6	2200
Folding	4	9	1800
Wrapping	1	2	400
Profit per length unit	12	8	

Each product must pass through the three mentioned stages. Determine the number of units produced to maximize the profit.

Solution:

Let x_1= the number units produced of A.

x_2= the number units produced of B.

The LP model is:

$$\begin{aligned}
max \quad & Z = 12x_1 + 8x_2 \\
S.t. \quad & 8x_1 + 6x_2 \leq 2200 \\
& 4x_1 + 9x_2 \leq 1800 \\
& x_1 + 2x_2 \leq 400 \\
& x_1, x_2 \geq 0
\end{aligned}$$

Then:

For $8x_1 + 6x_2 = 2200 \Rightarrow$ if $x_1 = 0$ then $x_2 = \frac{2200}{6} = 366.7 \Rightarrow$ (0, 366.7) is the intersection point with the $x_2 -$ axis.

And if $x_2 = 0$ then $x_1 = \frac{2200}{8} = 275 \Rightarrow$ (275, 0) is the intersection point with the $x_1 -$ axis.

For $4x_1 + 9x_2 = 1800 \Rightarrow$ if $x_1 = 0$ then $x_2 = \frac{1800}{9} = 200 \Rightarrow$ (0,200) is the intersection point with the $x_2 -$ axis.

And if $x_2 = 0$ then $x_1 = \frac{1800}{4} = 450 \Rightarrow$ (450, 0) is the intersection point with the $x_1 -$ axis.

For $x_1 + 2x_2 = 400 \Rightarrow$ if $x_1 = 0$ then $x_2 = \frac{400}{2} = 200 \Rightarrow$ (0, 200) is the intersection point with the $x_2 -$ axis.And if $x_2 = 0$ then $x_1 = 400 \Rightarrow$ (400,0) is the intersection point with the $x_1 -$ axis.The graphical representation is:

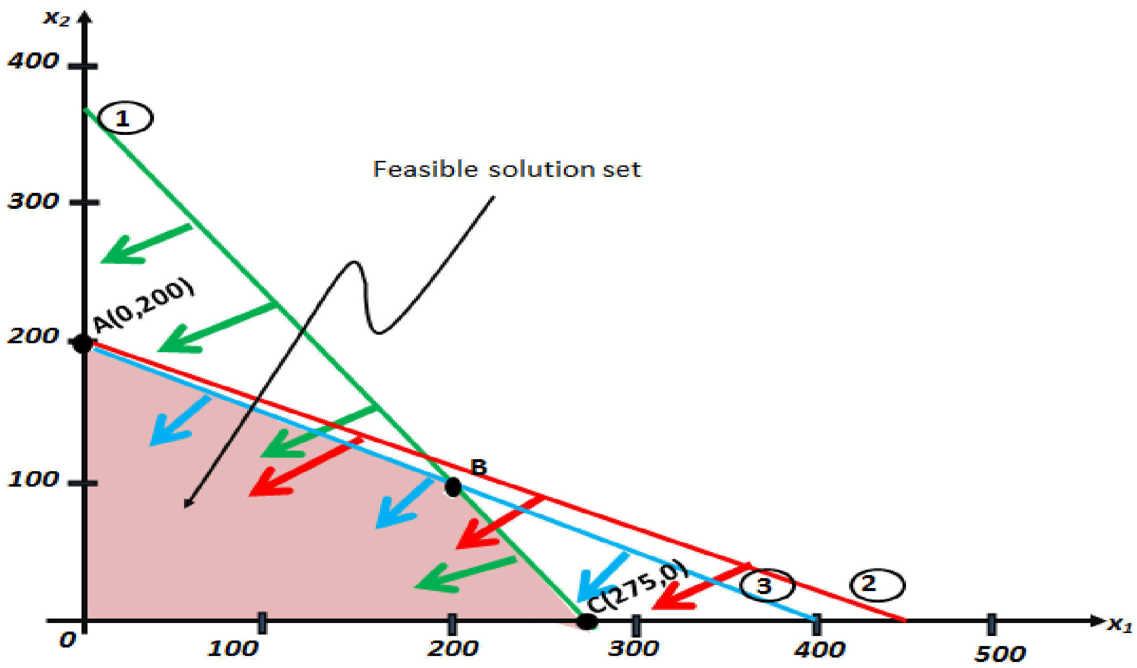

Figure (2.3)

The point B is resulting from the intersection of the lines representing the first and the third constraints, so we use these constraints to find the coordinates of B.

$$\left.\begin{aligned} 8x_1 + 6x_2 &= 2200 \\ x_1 + 2x_2 &= 400 \end{aligned}\right\} \xRightarrow{3\times(3)} - \begin{aligned} 8x_1 + \cancel{6x_2} &= 2200 \\ \mp 3x_1 \mp \cancel{6x_2} &= \mp 1200 \end{aligned}$$

$$\Rightarrow 5x_1 = 1000 \Rightarrow x_1 = 200 \overset{(3)}{\Rightarrow} x_2 = \frac{400-200}{2} = 100$$

The feasible solution region space is the shaded area OABC in figure (2.3) whose corners are the points $O(0,0)$, $A(0,200)$, $B(200,100)$, and $C(275,0)$.

Corner	Value of Z
O(0,0)	Z=0
$A(0,200)$	$Z = 12 \times 0 + 8 \times 200 = 1600$
$B(200,100)$	$Z = 12 \times 200 + 8 \times 100 = 3200$
$C(275,0)$	$Z = 12 \times 275 + 8 \times 0 = 3300$ *

From the table, we see that the optimum occurs at corner C, then $x_1 = 275$ units , $x_2 = 0$ units, and $Z_{max} = 3300$ monetary units.

Exercises 2.2 (In addition to the text book exercises)

1. The canonical form is the form in which the objective function is of maximization type and the constraints are of the ($\leq$) type (except the non-negativity restriction which is of ($\geq$) $type$), i.e. it has the form:

$$max. \quad Z = \sum_{j=1}^{n} c_j x_j$$

$$\text{S.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad i = 1,2, \dots, m$$

$$x_j \geq 0 \qquad j = 1, 2, \dots, n$$

Write the following LPP in canonical form then find the optimal solution of the canonical form (remember remark (2.1)):

$$\begin{aligned} min \quad & Z = -4x_1 - 2x_2 \\ S.t. \quad & -2x_1 - 4x_2 \geq -20 \\ & -2x_1 - 2x_2 \geq -12 \\ & -2x_1 + 2x_2 \geq -4 \\ & -2x_1 + 4x_2 \geq -2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

2. Find the optimal solution for the following LPP:

$$min \quad Z = 40x_1 + 20x_2$$

$$
\begin{aligned}
S.t. \quad & 2x_1 + 4x_2 \le 80 \\
& 6x_1 + 2x_2 \ge 60 \\
& 8x_1 + 6x_2 \ge 120 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

2.9 The Simplex Method

The graphical method cannot be applied when the number of variables in the LPP is more than three, or rather two, since even with three variables the graphical solution becomes tedious as it involves intersection of planes in three dimensions. The simplex method can be used to solve any LPP (for which the solution exists) involving any number of variables and constraints.

The computational procedure in the simplex method is based on a fundamental property that the optimal solution to an LPP, if it exists, occurs only at one of the corner points of the feasible region. The simplex method is an iterative method starts with initial basic feasible solution at the origin, i.e. Z=0. If the solution is not optimal, we move to the adjacent corner, until after a finite number of trials, the optimal solution, if it exists, is obtained.

The steps of the simplex method are as follows:

Step 1: Convert the given problem into the standard form. The Right Hand Side (RHS) of each constraint must be non-negative. Write the objective function in the form: $Z - \sum_{j=1}^{n} c_j x_j \;= 0$

Step 2: Set $x_1 = x_2 = \ldots = x_n = 0$, i.e. $x_1, x_2, \ldots, x_n$ are non-basic variables, thus $s_1, s_2, \ldots, s_m$ are the basic-variables.

Step 3: Construct the initial ***simplex table*** (or ***tableau***) with all slack variables in the BVS. The simplex table for the general LPP (in 2.4) is:

Basic variables	x_1	x_2	…	x_n	s_1	s_2	…	s_m	Solution
s_1	a_{11}	a_{12}	…	a_{1n}	1	0	…	0	b_1
s_2	a_{21}	a_{22}	…	a_{2n}	0	1	…	0	b_2
⋮	⋮	⋮	…	⋮	⋮	⋮	…	⋮	⋮
s_m	a_{m1}	a_{m2}	…	a_{mn}	0	0	…	1	b_m
Z	$-c_1$	$-c_2$	…	$-c_n$	0	0	…	0	*0*

Table (2.1)

The coefficients a_{ij} in the constraints (written under non-basic variables $x_1, x_2, \ldots, x_n$) is called the ***body matrix*** (or ***coefficient matrix***). The last column