Fourier Series

When we expand a given function $f(x)$ by Taylor series about a point a we write

$$f(x) \approx \sum_{n=0}^{\infty} \frac{f^{(n)}(a)\,(x-a)^n}{n!}$$

This expansion is locally, i.e in a neighbourhood of the point a but if we want to expand a function $f(x)$ on the interval $(-\pi, \pi)$ and satisfies

1) f is single valued
2) f is bounded
3) f has at most a finite numer of max. and min. points
4) f has only finite number of discontinuous points
5) $f(x+2\pi) = f(x)$ for x outside $[-\pi, \pi]$

then its Fourier series is

$$f(x) = \sum_{n=0}^{\infty} [a_n \cos nx + b_n \sin nx]$$

at the points of continuity and

$f(x) = \frac{1}{2}[f(x+0) + f(x-0)]$ at discontinuous points

Beautiful Properties of Fourier Series

1) Discontinuous function can be represented by Fourier series. Although derivatives do not exist. (This is not true for Taylor series)
2) Fourier series is useful for expanding periodic functions since outside the closed interval there exists a periodic expansion (extension) of the function.
3) Expansion of oscillatory function by Fourier series gives all modes of oscillation
4) Fourier series of a discontinuous function is not uniformly convergent at all points
5) Term by term integration of a convergent Fourier series is always (100%) valid and it may be valid if the series is not convergent ~~However~~ However term by term differentiation of Fourier series is not valid in most cases.

Notes

1) $\int_0^{2\pi} \sin nx\,dx = 0$ 2) $\int_0^{2\pi} \cos nx\,dx = 0$

3) $\int_0^{2\pi} \sin^2 nx\,dx = \pi$ 4) $\int_0^{2\pi} \cos^2 nx\,dx = \pi$

5) $\int_0^{2\pi} \sin nx \sin mx\,dx = 0$ 6) $\int_0^{2\pi} \cos nx \cos mx\,dx = 0$

7) $\int_0^{2\pi} \sin nx \cos mx\,dx = 0$ 8) $\int_0^{2\pi} \sin nx \cos nx\,dx = 0$

Finding the coefficients in Fourier series

$$f(x) = \sum_{n=0}^{\infty} [a_n \cos nx + b_n \sin nx] \quad \ldots (*)$$

1) Finding a_0

Integrate ~~multiply~~ both sides of Fourier series by from 0 to 2π

$$\int_0^{2\pi} f(x)\,dx = a_0 \int_0^{2\pi} dx + a_1 \int_0^{2\pi} \cos x\,dx + \ldots$$

$$+ b_1 \int_0^{2\pi} \sin x\,dx + b_2 \int_0^{2\pi} \sin 2x\,dx + \ldots$$

$$= a_0 x \Big]_0^{2\pi} = 2\pi a_0 \quad \begin{bmatrix}\text{remaining integrals} \\ \text{are equal to zero}\end{bmatrix}$$

$$\therefore a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\,dx$$

2) Finding a_n

Multiplying fourier series by $\cos nx$

and integrate from 0 to 2π

$$\int_0^{2\pi} f(x) \cos nx\,dx = \int_0^{2\pi} \sum_{n=0}^{\infty} [a_n \cos nx + b_n \sin nx] \cos nx\,dx$$

$$= a_n \int_0^{2\pi} \cos^2 nx\,dx = \pi a_n$$ (remaining integrals are equal to zero)

$$\boxed{\therefore a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx} \quad \forall n \geq 1$$

3) Finding b_n

Multiply both sides of Fourier series by $\sin nx$ and integrate from 0 to 2π

to get $\int_0^{2\pi} f(x)\sin nx\,dx = \int_0^{2\pi} b_n \sin^2 nx\,dx = \pi b_n$

$$\therefore b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \quad \forall n \geq 1$$

eg find Fourier series for

$f(x) = x, \quad 0 < x < 2\pi$

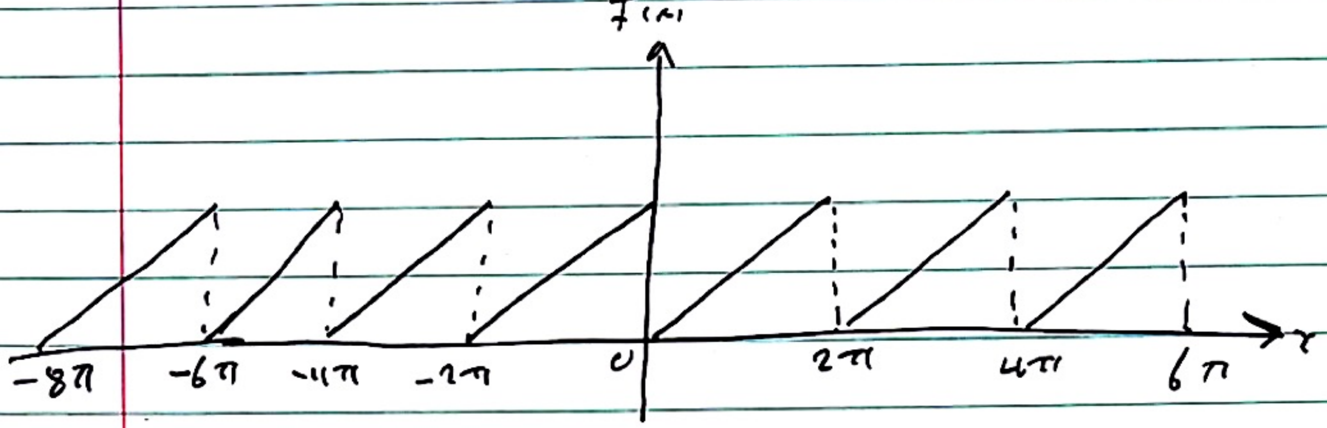
Sol $f(x) = \sum_{n=0}^{\infty} a_n \cos nx + b_n \sin nx$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\,dx = \cancel{2}\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx = \frac{-2}{\pi n} = -\frac{2}{n}$$

$$\therefore x = \pi - 2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right]$$



eg 2 find Fourier series for $f(x) = x + x^2$, $-\pi < x < \pi$

Sol $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\underbrace{\int_{-\pi}^{\pi} x\,dx}_{zero} + \int_{-\pi}^{\pi} x^2\,dx\right]$

$$= \frac{\pi^2}{3}$$

$$a_n = \frac{4(-1)^n}{n^2}, \quad b_n = \frac{2}{n}(-1)^{n+1}$$

(Check)

$$\therefore x + x^2 = \frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right]$$

$$+ 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right]$$

Suppose we want to get $\frac{\pi^2}{6} = ?$

$$\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] \quad \text{---(A)}$$

Also

$$-\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] \quad \text{..(B)}$$

$(A) + (B) \Longrightarrow$

$$2\pi^2 = \frac{2\pi^2}{3} + 8\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$

$$\frac{4\pi^2}{3} = 8\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\therefore \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

<u>ex</u> Find Fourier series

1) $f(x) = \pi - x, \quad 0 < x < 2\pi$

Ans. $2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right]$

2) $f(x) = e^{-x}, \quad 0 < x < 2\pi$

Ans. $\frac{1 - e^{-2\pi}}{\pi}\left[\frac{1}{2} + \frac{1}{2}\cos x + \frac{1}{5}\cos 2x\right.$

$+ \frac{1}{10}\cos 3x + \frac{1}{2}\sin x + \frac{2}{5}\sin 2x$

$\left. + \frac{3}{10}\sin 3x + \cdots\right]$