3-3-2 Separtion of Variables Method

This method is applicable for

$$a(x,y)\,u_{xx} + b(x,y)\,u_{yy} + c(x,y)\,u_x + d(x,y)\,u_y + e(x,y)\,u = 0 \quad \cdots\cdots \quad (3\cdot33)$$

ie it is applicable for the homag. and does not contain the term u_{xy}.

Q: What shall we do to apply this method if the term u_{xy} appears?

Ans. We must find a change of variables

$$\xi = \xi(x,y)\ ,\ \eta = \eta(x,y) \quad \text{s.t} \quad \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0 \quad \cdots (3\text{-}34)$$

to eleminate u_{xy}.

Note eq. (3-33) is of $\begin{pmatrix} \text{hyperbolic} \\ \text{ellep tic} \end{pmatrix}$ if $\begin{pmatrix} ab > 0 \\ ab < 0 \end{pmatrix}$ $\cdots (3\text{-}35)$

Q How to apply this method

Ans step 1 Since it is of separation of variables therefore we suppose $u(x,y) = X(x)\,Y(y)$ $\cdots (3\text{-}36)$

So equ. (3-33) becomes

$$a X''Y + bXY'' + cX'Y + dXY' + eXY = 0 \quad \ldots (3\text{-}37)$$

Step 2 Suppose $\exists$ a function $P(x,y)$ s.t. if equ. (3-37) is divided by $P(x,y)$ then it becomes:

$$A(x)\, X''Y + B(y)\, XY'' + C(x)\, X'Y + D(y)\, XY' + (E(x) + F(y))XY = 0 \quad \ldots (3\text{-}38)$$

Step 3
Divide equ. (3-38) by XY to get

$$A(x)\frac{X''}{X} + C(x)\frac{X'}{X} + E(x) = -\left[B(y)\frac{Y''}{Y} + \frac{D(y)\,Y'}{Y} + F(y)\right]$$

Step 4 Note that $\frac{d}{dx}\left[A\frac{X''}{X} + C\frac{X'}{X} + E\right] = 0 \quad \ldots (3\text{-}39)$

because $\frac{d}{dx}\left[A\frac{X''}{X} + C\frac{X'}{X} + E\right] = \underbrace{-\frac{d}{dx}\left[B\frac{Y''}{Y} + \frac{DY'}{Y} + F\right]}_{\text{zero}}$ ~~(3-40)~~

therefore

$$A\frac{X''}{X} + C\frac{X'}{X} + E = -\left[B\frac{Y''}{Y} + D\frac{Y'}{Y} + F\right] = \lambda \quad (3\text{-}40)$$

λ is called separation constant

Step 5 From equ (3.40) we get two 2^{nd} order ordinary diff equn

$$A X'' + C X' + [E - \lambda] X = 0 \quad \ldots\ldots (3.41)$$

$$B Y'' + D Y' + [F + \lambda] Y = 0 \quad \ldots\ldots (3.42)$$

$\boxed{\text{step. 6}}$ The sol. of equ. (3.33) is

$u(x,y) = X(x)\, Y(y)$ where $X(x)$ & $Y(y)$ are the solutions of (3.41) & (3.42) respectively.

The Last Question is How to evaluate λ.

The Answer is

The values of λ (called eigen values) are determined from the Eigen value problem constructed from equs (3.41) & (3.42) and the conditions (initial or boundary) as will be seen in the examples.

Def A value of λ is called eigen value iff the problem has non trivial solution.

Theorem If the coefficients of

$$a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + f u = 0$$

are constants then it can be solved by separation of variables.

$\boxed{\text{Proof}}$ Let $u(x,y) = X(x)\,Y(y)$ and we shall eleminate the term u_{xy}.

$$aX''Y + bX'Y' + cXY'' + dX'Y + eXY' + fXY = 0 \quad (7.43)$$

$a \neq 0$

$$\frac{X''}{X} + \frac{b}{a}\frac{X'}{X}\frac{Y'}{Y} + \frac{c}{a}\frac{Y''}{Y} + \frac{d}{a}\frac{X'}{X} + \frac{e}{a}\frac{Y'}{Y} + \frac{f}{a} = 0 \quad (7.44)$$

Differentiate (7.44) w.r.t x to get

$$\left(\frac{X''}{X}\right)' + \frac{b}{a}\left(\frac{X'}{X}\right)'\left(\frac{Y'}{Y}\right) + \frac{d}{a}\left(\frac{X'}{X}\right)' = 0$$

$$\frac{\left(\frac{X''}{X}\right)'}{\left(\frac{X'}{X}\right)'} + \frac{b}{a}\frac{Y'}{Y} + \frac{d}{a} = 0$$

$$\frac{\left(\frac{X''}{X}\right)'}{\frac{b}{a}\left(\frac{X'}{X}\right)} + \frac{Y'}{Y} + \frac{\frac{d}{a}}{\frac{b}{a}} = 0$$

<u>OR</u>

$$\frac{\left(\frac{X''}{X}\right)'}{\frac{b}{a}\left(\frac{X'}{X}\right)'} + \frac{d}{b} = -\frac{Y'}{Y} = \lambda$$

To get two O.D.E's

$$Y' + \lambda Y = 0 \quad \text{---} \quad (3.45)$$

and

$$\left(\frac{X''}{X}\right)' + \left(\frac{d}{a} - \lambda\right)\left(\frac{bX'}{aX}\right)' = 0 \quad \text{...} \quad (3.46)$$

Integrate (3.46) w.r.t. x to get

$$\frac{X''}{X} + \left(\frac{d}{a} - \lambda\right)\frac{bX'}{aX} = \beta \quad \text{---} \quad (3.47)$$

To get β return to (3.45) which is

$$\frac{Y'}{Y} = -\lambda$$

$$\therefore \frac{YY'' - Y'^2}{Y^2} = 0 \Rightarrow \frac{Y''}{Y} = \left(\frac{Y'}{Y}\right)^2 = \lambda^2$$

equ.(3.44) becomes

$$\frac{X''}{X} - \frac{b}{a}\lambda\frac{X'}{X} + \frac{c}{a}\lambda^2 + \frac{d}{a}\frac{X'}{X} - \frac{e}{a}\lambda + \frac{f}{a} = 0$$

$$\therefore X'' + \left(\frac{d}{a} - \frac{b}{a}\lambda\right)X' + \left(\frac{c}{a}\lambda^2 - \frac{e}{a}\lambda + \frac{f}{a}\right)X = 0$$

$$X'' + \left(\frac{d}{b} - \lambda\right)\frac{b}{a}X' + \underbrace{\left(\lambda^2 - \frac{e}{c}\lambda + \frac{f}{c}\right)}_{\beta}\frac{c}{a}X = 0 \quad \ldots\ldots(3.48)$$

Therefore $u(x,y)$ is sol. for

$$\left(\frac{X''}{X}\right)' + \frac{b}{a}\left(\frac{X'}{X}\right)'\left(\frac{Y'}{Y}\right) + \frac{d}{a}\left(\frac{X'}{X}\right) = 0$$

where $X(x)$ & $Y(y)$ are sols for (3.45) & (3.48) respectively