نظرية المعادلات التفاضلية 2/ المرحلة الرابعة علوم بنات / بغداد

2019-2020 د. سعد ناجي الجزائري

We notice (دائماً) six cases:-

① $\lambda_2 > \lambda_1 > 0$ ② $\lambda_1 > \lambda_2 > 0$ ③ $\lambda_2 < \lambda_1 < 0$

④ $\lambda_1 < \lambda_2 < 0$ ⑤ $\lambda_1 < 0 < \lambda_2$ (6) $\lambda_2 < 0 < \lambda_1$

<u>Case 1</u> $\lambda_2 > \lambda_1 > 0$

$$\therefore \lim_{t\to\infty} x(t) = x_0 \lim_{t\to\infty} e^{\lambda_1 t} = \infty$$

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y_0 e^{\lambda_2 t} = y_0 \lim_{t\to\infty} e^{\lambda_2 t} = \infty$$

$\therefore (0,0)$ is unstable (source)

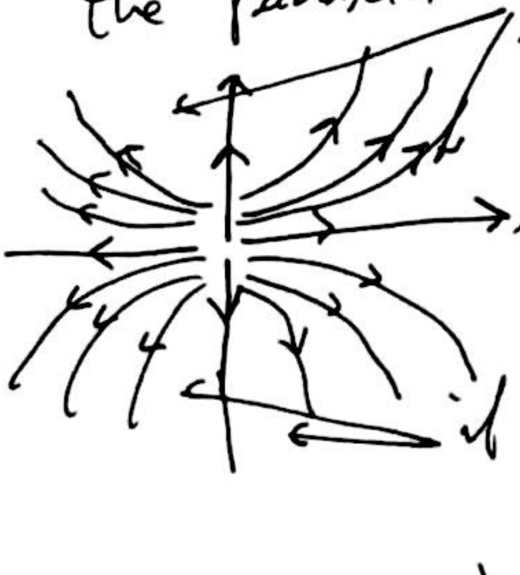
To draw the plane portrait we

go to $y = c\, x^{\frac{\lambda_2}{\lambda_1}}$

$y = c\, x^{\frac{\lambda_2}{\lambda_1}}$ looks like يشبه

since $\lambda_2 > \lambda_1 \Rightarrow$

the parabola القطع المكافئ



if $c > 0$

x

if $c < 0$

لاحظ أن الحلول لا تقطع المحورين x, y لأن المحورين هما حلول أيضاً ممثلات المتجهات الذاتية المرافقة لـ λ_1 و λ_2 .

<u>Case 2</u>: $0 < \lambda_2 < \lambda_1$

We have

$$\dot{x} = \lambda_1 x \ , \ \dot{y} = \lambda_2 y$$

$$\lim_{t \to \infty} x(t) = x_0 \lim_{t \to \infty} e^{\lambda_1 t} = \infty$$

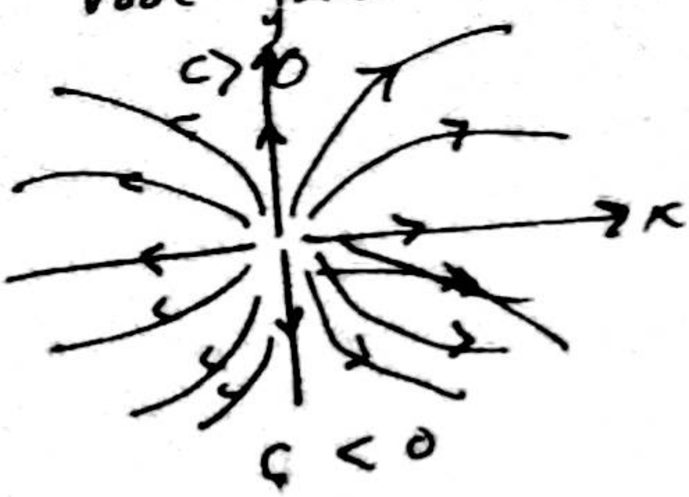
$$\lim_{t \to \infty} y(t) = y_0 \lim_{t \to \infty} e^{\lambda_2 t} = \infty$$

$\therefore (0,0)$ is unstable (source)

To draw the phase portrait we go to $y = c\, x^{\frac{\lambda_2}{\lambda_1}}$

since $\lambda_2 < \lambda_1 \longrightarrow \frac{\lambda_2}{\lambda_1} < 1$

$\therefore y(x) = c\, x^{\frac{\lambda_2}{\lambda_1}}$ looks like the root function

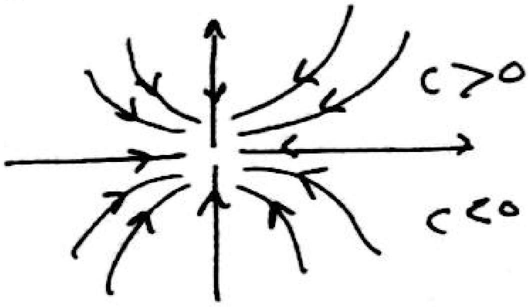


Case 3 $\lambda_2 < \lambda_1 < 0$

$$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} y(t) = 0$$

$\therefore (0,0)$ is asymptotically stable (sink)

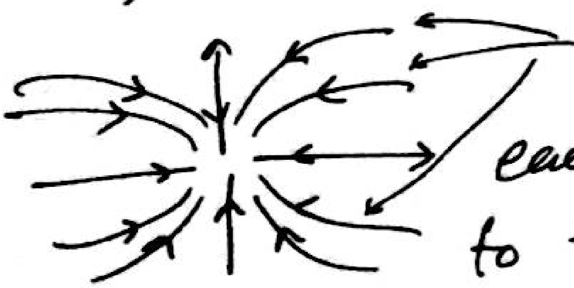
$$y(x) = c\, x^{\frac{\lambda_2}{\lambda_1}} \qquad \left(\frac{\lambda_2}{\lambda_1} > 1 \right)$$



Case 4 $\lambda_1 < \lambda_2 < 0$

As in case 3, $(0,0)$ is asymptotically stable

$$y(x) = c\, x^{\frac{\lambda_2}{\lambda_1}} \quad , \quad \frac{\lambda_2}{\lambda_1} < 1$$



solutions

each solution goes to the equilibrium point $(0,0)$

بمعنى أنه أي حل من الحلول مهما كانت
نقطة بدايته بعيدة عن نقطة الاتزان $(0,0)$ سوف
يتجه الى نقطة الاتزان أي أنها نقطة جاذبة

Case 5 $\lambda_1 < 0 < \lambda_2$

As we see $x(t) = x_0 e^{\lambda_1 t}, \; y(t) = y_0 e^{\lambda_2 t}$

$\lim_{t \to \infty} x(t) = 0 \quad [\lambda_1 < 0]$

$\lim_{t \to \infty} y(t) = \infty \quad [\lambda_2 > 0]$

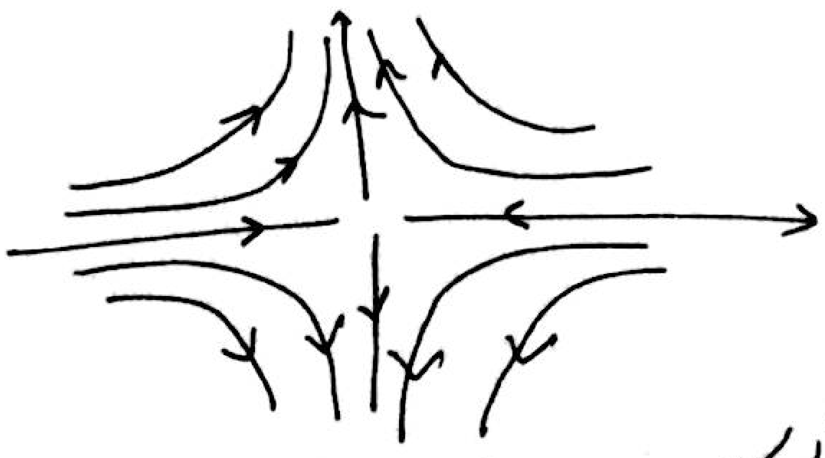
نلاحظ أن الحل يقترب من (0,0) من جهة محور x
ويبتعد عنها من جهة محور y

$\therefore (0,0)$ in unstable (saddle point)
نقطة سرجية

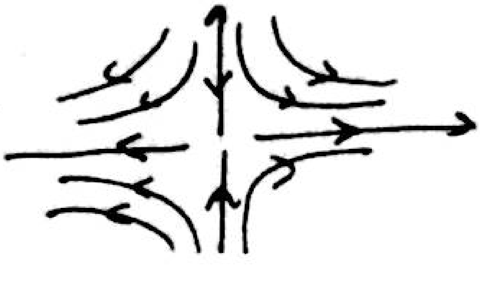
$$y(x) = c \, x^{\frac{\lambda_2}{\lambda_1}} \quad \left[\frac{\lambda_2}{\lambda_1} < 0 \right]$$

$$= \frac{c}{x^{-\frac{\lambda_2}{\lambda_1}}}$$

looks like the hyperbola القطع الزائد



Case 6 As above



(0,0) in unstable (saddle)

نظرية الانظمة الديناميكية / الرابع / رياضيات تطبيقية
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(2) $A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\dot{x} = \lambda x$, $\dot{y} = \lambda y$

$\lambda < 0$

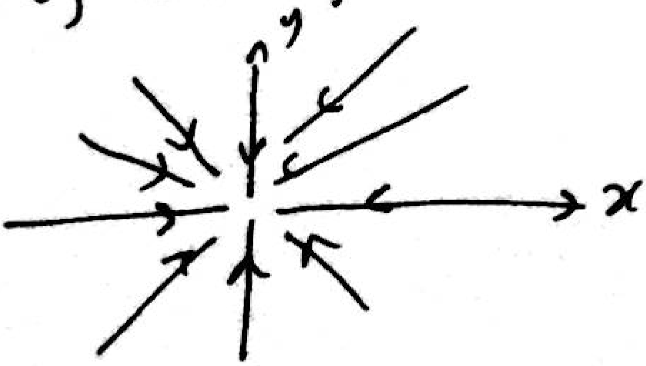
$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} y(t) = 0$

(0,0) is asymptotically stable (sink)
and called also star node نقطة نجمية

$$\frac{\dot{y}}{\dot{x}} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} = \frac{y}{x}$$

$\therefore y = cx$

they are rays أشعة



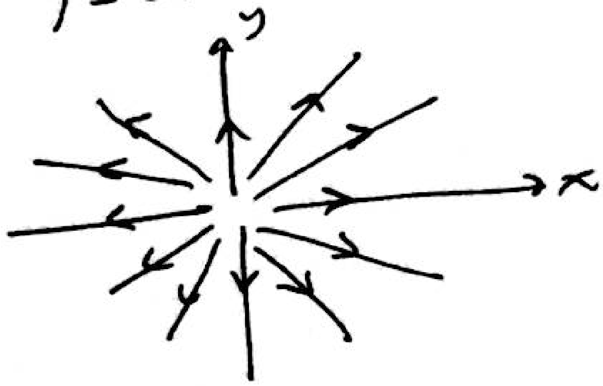
$\lambda > 0$

$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} y(t) = \infty$

(0,0) is unstable (source) star node

Also

$y = cx$



نظرية المعادلات التفاضلية ٢/ المرحلة الرابعة علوم حاسبات/ الصباحي

د. سعد ناجي الجزائري 2019-2020

(3) A is similar to $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

هنا المصفوفة A ليست قطرية - لكن قيمها الذاتية متشابهة

فيما سبق كان المحورين x و y هما متجهات ذاتية وربما عندها

الحالة لدينا متجه ذاتي واحد وكما لاحظنا بالمعادلة سيكون

لدينا حل ونضع الثاني وفق العلاقة

$$x_1(t) = v_1 e^{\lambda t}$$

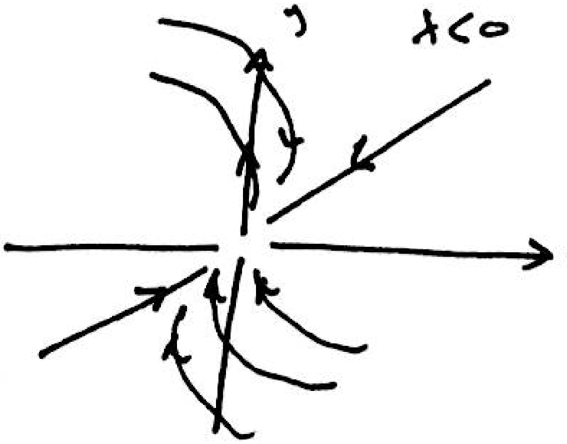
$$x_2(t) = k e^{\lambda t} + L t e^{\lambda t}$$

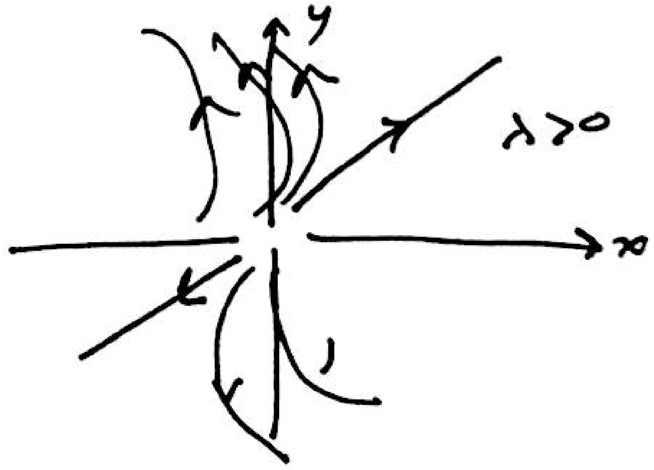
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1\alpha_1 + c_2\beta_1 + c_2\beta_2 t \\ c_1\alpha_2 + c_2\beta_2 + c_2\beta_2 t \end{bmatrix} e^{\lambda t}$$

Clearly

$$\lim_{t\to\infty} X(t) \xrightarrow{\lambda>0} \infty : (0,0) \text{ unstable}$$

$$\xrightarrow{\lambda<0} 0 : (0,0) \text{ asymptotically stable}$$





Finally

④ A is similar to $\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ or $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$

$$\dot{X} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} X$$

i.e

$$\left.\begin{aligned} \dot{x} &= \alpha x - \beta y \\ \dot{y} &= \beta x + \alpha y \end{aligned}\right\} (*)$$

We use polar coordinates for simplicity

i.e let $x = r\cos\theta$ & $y = r\sin\theta$

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$$

$$\dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$$

using in $(*)$

$$\dot{r}\cos\theta - r\sin\theta\,\dot{\theta} = \alpha r\cos\theta - \beta r\sin\theta \quad \ldots (1)$$

$$\dot{r}\sin\theta + r\cos\theta\,\dot{\theta} = \beta r\cos\theta + \alpha r\sin\theta \quad \ldots (2)$$

$(1)\cos\theta + (2)\sin\theta \longrightarrow$

$\boxed{\dot{r} = \alpha r}$ $\xrightarrow{\alpha > 0}$ r is increasing

$\xrightarrow{\alpha < 0}$ r is decreasing

$(1)(-\sin\theta) + (2)\cos\theta \longrightarrow$

$\boxed{\dot{\theta} = \beta}$ $\xrightarrow{\beta > 0}$ θ increasing (counter clock wise)

$\xrightarrow{\beta < 0}$ θ decreasing (clock wise)

ا.د سعد ناجي العزاوي

In the previous pages we study the phase portrait of $\dot{X}_{2\times1} = A_{2\times2} X_{2\times1}$ where

$|A| \neq 0$

Now we study the case $|A| = 0$, therefor

$AX = 0$ has infinite number of solutions, ie $\exists$ infinite number of equilibrium points for $\dot{X} = AX$

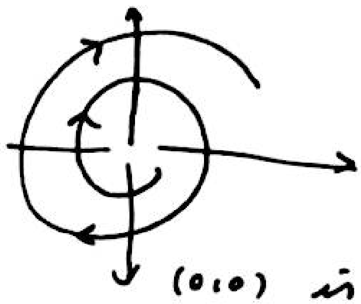
$|A| = 0 \Longrightarrow$ at least one of the eigen values of A is zero

① If the two eigen values of A are zero then A is similar to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

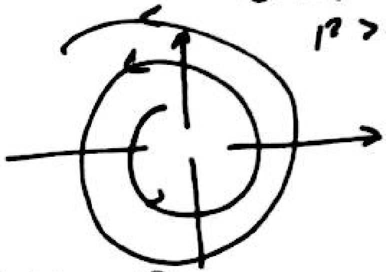
ie every point of the xy plane is equil. pt

② If one of the eigen values is zero, then

① $\alpha > 0$, $\beta < 0$

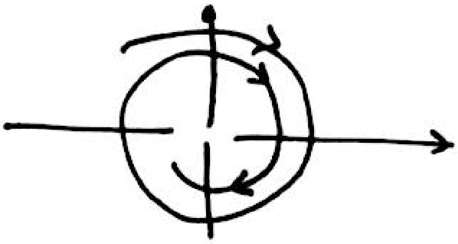


① $\alpha > 0$ $\beta > 0$

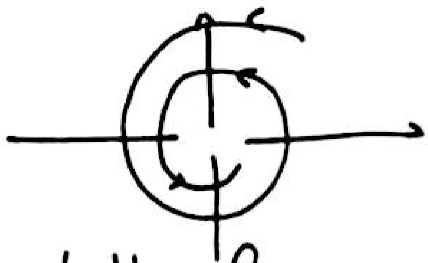


$(0,0)$ is unstable focus

③ $\alpha < 0$, $\beta < 0$



④ $\alpha < 0$ $\beta > 0$

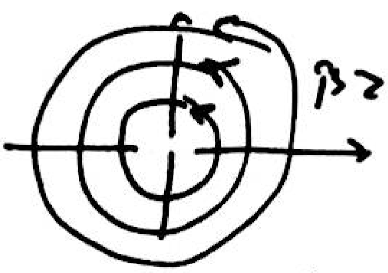


$(0,0)$ asymptotically stable focus

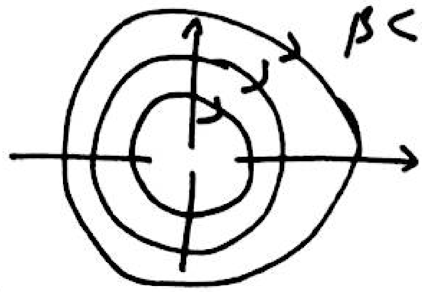
Note if $\lambda_{1,2} = \mp i\beta$ $(\alpha = 0)$

$\dot{x} = -\beta y$, $\dot{y} = \beta x$

$\frac{dy}{dx} = \frac{x}{y} \longrightarrow x^2 + y^2 = c^2$



$\beta > 0$



$\beta < 0$

$(0,0)$ is stable (center)

In the previous pages we study the phase portrait of $\dot{\underset{2\times1}{X}} = \underset{2\times2}{A}\,\underset{2\times1}{X}$ where

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Now we study the case $|A|=0$, therefor

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