نقطة الاستقرار [illegible]

e.g) Find if possible the equili. pts of

① $\dot{x} = \sin x$, $\dot{y} = \cos y$

② $\dot{x} = 3x - 5y + 2$, $\dot{y} = x + 4y - 3$

③ $\dot{x} = x^2 + y^2 + 4$, $\dot{y} = 3x - 4y$

④ $\dot{x} = -2xy$, $\dot{y} = -x + y + xy - y^3$

<u>Sol.(1)</u> To find the equilibrium points we solve

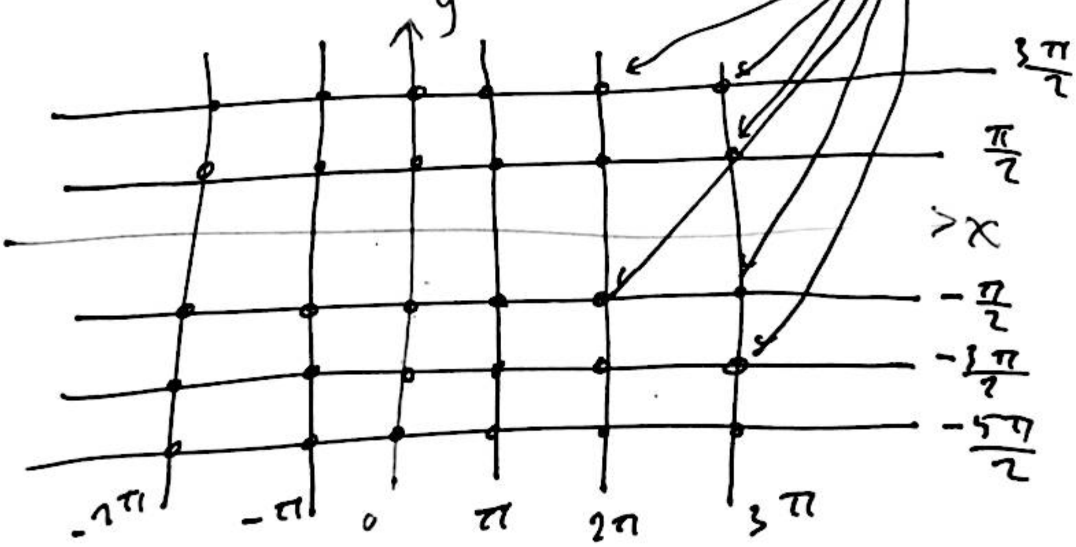
$\sin x = 0 \longrightarrow x = n\pi \quad , n = 0, \mp 1, \mp 2, \ldots$

$\cos y = 0 \longrightarrow y = \frac{2m+1}{2}\pi \quad , m = 0, \mp 1, \mp 2, \ldots$

$\therefore$ This system has infinite number of equilibrium points

$(x_n, y_m) = \left(n\pi, \frac{2m+1}{2}\pi\right)$

نقاط اتزان

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Sol (2)   $\dot{x} = 3x - 5y + 2$   [illegible]

$\dot{y} = x + 4y - 3$

$$3x - 5y + 2 = 0 \rightarrow 3x - 5y = -2$$
$$x + 4y - 3 = 0 \rightarrow (x + 4y = 3) \quad \times(-3)$$

$$-17y = -11 \rightarrow y = \frac{11}{17}$$
$$3x - 5\left(\frac{11}{17}\right) = 2 \Rightarrow$$
$$3x = \frac{55}{17} + 2 = \frac{89}{17}$$
$$\therefore x = \frac{89}{51}$$

This system has on equilibrium point $\left(\frac{89}{51}, \frac{11}{17}\right)$

Sol (3)   $x^2 + y^2 + 4 = 0$ C! $[x^2 + y^2 + 4 \geq 4]$

$3x - 4y = 0$

$\therefore \nexists$ equilibrum point

Sol.(4)   $-2xy = 0 \rightarrow x = 0,\ y = 0$

$-x + y + xy - y^3 = 0$

If $x = 0 \rightarrow 0 = -x + y + xy - y^3 = y - y^3 = y(1 - y^2)$

$\therefore y = 0$ or $y = 1$ or $y = -1$

If $y = 0 \rightarrow x = 0 \Rightarrow$ three equil. pts $(0,0)$ $(0,1)$ & $(0,-1)$

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نقط التعادل لمعادلات 2D ذات بعدين

We now see the influnce (تأثير) of the nature (طبيعة) of the equili pt on the solutions near it.

e.g Draw the phase portrait of
ارسم صور الطور

$$\dot{X} = \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}}_{A} X$$

Sol The eigen values of $A$ are determined by

$$0 = |A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = \lambda^2 - 7$$

$$\therefore \lambda_1 = \sqrt{7} \ \& \ \lambda_2 = -\sqrt{7}$$

$$\therefore A \approx \begin{bmatrix} \sqrt{7} & 0 \\ 0 & -\sqrt{7} \end{bmatrix}$$

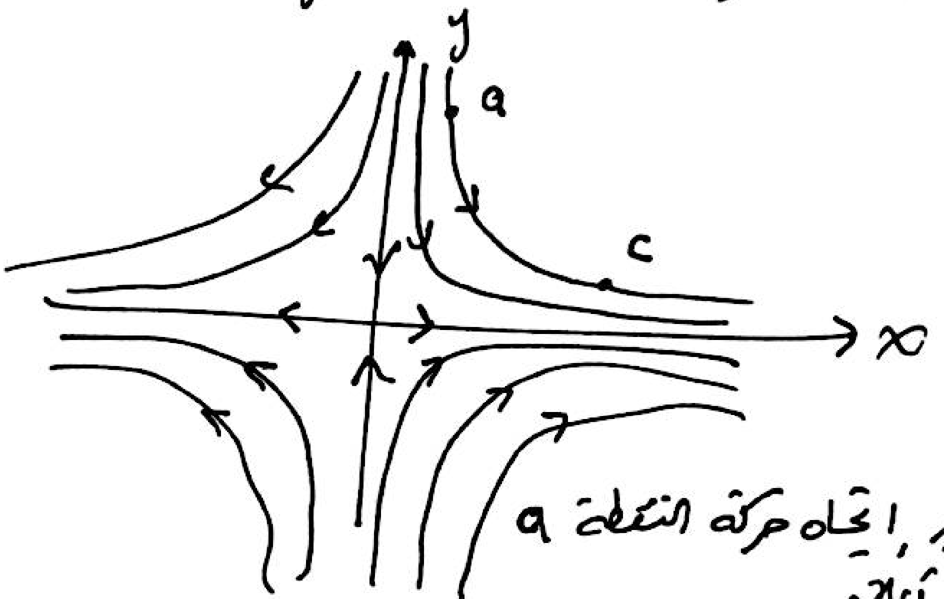
ie $$\dot{X} \approx \begin{bmatrix} \sqrt{7} & 0 \\ 0 & -\sqrt{7} \end{bmatrix} X$$

$$\dot{x} = \sqrt{7}\, x \ \& \ \dot{y} = -\sqrt{7}\, y$$

نقسم المعادلات المعطاة الأولى بالثانية وبعد تكامل

$$\frac{dy}{dx} = \frac{-y}{x} \longrightarrow \frac{dy}{y} = \frac{-dx}{x} \longrightarrow \ln y = -\ln x + \ln c$$

$$\therefore\ y = cx^{-1} \quad \text{family of hyperbolas}$$



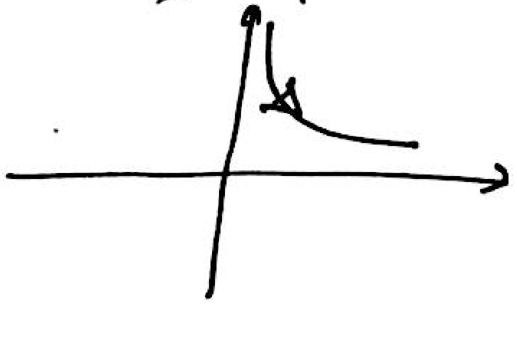

الآن نريد اتجاه حركة النقطة a
على المنحنى أعلاه.

$$\frac{dx}{dt} = \sqrt{7}\,x \rightarrow x(t) = l\, e^{\sqrt{7}t},\ l > 0$$

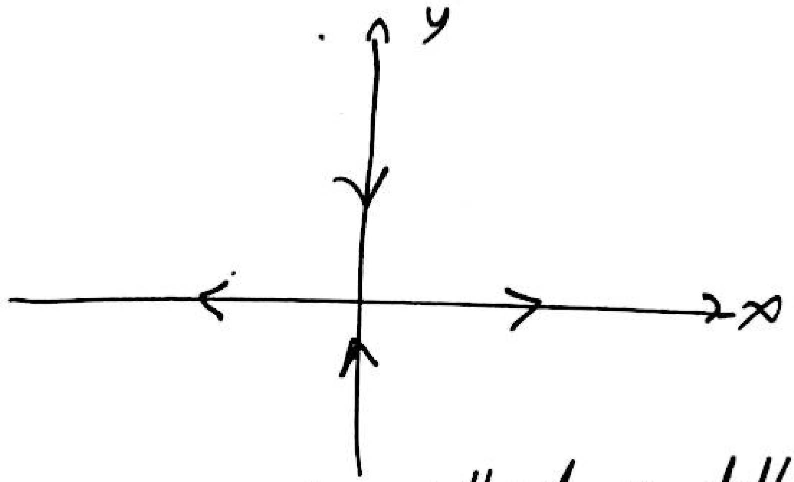
لأن النقطة بالربع الأول $\Leftarrow$ $x(t)$ متزايدة

$$\frac{dy}{dt} = -\sqrt{7}\,y \longrightarrow y \text{ is decreasing}$$

نلاحظ أن النقطة a تنزل مع محور y وتسير
مع محور x فيكون الرسم والاتجاه



If $a \in y$-axis then $a$ goes to $(0,0)$ & if $a \in x$-axis then $a$ goes away from (يبتعد عن) $(0,0)$




So $(0,0)$ is called saddle point.

(0,0) atracts (تجذب) the points in $y$-direction and repels تطرد the points in $x$-direction

Therefore we call it unstable غير مستقرة

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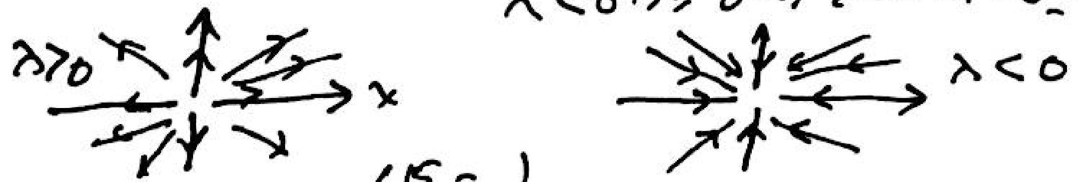
for $\dot{X} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} X \Rightarrow y = cx \,,\ c \in \mathbb{R}$

ونتصور أن هذه الحلول مستقيمات تمر بنقطة الأصل لكن هذا التصور غير صحيح لأن نقطة الأصل حل ولا يتقاطع حلان وبالتالي تكون الحلول أشعة منطلقة من قريب نقطة الأصل إذا $\lambda > 0$ وأشعة متجهة الى نقطة الأصل إذا $\lambda < 0$



We notice that the solutions either stay near (around) the equilibrium point or goes to the equil. pt or goes away from the equil. pt. or converge to it than diverges from it. These cases lead to the concept of stability.

## STABILITY الاستقرارية

Def. (1) An equil. pt. $X_0$ of a dynamical system $\underset{n\times 1}{\dot{X}} = \underset{n\times 1}{F(X)}$ is called stable if $\forall$ sol. $X(t)$ of $\dot{X} = F(X)$ starting near $X_0$ then it stays near $X_0$.

i.e. $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t

$$\|x(t_0) - X_0\| < \delta \Rightarrow \|X(t) - X_0\| \leq \varepsilon \quad \forall t \geq t_0$$

**Def** An equil. pt $X_0$ is called مستقرة بالتقارب asymptotically stable if $\exists\, r>0$ s.t if $\|X(t_0)-X_0\|<r$ then $\|X(t)-X_0\| \Rightarrow 0$ as $t\to\infty$

i.e $\lim\limits_{t\to\infty} X(t) = X_0$


**Def** If $X_0$ is not stable then it is called unstable.

$$\boxed{\text{Stability of the equili pt(s) of}}$$

$$\dot{X}_{n\times 1} = A_{n\times n} X_{n\times 1}$$

Remember that if $|A| \neq 0$ then $\dot{X} = AX$ has one equil. pt. which is $X=0$

and if $|A|=0$ then $\exists$ infinite number of equil. pts.

<u>Th.</u> The equili. pt(s) of $\dot{X}=AX$ is:-

(1) <u>stable</u> if the simple eigen values have nonpositive real parts and the multiple eigen values have negative real parts

(2) asymptotically stable if all the eigen values have negative real parts

(3) <u>unstable</u> if (1) is not satisfied

<u>Now</u> we study in full details

$$\underset{2\times 1}{\dot{X}}=AX\,,\ A \text{ is constant},\ |A|\neq 0$$

i.e

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}\ ;\ \cdot=\frac{d}{dt}$$

Since $|A|\neq 0$ then $\begin{bmatrix}0\\0\end{bmatrix}=(0,0)$ is the unique equili. pt.

Let $\lambda_1$ & $\lambda_2$ be the eigen values of $A$.

Since $|A| \neq 0 \Rightarrow \lambda_1 \lambda_2 \neq 0 \Rightarrow \lambda_1 \neq 0$ & $\lambda_2 \neq 0$

where $\lambda_1$ & $\lambda_2$ may be real or complex

(1) $(0,0)$ is stable if $\lambda_1, \lambda_2$ are pure imaginary

(2) ~~X~~ $(0,0)$ is asymptotically stable if $\lambda_1 < 0$ & $\lambda_2 < 0$ as a real eigen values

or $\alpha < 0$ where $\lambda_1 = \alpha + i\beta$ & $\lambda_2 = \alpha - i\beta$

(3) $(0,0)$ is unstable if $\lambda_1 > 0$ or $\lambda_2 > 0$ or $\alpha > 0$

## Types of the equil. pts

Def (1) An equil. pt $X_0$ is called sink if the eigen values are negtive real numbers

Def (2) An equil. pt $X_0$ is called source if the eigen values are positive real number

Def 3. $X_0$ is called center if the eigen values are pure imaginary.

<u>Def 4</u> $X_0$ is called focus بؤرة of the eigen values are complex numbers and $\alpha \neq 0$ $(\lambda_{1,2} = \alpha \mp i\beta)$

In the following we study

$\dot{X} = AX$, $A_{2\times 2}$ constant matrix, $|A| \neq 0$

to draw the phase portrait صورة الطور

$A$ is similar to $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ & $\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$. $(0,0)$ is the unique equil pt.

① $A$ is similar to $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

where $\lambda_1$ & $\lambda_2$ are real

So we study $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

ie $\dot{x} = \lambda_1 x \longrightarrow x(t) = x_0 e^{\lambda_1 t}$

$\dot{y} = \lambda_2 y \longrightarrow y(t) = y_0 e^{\lambda_2 t}$

Also $\dfrac{dy}{dx} = \dfrac{\lambda_2}{\lambda_1} \dfrac{y}{x}$ the sol

$\therefore \dfrac{dy}{y} = \dfrac{\lambda_2}{\lambda_1} \dfrac{dx}{x} \Longrightarrow y = c\, x^{\frac{\lambda_2}{\lambda_1}}$