Sol. (3) $\dot{x} = \alpha x - \beta x y \ , \dot{y} = (kx - d)y$

$\alpha x - \beta x y = 0 \longrightarrow (\alpha - \beta y)x = 0$ → $x = 0$, → $y = \frac{\alpha}{\beta}$

$(kx - d)y = 0$ → $y = 0$, → $x = \frac{d}{k}$

The equil pts are $(0,0), \left(\frac{d}{k}, \frac{\alpha}{\beta}\right)$

Sol. (4) exercise

$(0,0) , (0,2) , (1,0) \ \& \ \left(\frac{1}{2}, \frac{1}{2}\right)$

example Find the equilibrium points of

$$\ddot{\theta} + w^2 \sin\theta = 0$$

Sol. We reduce this 2^{nd} order equation to a system as follows

Let $\dot{\theta} = u$

$\therefore \dot{u} = \ddot{\theta} = -w^2 \sin\theta$

$$\begin{bmatrix} \dot{\theta} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -w^2 \sin\theta & 0 \end{bmatrix} \begin{bmatrix} \theta \\ u \end{bmatrix}$$

equil. pts (μ, θ) are determined by

$$\left.\begin{array}{l} \mu = 0 \\ \& \; -w^2 \sin\theta = 0 \end{array}\right\} \Longrightarrow \begin{array}{l} \mu = 0 \\ \theta_n = n\pi \;, \; n = 0, \mp 1, \mp 2, - \end{array}$$

$$(\mu, \theta_n) = (0, n\pi)$$

Note ~~(Stability)~~

let (x_0, y_0) be an equili. pt. of

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

Then the stability of (x_0, y_0) is the same of the stability of $(0, 0)$ for

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \dot{X} = \left.\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}\right|_{(x_0, y_0)} \begin{bmatrix} x \\ y \end{bmatrix}$$

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A is similar to $\begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix}$, $\lambda \neq 0$

(I) A is similar to $\begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}$

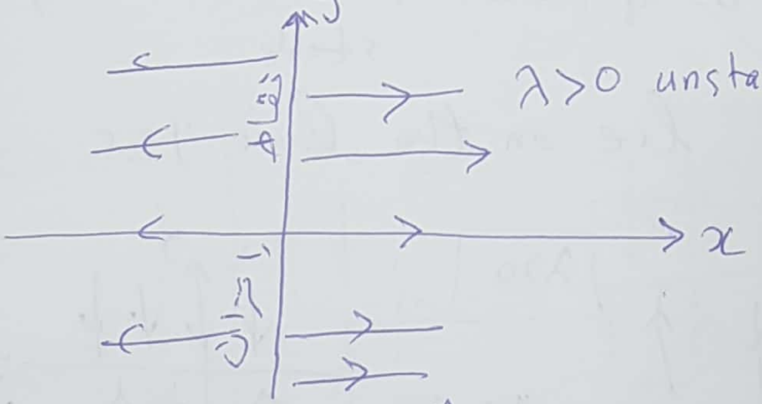
$$\dot{X} = AX \approx \dot{X} = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} X$$

i.e. $\left.\begin{aligned} \dot{x} &= \lambda x \\ \dot{y} &= 0 \end{aligned}\right\} \Rightarrow$ the equilibrium points is $x = 0$ (y-axis)

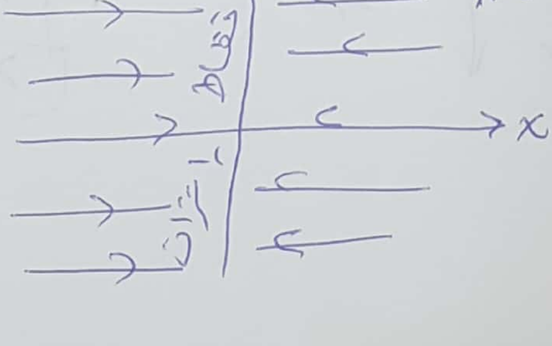
Now the sol. of $\dot{x} = \lambda x \overset{\text{is}}{\Longrightarrow} x(t) = x_0 e^{\lambda t}$

$\dot{y} = 0 \longrightarrow y = \text{constant}$

$$x(t) = x_0 e^{\lambda t} \begin{cases} \nearrow \infty & \text{if } \lambda > 0 \\ \searrow 0 & \text{if } \lambda < 0 \end{cases}$$



$\lambda > 0$ unstable equil. pts



$\lambda < 0$ asymptotically stable equil. pts

نظرية المعادلات التفاضلية رابع رياضيات علوم بنات

د. سعد ناجي العزاوي

وبنفس الاسلوب للحالة الثالثة والرابعة

$$A \simeq \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix}$$

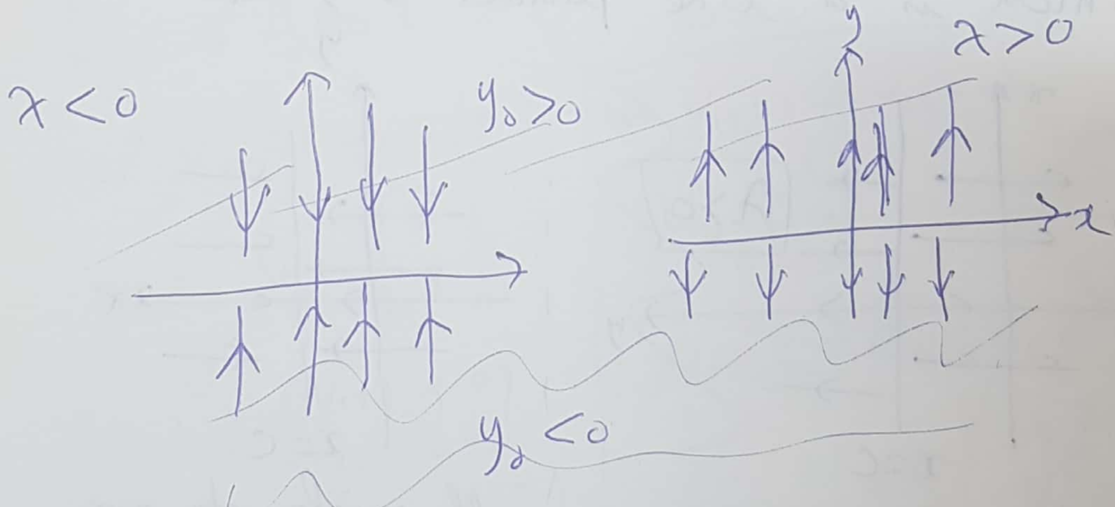
$$\dot{X} = AX \approx \dot{X} = \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix} X \;, \quad \lambda \neq 0$$

$$\left.\begin{array}{l} \dot{x} = 0 \\ \dot{y} = \lambda y \end{array}\right\} \longrightarrow \text{the equil. pts is } y = 0 \ (x\text{-axis})$$

$$y(t) = y_0 e^{\lambda t} \begin{cases} \to \infty & \text{if } \lambda > 0 \\ \to 0 & \text{if } \lambda < 0 \end{cases}$$

$$x(t) = \text{constant}$$

∴ نقاط الاتزان التي هي كل محور x تكون مستقرة لاتقاربي عندما $\lambda<0$ وغير مستقرة عندما $\lambda>0$



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Collecting all the diagrams in One diagram

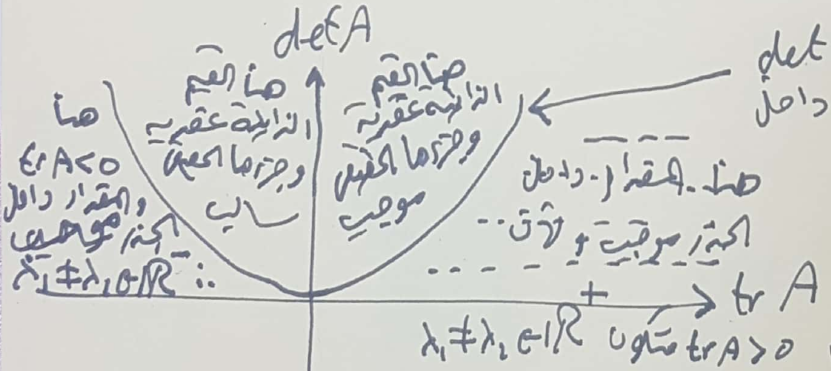
$$\dot{X} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} X$$

the eigen values of A are given by

$$0 = |A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21}$$

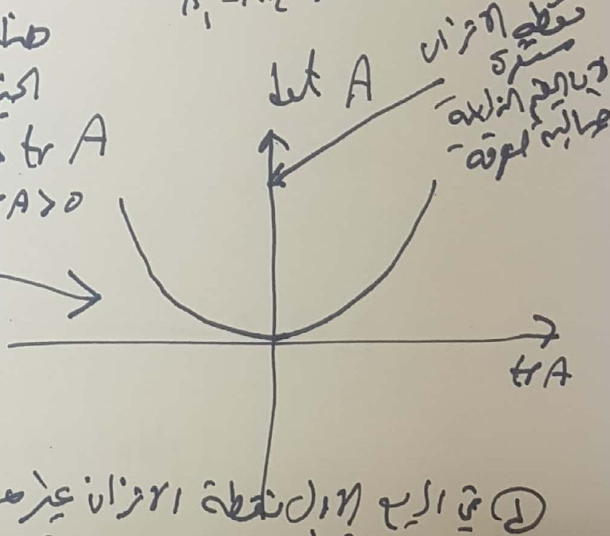
$$= \lambda^2 - \underbrace{(a_{11}+a_{22})}_{tr(A)}\lambda + \underbrace{a_{11}a_{22} - a_{12}a_{21}}_{det(A)}$$

$$\lambda_{1,2} = \frac{trA \mp \sqrt{(trA)^2 - 4\det A}}{2}$$

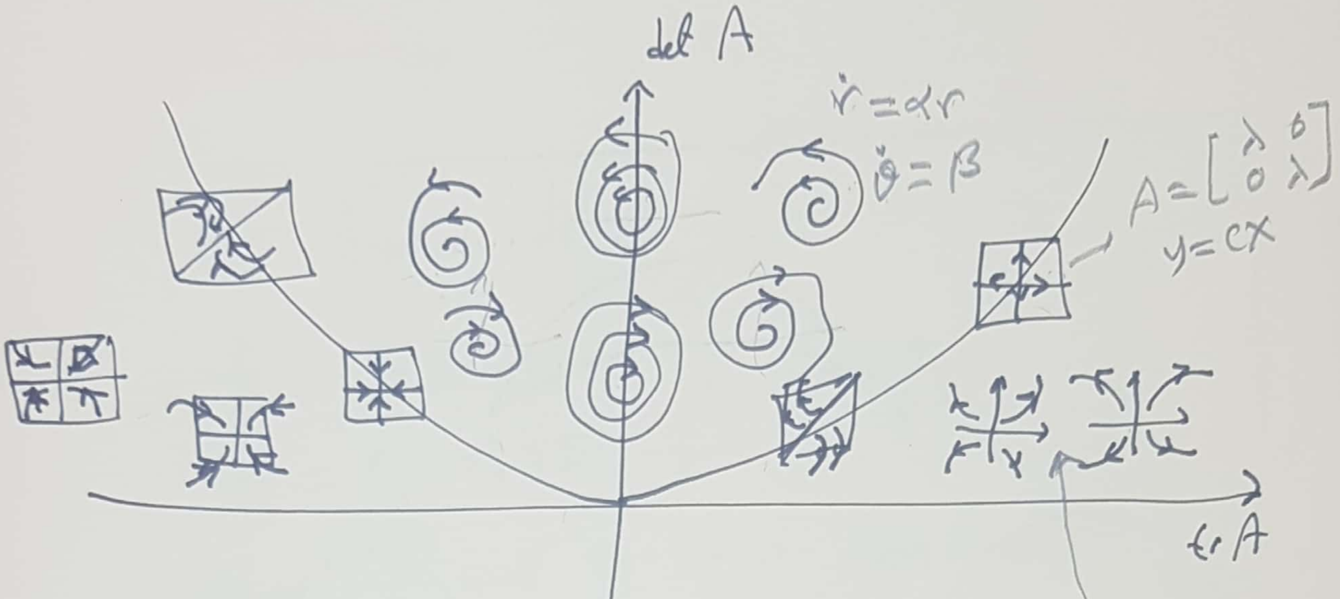


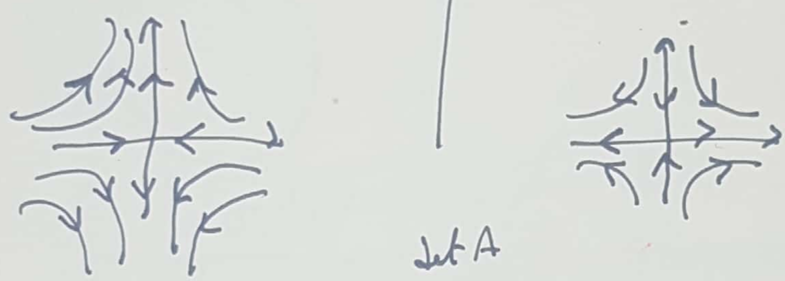
$\det A = \frac{1}{4}(trA)^2$ على هذا المنحنى يكون المقدار داخل الجذر صفر فتكون $\lambda_1 = \lambda_2$

في الربعين الثالث والرابع تكون قيمة ذاتية موجبة والاخرى سالبة



1. في الربع الاول نقطة الاتزان غير مستقرة
2. " " الثاني " " مستقرة بانتظام
3. " الربعين الثالث والرابع نقطة الاتزان سرجية (غير مستقرة)
4. على الجزء الموجب من محور $\det A$ تكون نقطة الاتزان مستقرة





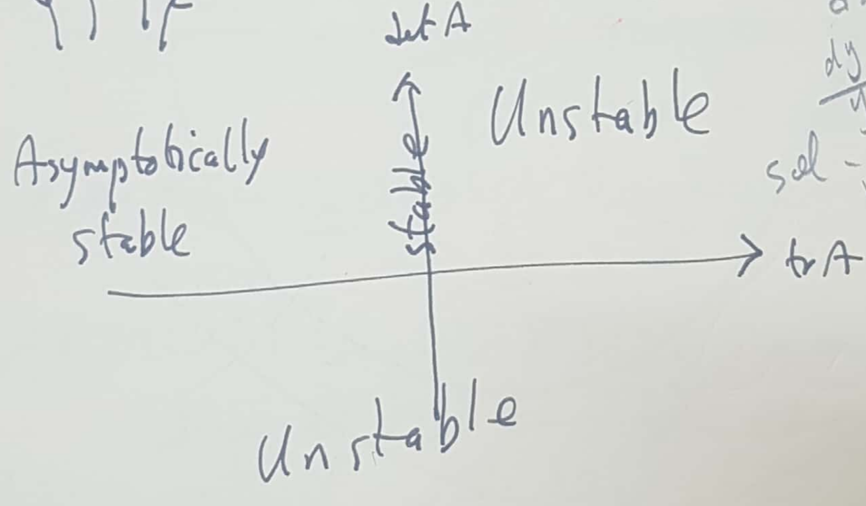
$\dot{x} = \lambda_1 x$

$\dot{y} = \lambda_2 y$

$\frac{dy}{dx} = \frac{\lambda_2 y}{\lambda_1 x}$

$\frac{dy}{y} = \frac{\lambda_2}{\lambda_1} \frac{dx}{x}$

sol - $y(x) = C\, x^{\frac{\lambda_2}{\lambda_1}}$



All phase portraits are collected in one diagram

سنعود ثاني
علاماتك ر(رابع رياضيات)

طريقة ليبانوف

Lyapunov Method

هي طريقة أخرى لاختبار استقرارية نقاط الاتزان للأنظمة خاصة عندما لا يكون للأنظمة جزء خطي مثلا

$$\dot{x} = x^3 + y^2 = f(x,y)$$

$$\dot{y} = xy^2 + y^4 + x^3 = g(x,y)$$

من الواضح أن $(0,0)$ نقطة اتزان لكن لا يوجد لهذا النظام جزء لاخطي كي نستخدمه لمعرفة الاستقرارية

تتلخص طريقة ليبانوف كالآتي

Th. Consider

$$\dot{x} = f(x,y) \qquad \cdot = \frac{d}{dt}$$

$$\dot{y} = g(x,y)$$

Let $(0,0)$ be an equilibrium point,

Let $V(x,y)$ be a function s.t

محاضرة 8 علوم تطبيقية / [illegible]

سهام علي

<u>طريقة ليابونوف</u>

$V(0,0)=0$

$V(x,y)>0 \;\; \forall\, (x,y)\neq(0,0)$ &

$(x,y)\in D\subseteq \mathbb{R}^2$, $(0,0)\in D$

If

(1) $\dot V\le 0$ then $(0,0)$ is stable.

(2) $\dot V<0$ " " " asymptotically stable

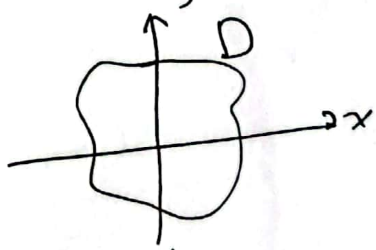
(3) $\dot V>0$ " " " unstable.

<u>eg</u> Use Lyapunov theorem to check the stability of $(0,0)$ for

$$\dot x = x^3+xy^2$$
$$\dot y = yx^2+y^3$$

<u>Sol.</u> Clearly $(0,0)$ is equili pt.

Let $V(x,y)=x^2+y^2$



Now $V(0,0)=0$ ✓

$V(x,y)>0 \;\; \forall\, (x,y)\in D$ s.t $(0,0)\in D$

$$\dot{V} = 2x\dot{x} + 2y\dot{y} = 2x[x^3 + xy^2] + 2y[yx^2 + y^3]$$

$$= 2x^4 + 2x^2y^2 + 2y^2x^2 + 2y^4 > 0$$

$\therefore (0,0)$ is unstable.

<u>eg2</u> Check the stability of the equil pt of

[illegible]

عدد نقاط الاتزان [illegible]

$$\dot{x} = -2xy^2$$
$$\dot{y} = x^2y - y^3$$

by Lyapunov theorem.

<u>Sol.</u> $f(x,y) = -2xy^2$, $g(x,y) = x^2y - y^3$

$x^2 \; 0 - 0 = 0$

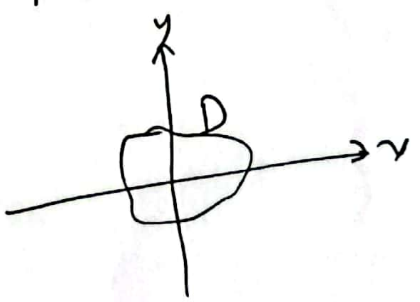
x^2 arbitrary

$f = 0 \Rightarrow x = 0$, $y = 0$

$g = 0$ if $x = 0$ then $y = 0$

If $y = 0$ then x arbitrary

$\therefore$ x-axis is equil ptr



For $(0,0)$

Let $V(x,y) = x^2 + y^2$

1) $V(0,0) = 0$ ✓

(2) $V(x,y) > 0 \;\forall\, (x,y) \in D$

(3) $\dot{V} = 2x\dot{x} + 2y\dot{y} = 2x(-2xy^2) + 2y(x^2y - y^3)$

$$= -2x^2y^2 - 2y^4 \leq 0 \Rightarrow (0,0) \text{ is stable.}$$

Perturbation Method طريقة الاضطراب

Perturbation method are aims of finding approximate analytic Solution to the problems whose exact analytic solution can not be found. It's applicable where family of equation, $p(\varepsilon)$ exist and depending on a parameter $\varepsilon > 0$ and where $p(0)$ has a known solution. perturbation method are design to construct Sol. to $p(\varepsilon)$ by adding small correction to known sol. of $p(0)$.

Now, look for this O.D.E

$$y'' + y = 0 \Rightarrow y(0) = 1 \ , \ y'(0) = 1$$

$$m^2 + 1 = 0 \rightarrow m = \mp i \Rightarrow y(t) = A\cos t + B\sin t$$

$$y(0) = A = 1 \ , \ y(t) = \cos t + B\sin t$$

$$y'(t) = -\sin t + B\cos t$$

$$y'(0) = 1 = B$$

$\therefore y(t) = \cos t + \sin t$

$$y(t) = \left(1 - \frac{t^2}{2!} + \dots\right) + \left(t - \frac{t^3}{3!} + \dots\right)$$

Now, if $y'' + \varepsilon y = 0$ --- (*) , $y(t) = \sum_{k=0}^{\infty} \varepsilon^k y_k$

$$\boxed{y(t) = \varepsilon^0 y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots}$$

subtitute this eq. in o.d.e (*) to get