## 3.2.3 Equations of Parabolic Type

معادلات من نمط القطع المكافئ

In such kind of equations $B^2 - AC = 0$

therefore equations (3.14) & (3.15) are the same

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - AC}}{A} = \frac{B}{A}$$

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - AC}}{A} = \frac{B}{A}$$

so we get one characteristic curve $\xi$ or $\eta$

since $B^2 - AC = 0$ & $A^* = 0$ then $B^* = 0$

then equ. (3.9) becomes

$$u_{\eta\eta} = \frac{H^*}{C^*} \quad \text{----} \quad (3\text{-}21)$$

which is the canonical form of the equation of parabolic type

or in the form

$$u_{\xi\xi} = \frac{H^*}{A^*} \quad \text{....} \quad (3.22)$$

**Note** We get one of $\xi$ or $\eta$ say $\xi$

so we choose $\eta(x,y)$ s.t. $\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0$

say $\eta(x,y) = y$ or any function

e.g Solve if possible

$$u_{xx} - 2y u_{xy} + y^2 u_{yy} - 3u = 10 - u_x$$

sol At first we find its canonical form

$$A = 1,\ B = -y \ \& \ C = y^2$$

$$B^2 - AC = 0$$

$\therefore$ It is of parabolic type

$$\frac{dy}{dx} = B = -y$$

$$\therefore \xi(x,y) = \int \left(\frac{dy}{y} + dx\right) = \ln|y| + x$$

Choose $\eta(x,y) = y$

Now $J(\xi,\eta) = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} 1 & \frac{1}{y} \\ 0 & 1 \end{vmatrix} \neq 0$

$$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y = \frac{1}{y} u_\xi + u_\eta$$

For the student evaluate $u_{xx}$, $u_{xy}$ & $u_{yy}$

After substituting these five partial derivatives in the given pde.

$$\eta^2 u_{\eta\eta} + 3u = 10$$

We can look to this pde as ode

i.e $\eta^2 u'' + 3u = 10$

① If we want to find the solution in a neighbourhood of $\eta = 0$ then we use

Frobenuis method

$$u(\eta) = \sum_{n=0}^{\infty} c_i \eta^{i+r}$$

③ طريقة ثالثة بحله

Let $u = v + \frac{10}{3}$

$\eta^2 v'' + (3v + 10) = 10$

$\eta^2 v'' + 3v = 0$ Euler eq.

Let $\eta = e^x \rightarrow x = \ln \eta$

$$\frac{dv}{d\eta} = \frac{1}{\eta}\frac{dv}{dx}$$

$$\frac{d^2v}{d\eta^2} = \frac{1}{\eta^2}\left(\frac{d^2v}{dx^2} - \frac{dv}{dx}\right)$$

$\eta^2 v'' + 3v = 0$ becoms

$$\frac{d^2v}{dx^2} - \frac{dv}{dx} + 3v = 0$$

i.e $m^2 - m + 3 = 0$

$$m_{1,2} = \frac{1 \mp \sqrt{-11}}{2} = \frac{1}{2} \mp \frac{\sqrt{11}}{2} i$$

$$v(x) = e^{\frac{1}{2}x}\left[A \cos \frac{\sqrt{11}}{2} x + B \sin \frac{\sqrt{11}}{2} x\right]$$

......

② If we want to find the sol. in a neighbourhood of $\eta = a \neq 0$ then we suppose

$$u(\eta) = \sum_{i=0}^{\infty} c_i (\eta - a)^i$$

<u>Another sol.</u> by taking $\eta(x,y) = x$

$$J(\xi,\eta) = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} 1 & \frac{1}{y} \\ 1 & 0 \end{vmatrix} \neq 0 \ \forall y \neq 0$$

$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$

$u_y = u_\xi \xi_y + u_\eta \eta_y = \frac{1}{y} u_\xi$

$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$u_{xy} = \frac{1}{y} u_{\xi\xi} + \frac{1}{y} u_{\eta\xi}$

$u_{yy} = \frac{1}{y^2} u_{\xi\xi} - \frac{1}{y^2} u_\xi$

<u>ex.</u> Complete the steps.

## 3-2-4 Equations of Elleptic Type
معادلات من نمط القطع الناقص

In this case $B^2 - AC < 0$ so we rewrite equ. (3.14) & (3-15) in the form

$$\left.\begin{aligned} \frac{dy}{dx} &= \frac{B}{A} + \frac{\sqrt{AC-B^2}}{A}\, i \\ \frac{dy}{dx} &= \frac{B}{A} - \frac{\sqrt{AC-B^2}}{A}\, i \end{aligned}\right\} \quad (i = \sqrt{-1}) \quad \ldots \cancel{3.2}$$

The characteristic curves are

$$\xi(x,y) = y - \int \frac{B}{A}\,dx + \int \frac{\sqrt{AC-B^2}}{A}\,dx\; i$$

$$\eta(x,y) = y - \int \frac{B}{A}\,dx - \int \frac{\sqrt{AC-B^2}}{A}\,dx\; i$$

<u>Note</u> that $\xi = \bar{\eta}$ or $\eta = \bar{\xi}$ (conjugate)

i.e $\xi$ & $\eta$ are complex functions

Define the two real functions

$$\left.\begin{aligned} \alpha &= \frac{1}{2}(\xi + \eta) \\ \& \quad \beta &= \frac{1}{2i}(\xi - \eta) \end{aligned}\right\} \quad \ldots (3\text{-}21)$$

By substituting these functions $\alpha$ & $\beta$ in equ. (3-8) we get the equ.

$$A^{**}(\alpha,\beta)u_{\alpha\alpha}+2B^{**}(\alpha,\beta)u_{\alpha\beta}+C^{**}(\alpha,\beta)u_{\beta\beta}$$
$$=H^{**}(\alpha,\beta,u_\alpha,u_\beta,u) \quad \ldots(3\text{-}24)$$

Since $A^*=B^*=0$ and since the coefficients in (3-24) look like those in (3-8) so we get

$$\left.\begin{aligned} A^{**}-C^{**}+i\,B^{**}&=0\\ A^{**}-C^{**}-i\,B^{**}&=0\end{aligned}\right\} \Rightarrow A^{**}-C^{**}=0 \text{ i.e. } A^{**}=C^{**} \text{ and } B^{**}=0$$

$\therefore$ equ. (3-24) becomes

$$u_{\alpha\alpha}+u_{\beta\beta}=\frac{H^{**}}{A^{**}} \quad \ldots(3.25)$$

which is called the canonical form of elliptic type pdes.

<u>eg</u> Find the canonical form of

$$u_{xx}-4u_{xy}+8u_{yy}+yu_x-xu_y-u=12$$

<u>Sol</u> $B^2-AC=-4<0$

$\therefore$ It is of elliptic type

$$\frac{dy}{dx} = -2+2i \quad \& \quad \frac{dy}{dx} = -2-2i$$

The characteristic curves are

$$\xi(x,y) = y+2x-2ix$$

$$\eta(x,y) = y+2x+2ix$$

Let $\alpha = \frac{1}{2}(\xi+\eta)$ & $\beta = \frac{1}{2i}(\xi-\eta)$

So $\alpha = y+2x$ & $\beta = -2x$

$\therefore x = -\frac{1}{2}\beta, \; y = \alpha+\beta$

$\alpha_x = 2, \; \alpha_y = 1, \; \beta_x = -2, \; \beta_y = 0 = \alpha_{xx} = \alpha_{xy} = \alpha_{yy}$

also $\beta_{xx} = \beta_{xy} = \beta_{yy} = 0$

After substituting in the given equ. (ex.)

we get

$$u_{\alpha\alpha} + u_{\beta\beta} = \frac{2(\alpha+\beta)u_\beta - (2\alpha+\frac{5}{2}\beta)u_\alpha + u + 12}{4}$$