جامعة بغداد

كلية التربية للعلوم الصرفة ابن الهيثم

قسم الرياضيات/ المرحلة الاولى

مادة التفاضل والتكامل

الفصل الاول

المتراجحات

Inequalities

اعضاء التدريس

- أ.م.د. ايمان محمد نعمه
- أ.م اريج صلاح محمد
- م. حنان فاروق قاسم

CHAPTER ONE: The Real Numbers $\mathbb{R}$

The subsets of:

1. Natural Numbers (denoted by $\mathbb{N}$) such that:

$$\mathbb{N} = \{1,2,3,4, \ldots\}$$

2. Intager Numbers (denote by $\mathbb{I}$ or $\mathbb{Z}$) such that:

$$\text{I or } \mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

3. Rational Numbers (denoted by $\mathbb{Q}$) : it is all numbers of the form $\frac{\mathrm{p}}{\mathrm{q}}$, such that p and q are integers and $q \neq 0$:

$$\mathbb{Q} = \left\{x \in \mathbb{R}: x = \frac{p}{q}, \text{ where } p, q \in \mathbb{Z} \text{ and } q \neq 0\right\}$$

Example: $\frac{1}{2}, \frac{5}{3}, 0, \frac{50}{10}, \ldots$

Note (1): The rational Numbers can be written as decimal form

$$\left(\frac{1}{3} = 0.333, \frac{1}{4} = 0.25, \ldots\right).$$

4. Irrational Numbers (denoted by $\mathbb{Q}'$): A number which is not rational is said to be irrational.

Example: $\{\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \pi = 3.14, \ldots\}$

Note (2): $\emptyset \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$ and $\mathbb{Q} \cup \mathbb{Q}' = \mathbb{R}$

Properties of Real Numbers with Addition: $(\mathbb{R}, +)$
Let $\mathrm{a, b, c} \in \mathrm{R}$, then:

1. $a + b \in \mathbb{R}$ (Closure)
2. $a + b = b + a$ (Commutative)
3. $a + (b + c) = (a + b) + c$ (Associative)
4. $a + 0 = 0 + a = a$ (Identity Element)
5. $\exists(-a) \in \mathbb{R}$ such that $a + (-a) = (-a) + a = 0$ (Additive Inverse)

Properties of Real Numbers with Multiplication: $(\mathbb{R}, .)$

Let $a, b, c \in \mathbb{R}$, then:

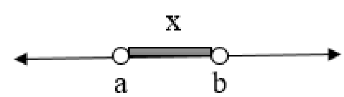
1. $a.b \in \mathbb{R}$ (Closure)
2. $a.b = b.a$ (Commutative)
3. $a.(b.c) = (a.b).c$ (Associative)
4. $1.a = a.1 = a$ (Multiplicative Identity)
5. $a.(b+c) = a.b + a.c$ (Distributive)

   $(b+c)\cdot a = b\cdot a + c\cdot a$
6. $\exists a^{-1} \in$ R such that $a\cdot a^{-1} = a\cdot\frac{1}{a} = 1$ (Multiplication Inverse)

Intervals:-

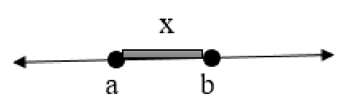
1. **Finite intervals**:- Let $a, b \in R$ such that $a < b$ then:

(a) **Open Interval:** $\{x \in R: a < x < b\} = (a, b)$



(Note: $a \notin (a,b)$ and $b \notin (a,b)$)

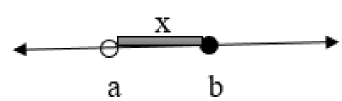
(b) **Closed Interval:** $\{x \in R: a \le x \le b\} = [a, b]$



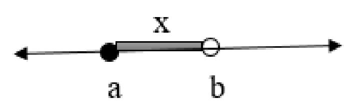
(Note: $\in [a,b]$ and $b \in [a,b]$)

(c) **The Half Open Interval:** $\{x \in R: a < x \le b\} = (a, b]$ $(Note: b \in (a,b]$ and $a \notin (a,b])$

OR:



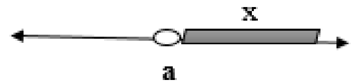
The Half Open Interval: $\{x \in R: a \le x < b\} = [a, b)$



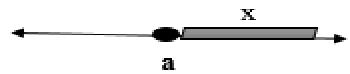
(Note: $\notin [a,b)$ and $a \in [a,b)$)

2. **Infinite intervals**: Let each of $a, b \in R$ such

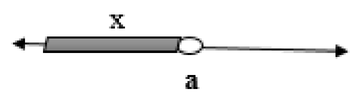
(a) $\{x \in R\,;\ a < x < \infty\ (or\ x > a)\} = (a, \infty)$



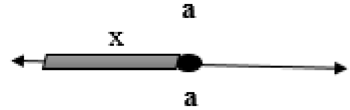
(b) $\{x \in R\,;\ a \le x < \infty\ (\ or\ x \ge a)\} = [a, \infty)$



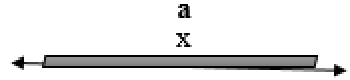
(c) $\{x \in R\,;\ -\infty < x < a\ (or\ x < a)\} = (-\infty, a)$



(d) $\{x \in R\,;\ -\infty < x \le a\ (or\ x \le a)\} = (-\infty, a]$



(e) $\{x \in R\,;\ -\infty < x < \infty\} = (-\infty, \infty) = R$



<u>Inequalities:-</u>

Let $a, b \in R, b$ is greater than a and denoted by $b > a$ if $b - a > 0$.

<u>Solving Inequalities</u>:-

Solving the inequalities means obtaining all values of x for which the inequality is true.

<u>Properties of Inequalities:-</u>

Let $a, b, c \in \mathbb{R}$, then:

1. If $a < b$, then $a + c < b + c$
2. If $a < b$ and $c > 0$, then $a.c < b.c$
3. If $a < b$ and $c < 0$, then $a.c > b.c$
4. If $a, b \in \mathbb{R}\ or\ a, b \in \mathbb{R}^+$, $a < b$, then $\frac{1}{a} > \frac{1}{b}$

Note: In general, we have linear and non-linear inequalities.

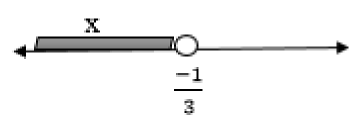
<u>Linear Inequalities Examples:-</u>

Example (1): Solve the following inequality: $3(x + 2) < 5$?

Solution:

$$3(x+2) < 5 \longrightarrow 3x + 6 < 5 \longrightarrow 3x + 6 - 6 < 5 - 6 \longrightarrow 3\text{x} < -1 \rightarrow \text{x} < \frac{-1}{3}$$

Hence, the solution set $S = \left\{x \in \mathbb{R}: x < \frac{-1}{3}\right\} = \left(-\infty, \frac{-1}{3}\right)$.

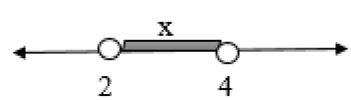


Example (2): Solve the following inequality: $7 < 2x + 3 < 11$?

Solution:

$$7 < 2x + 3 < 11 \longrightarrow 7 - 3 < 2x + 3 - 3 < 11 - 3 \longrightarrow 4 < 2x < 8 \longrightarrow \ 2 < \text{x} < 4$$

Hence, the solution set $S = \{x \in \mathbb{R}: 2 < x < 4\} = (2,4)$.



<u>Non-Linear Inequalities Examples:-</u>

Example (1): Solve the following inequality: $x^2 < 25$?

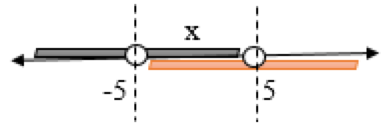
Solution:

$x^2 < 25 \ \rightarrow \ x^2 - 25 < 25 - 25 \ \rightarrow \ (x-5)(x+5) < 0$

Since the result is negative, then there are two possibilities: $+.-< 0\ or\ -.+< 0$

Either:

$(x+5) > 0$ and $(x-5) < 0 \longrightarrow x > -5$ and $x < 5$



So, the solution set is $S_1 = (-5,5)$

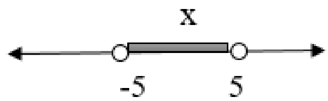
Or:

$(x+5) < 0$ and $(x-5) > 0 \longrightarrow\ x < -5$ and $\ x > 5$

So, the solution set is $S_2 = \emptyset$

Therefore, the solution set for the inequality is:

$S = S_1 \cup S_2 = (-5,5) \cup \emptyset = (-5,5)$



Example (2): Solve the following inequality: $\ x^2 - 5x > 6$?

Solution:

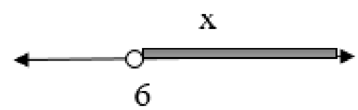
$$x^2 - 5x > 6 \ \rightarrow\ x^2 - 5x - 6 > 6 - 6 \ \rightarrow\ (x-6)(x+1) > 0$$

Since the result is Positive, then there are two possibilities: $\quad +.+> 0 \ or\ -.-> 0$

Either:

$(x-6) > 0$ and $(x+1) > 0 \longrightarrow x > 6$ and $x > -1$

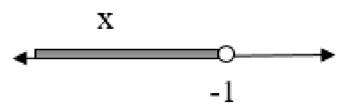
So, the solution set: $S_1 = \{x \in \mathbb{R}: x > 6\} = (6,\infty)$



Or:

$(x-6) < 0$ and $(x+1) < 0 \longrightarrow\ x < 6$ and $\ x < -1$

So, the solution set: $S_2 = \{x \in \mathbb{R} : x < -1\} = (-\infty,-1)$



Therefore, the solution set for the inequality is:

$S = S_1 \cup S_2 = (6,\infty) \cup (-\infty,-1) = \mathbf{R} \setminus [-1,6]$

<u>Absolute Value:-</u>

The absolute value of a real number x is denoted by $|x|$ and defined as follows:

$$|x| = \sqrt{x^2} = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

Examples: $|-8| = 8$, $\left|\frac{-2}{3}\right| = \frac{2}{3}$, $\ |9| = 9$, $\ |0| = 0$, etc.

<u>Properties of Absolute Value:-</u>

1. $|-a| = |a|$

   proof: $|-a| = \sqrt{(-a)^2} = \sqrt{a^2} = |a|$

2. $\| a \| = |a|$

   proof: $\| a|| = \sqrt{|a|^2} = \sqrt{a^2} = |a|$

3. $|a.b| = |a| \cdot |b|$
   proof: $|a \cdot b| = \sqrt{(a \cdot b)^2} = \sqrt{a^2 \cdot b^2} = \sqrt{a^2} \cdot \sqrt{b^2} = |a| \cdot |b|$

4. $\left|\frac{a}{b}\right| = \left|\frac{|a|}{|b|}\right|$; $b \neq 0$
   proof: $\left|\frac{a}{b}\right| = \sqrt{\left(\frac{a}{b}\right)^2} = \sqrt{\frac{a^2}{b^2}} = \frac{\sqrt{a^2}}{\sqrt{b^2}} = \frac{|a|}{|b|}$

5. $|a + b| \leq |a| + |b|$

Solving Absolute Value Inequalities:-
The absolute value of x can be written as follows:

$$|x| = \sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The above definition means the absolute value of any real number is a real non-negative number.

Geometrically, the absolute value of unmber x is the distance point between " x " and the origin point " 0 ". In general, $|a - b|$ is the distance between a and b on the real number line " $\mathbb{R}$ ".

Remarks:

1. To solve the inequality $|x| < a$ where $a, x \in \mathbb{R}$.

Case (1): If $x \geq 0 \Longrightarrow |x| = x,$ but $|x| < a \Longrightarrow x < a \Longrightarrow S_1 = (-\infty, a)$
Case (2): If $x < 0 \Longrightarrow |x| = -x,$ but $|x| < a \Longrightarrow -x < a \Longrightarrow x > -a. \Longrightarrow S_2 = (-a, \infty)$
Since, $S = S_1 \cap S_2$

$$\Longrightarrow \{x \in \mathbb{R}: |x| < a\} = \{x \in \mathbb{R}: -a < x < a\} = (-a, a)$$

Similarly,

$$\Longrightarrow \{x \in \mathbb{R}: |x| \leq a\} = \{x \in \mathbb{R}: -a \leq x \leq a\} = [-a, a]$$

2. To solve the inequality $|x| > a$ where $a, x \in \mathbb{R}$.

Case (1): If $x \geq 0 \Longrightarrow |x| = x,$ but $|x| > a \Longrightarrow x > a \Longrightarrow S_1 = (a, \infty)$

Case (2): If $x < 0 \Longrightarrow |x| = -x,$ but $|x| > a \Longrightarrow -x > a \Longrightarrow x < -a. \Longrightarrow S_2 = (-\infty, -a)$

Since, $S = S_1 \cup S_2$

$$\Longrightarrow \{x \in \mathbb{R}: |x| > a\} = \{x \in \mathbb{R}: x > a \ or \ x < -a\} = (a, \infty) \cup (-\infty, -a) = \mathbb{R} \setminus [-a, a]$$

Similarly,

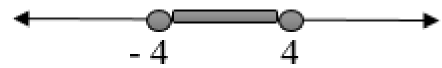
$$\Longrightarrow \{x \in \mathbb{R}: |x| \geq a\} = \{x \in \mathbb{R}: x \geq a \text{ } or \text{ } x \leq -a\} = [a, \infty) \cup (-\infty, -a] = \mathbb{R} \setminus (-a, a)$$

Examples: Find the solution set for the following inequalities?

- $|x| > 3$
  Solution:

$$\{x \in \mathbb{R}: |x| > 3\} = \{x \in \mathbb{R}: x > 3 \text{ or } x < -3\} = (3, \infty) \cup (-\infty, -3) = \mathbb{R} \setminus [-3,3]$$

- $|x| \leq 4$
  Solution:

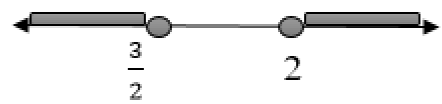


$$\{x \in \mathbb{R}: |x| \leq 4\} = \{x \in \mathbb{R}: -4 \leq x \leq 4\} = [-4,4]$$

- $|x-4| < 5$
  Solution:

$$\{x \in \mathbb{R}: |x-4| < 5\} = \{x \in \mathbf{R}: -5 < x-4 < 5\} = \{x \in \mathbb{R}: -1 < x < 9\} = (-1,9)$$

- $|7-4x| \geq 1$
  Solution:



$$\{x \in \mathbb{R}: |7-4x| \geq 1\} = \{x \in \mathbb{R}: 7-4x \geq 1 \text{ or } 7-4x \leq -1\}$$

$$= \{x \in \mathbb{R}: -4x \geq -6 \text{ or } -4x \leq -8\}$$

$$= \left\{x \in \mathbb{R}: x \leq \frac{3}{2} \text{ or } x \geq 2\right\}$$

$$= \left(-\infty, \frac{3}{2}\right] \cup [2, \infty)$$

$$= \mathbb{R} \setminus \left(\frac{3}{2}, 2\right)$$

Problems 1.1:

1. Write the following sets equivalent interval, and test of these intervals whether they are Open, Close or Half Open Intervals:

   (a) $\{x: -20 \leq x \leq -12\}$

   (b) $\{x: -3 \leq x < 4\}$

   (c) $\{x: -1 < x < 10\}$

   (d) $\{x: -2 < x \leq 0\}$

2. Give a description of the following intervals as sets:

   (a) $(3,5)$

   (b) $(-3,0)$

   (c) $[2,7]$

   (d) $[-5,-1)$

   (e) $(-4,4)$

   (f) $(-0,7]$

3. Find the solution set of the following inequalities:

   (a) $x(x-3) > 4$

   (b) $2 < \frac{1}{\mathrm{x}}; x \neq 0$

   (c) $x^2 \geq 25$

   (d) $x^2 - 2x - 24 < 0$

   (e) $-7 \leq -3x + 5 \leq 14$

   (f) $\frac{x}{x-3} < 4$

   (g) $\frac{x^2+2x-35}{x+2} > 0$

   (h) $6x - 4 > 7x + 2$

   (i) $x^2 \leq 16$

   (j) $3x^2 > 2x + 5$

   (k) $x^2 > 5x + 6$

   (1) $\frac{x-3}{x+2} < 5$

   (m) $\frac{1}{x-2} > \frac{2}{\mathrm{x}+3}$

   (n) $\frac{x-2}{x+3} < \frac{1}{2}$

4. Find the solution set of the following inequalities:

   (a) $|x| \geq 5$

   (b) $|x| < 2$

   (c) $|3x+3| \geq 2$

   (d) $1 \leq \left|\frac{x-3}{1-2\mathrm{x}}\right| \leq 2$

   (e) $\left|\frac{2-x}{x-3}\right| \geq 4$

   (g) $\frac{|2-x|}{3x} \leq 1$

   (h) $\left|\frac{3+2x}{3x}\right| \leq 1$

   (i) $|x-1| \geq 6$

   (j) $|2-2x| \leq 7$

   (k) $\left|\frac{4}{2x+1}\right| \leq 3$

   (f) $|x+1| < |3x+4|$