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الفصل الثالث

الغايات و الاستمرارية

Limits and Continuity

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CHAPTER THREE: Limits and Continuity

Definition: If the values of $f(x)$ approaches the value L as x approaches c, we say that f has **limit** equal to L as x approaches c, and we write it as:

$$\lim_{x\to c} f(x) = L$$

Example: Let $f(x) = x^2 + 3$, find the limit of $f(x)$ as x approaches 2 .

$x \to 2^+$	x	3	2.5	2.3	2.1	2.01	2.001	2.0001	
(from the right)	$f(x)$	12	9.25	8.25	7.44	7.040	7.004	7.0007	$\simeq 7$
$x \to 2^-$	x	1	1.2	1.4	1.5	1.9	1.99	1.999	
	$f(x)$	4	4.44	4.96	5.95	5.98	6.98	6.999	$\simeq 7$

From the table, we notice that:

- When x approaches 2 from the right, $f(x)$ approaches 7

  (i.e., $\lim_{x\to 2+} f(x) = 7$).

- When x approaches 2 from the left, $f(x)$ approaches 7

  (i.e., $\lim_{x\to 2^-} f(x) = 7$).

Properties of Limits: Let $\lim_{x\to c} f_1(x) = \mathrm{L}_1$ and $\lim_{x\to c} f_2(x) = \mathrm{L}_2$

where $c, K, L_1, L_2 \in \mathbb{R}$, then:

1. $\lim_{x\to c} x = c$
2. $\lim_{x\to c}[f_1(x) \mp f_2(x)] = \lim_{x\to c} f_1(x) \mp \lim_{x\to c} f_2(x) = L_1 \mp L_2$
3. $\lim_{x\to c}[f_1(x) * f_2(x)] = \lim_{x\to c} f_1(x) * \lim_{x\to c} f_2(x) = L_1 * L_2$
4. $\lim_{x\to c}\left[K * f_1(x)\right] = K \lim_{x\to c} f_1(x) = K * L_1$
5. $\lim_{x\to c} \frac{f_1(z)}{f_2(x)} = \frac{\lim_{x\to c} f_1(z)}{\lim_{x\to c} f_2(x)} = \frac{L_1}{L_2}$, where $L_2 \neq 0$

6. $\lim_{x\to c}[f_1(x)]^n = \left[\lim_{x\to c} f_1(x)\right]^n = {L_1}^n$

7. $\lim_{x\to c} \sqrt[n]{f_1(x)} = \sqrt[n]{\lim_{x\to c} f_1(x)} = \sqrt[n]{L_1}$

Examples: Evaluate the following limits:

1. $\lim_{n\to 5} \frac{\sqrt{4+n}-2}{n} = \frac{\sqrt{4+5}-2}{5} = \frac{1}{5}$

2. $\lim_{x\to 2} \frac{x^2+2x+4}{x+2} = \frac{2^2+2.2+4}{2+2} = \frac{12}{4} = 3$

3. $\lim_{x\to 5} \frac{x^2-25}{3(x-5)} = \lim_{x\to 5} \frac{(x+5)(x-5)}{3(x-5)} = \lim_{x\to 5} \frac{x+5}{3} = \frac{5+5}{3} = \frac{10}{3}$

4. $\lim_{h\to 0} \frac{(2+h)^2-4}{h} = \lim_{h\to 0} \frac{4+4h+h^2-4}{h} = \lim_{h\to 0} \frac{h(4+h)}{h}$
$= \lim_{h\to 0}(4+h) = 4+0 = 0$

5. $\lim_{n\to 0} \frac{\sqrt{4+n}-2}{n} = \lim_{n\to 0} \frac{\sqrt{4+n}-2}{n} \cdot \frac{\sqrt{4+n}+2}{\sqrt{4+n}+2}$
$= \lim_{n\to 0} \frac{4+n-4}{n(\sqrt{4+n}+2)} = \lim_{n\to 0} \frac{n}{n(\sqrt{4+n}+2)}$
$= \lim_{n\to 0} \frac{1}{(\sqrt{4+n}+2)} = \frac{1}{\sqrt{4+0+2}} = \frac{1}{2+2} = \frac{1}{4}$

<u>Right and Left Hand-Side Limits:</u>

Sometimes the value of a function $f(x)$ lend to different limits as x approaches c from different sides.

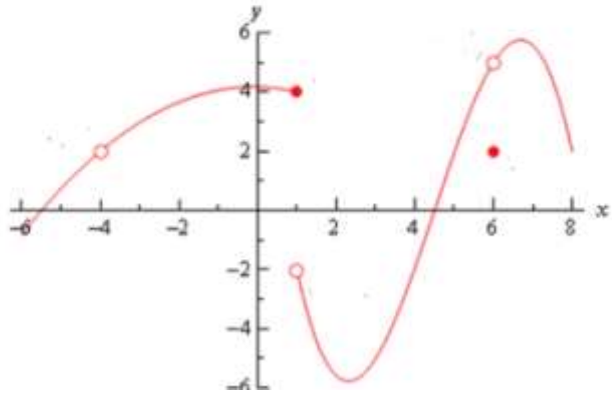
<u>Theorem:</u> Suppose that $f(x)$ is defined on an open interval that containing c. Then $\lim_{x\to c} f(x)$ is defined if and only if $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$ are both defined and equal.

i.e.

$$\lim_{x\to c} f(x) = L \Leftrightarrow \lim_{x\to c^+} f(x) = \lim_{x\to c-} f(x) = L$$

Note: If $\lim_{x \to c^+} f(x) \neq \lim_{x \to c^-} f(x) \Longrightarrow \lim_{x \to c} f(x)$ "**Does Not Exist**"

Example 1: Evaluate the following, where $f(x)$ is defined as shown below:



- $f(6) = 2$
- $f(1) = 4$
- $\lim_{x \to 6^-} f(x) = 5$, $\lim_{x \to 6^+} f(x) = 5 \quad \Longrightarrow \lim_{x \to 6} f(x) = 5$
- $\lim_{x \to 1^-} f(x) = 4$, $\lim_{x \to 1^+} f(x) = -2 \Longrightarrow \lim_{x \to 1} f(x)$ "Does Note Exists"

Example 2: Let $f(x) = \begin{cases} x^2 - 4 & \text{if } x \leq 3 \\ 5 & \text{if } x > 3 \end{cases}$

Find $\lim_{x \to 3^+} f(x)$, $\lim_{x \to 3^-} f(x)$, and $\lim_{x \to 3} f(x)$

Solution:-

- $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 5 = 5$
- $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x^2 - 4 = 3^2 - 4 = 9 - 4 = 5$
- $\lim_{x \to 3} f(x) = ?$

  $\because \lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x) = 5 \Longrightarrow \lim_{x \to 3} f(x) = 5$

<u>Example 3</u>: Lef $g(x) = \begin{cases} \sqrt{x+4} - 1 & \text{if } x < 0 \\ -2 & \text{if } x = 0 \\ \frac{x}{x+3} & \text{if } x > 0 \end{cases}$

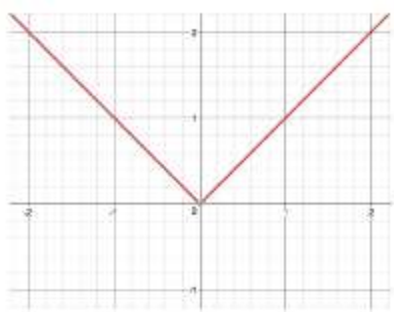
Find $\lim\limits_{x \to 0^+} g(x)$, $\lim\limits_{x \to 0^-} g(x)$, and $\lim\limits_{x \to 0} g(x)$

Solution:-

- $\lim\limits_{x \to 0^+} g(x) = \lim\limits_{x \to 0^+} \frac{x}{x+3} = \frac{0}{0+3} = 0$
- $\lim\limits_{x \to 0^-} g(x) = \lim\limits_{x \to 0^+} \sqrt{x+4} - 1 = \sqrt{0+4} - 1 = 2 - 1 = 1$
- $\lim\limits_{x \to 0^+} g(x) = 1 \neq 0 = \lim\limits_{x \to 0^-} g(x) \Longrightarrow \lim\limits_{x \to 0} g(x)$ " **Does Not Exists**"

<u>Example 4</u>: Evaluate $\lim\limits_{x \to 0} |x|$?

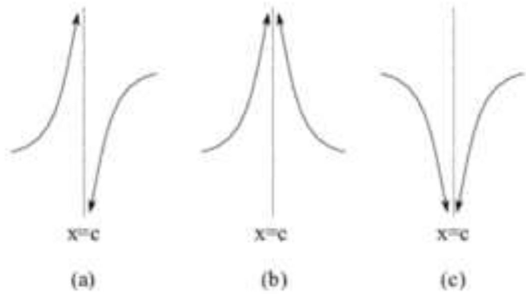
Solution :



$\lim\limits_{x \to 0} |x| = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$

- $\lim\limits_{x \to 0^+} |x| = \lim\limits_{x \to 0^+} (x) = 0$
- $\lim\limits_{x \to 0^-} |x| = \lim\limits_{x \to 0^-} (-x) = 0$
- $\because \lim\limits_{x \to 0^+} |x| = \lim\limits_{x \to 0^-} |x| = 0 \Longrightarrow \lim\limits_{x \to 0} |x| = 0$

<u>Infinite ($\mp\infty$) limits</u>: Let $f(x)$ be defined as follows, then:



In (a): $\lim\limits_{x\to c^+} f(x) = -\infty$ and $\lim\limits_{x\to c^-} f(x) = +\infty \quad \Longrightarrow \lim\limits_{x\to c} f(x)$ "Does Not Exist"

In (b): $\lim\limits_{x\to c^+} f(x) = +\infty$ and $\lim\limits_{x\to c^-} f(x) = +\infty \quad \Longrightarrow \lim\limits_{x\to c} f(x) = +\infty$

In (c): $\lim\limits_{x\to c^+} f(x) = -\infty$ and $\lim\limits_{x\to c^-} f(x) = -\infty \quad \Longrightarrow \lim\limits_{x\to c} f(x) = -\infty$

<u>Remark:</u>

$$\frac{0}{(\mp)\text{ value}} = 0 \quad , \quad \frac{(+)\text{ value}}{0} = +\infty \quad , \quad \frac{(-)\text{ value}}{0} = -\infty$$

<u>Example 1:</u> Evaluate $\lim\limits_{x\to 0} \frac{1}{x^2}$?

Solution:-

- $\lim\limits_{x\to 0^+} \frac{1}{x^2} = \frac{1}{\text{"a positive value that is very close to zero"}} = +\infty$
- $\lim\limits_{x\to 0^-} \frac{1}{x^2} = \frac{1}{\text{a positive value that is very close to zero}} = +\infty$

$$\because \lim_{x\to 0^+} \frac{1}{x^2} = \lim_{x\to 0^-} \frac{1}{x^2} = +\infty \Rightarrow \lim_{x\to 0^-} \frac{1}{x^2} = +\infty$$

<u>Example 2:</u> Evaluate $\lim\limits_{x\to 2} \frac{1}{x-2}$?

Solution:-

- $\lim\limits_{x\to 2^+} \frac{1}{x-2} = \frac{1}{\text{"a positive value that is very close to zero"}} = +\infty$
- $\lim\limits_{x\to 2^-} \frac{1}{x-2} = \frac{1}{\text{"a negative value that is very close to zero"}} = -\infty$

  $\because \lim\limits_{x\to 2^+} \frac{1}{x-2} \neq \lim\limits_{x\to 2^-} \frac{1}{x-2} \Longrightarrow \lim\limits_{x\to 2} \frac{1}{x-2}$ DOES NOT EXIST"

Example 3: Evaluate $\lim_{x\to -1} \frac{x}{1+x}$?

Solution:-

- $\lim_{x\to -1^+} \frac{x}{1+x} = \frac{\text{negative value}}{\text{"a positive value that is very close to zero "}} = -\infty$

- $\lim_{x\to -1^-} \frac{x}{1+x} = \frac{\text{negativevalue}}{\text{"a negative value that is very close to zero "}} = +\infty$

  $\because \lim_{x\to -1^+} \frac{x}{1+x} \neq \lim_{x\to -1^-} \frac{x}{1+x} \Longrightarrow \lim_{x\to -1} \frac{x}{1+x}$ "DOES NOT EXIST"

Evaluating Limits at Infinite ($\pm\infty$):

Remark (1): $\frac{1}{\pm\infty} = 0$

Example: $\lim_{x\to\infty} \frac{1}{\mathrm{x}} = \frac{1}{\infty} = 0 = \lim_{x\to -\infty} \frac{1}{x} = \frac{1}{-\infty} = 0$

Remark (2): To find the limit of a rational function as $x \to \infty$ (when the limit exists), we divided the numerator and denominator by the highest power of x in the denominator.

Examples:

- $$\lim_{x\to\infty} \frac{x^2+2x+1}{5x^2+2} = \lim_{x\to\infty} \frac{\frac{x^2+2x+1}{x^2}}{\frac{5x^2+2}{x^2}}$$

$$\lim_{x\to\infty} \frac{\frac{x^2}{x^2}+\frac{2x}{x^2}+\frac{1}{x^2}}{\frac{5x^2}{x^2}+\frac{2}{x^2}} = \lim_{x\to\infty} \frac{1+\frac{2}{x}+\frac{1}{x^2}}{\frac{5}{1}+\frac{2}{x^2}}$$

$$= \lim_{x\to\infty} \frac{1+0+0}{5+0} = \frac{1}{5}$$

- $\lim\limits_{x\to\infty} \frac{x-2}{2x^2-7x+5} = \lim\limits_{x\to\infty} \frac{\frac{x}{x^2}-\frac{2}{x^2}}{\frac{2x^2}{x^2}-\frac{7x}{x^2}+\frac{5}{x^2}}$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}-\frac{2}{x^2}}{\frac{2}{1}-\frac{7}{x}+\frac{5}{x^2}} = \lim_{x\to\infty} \frac{0}{2-0+0} = 0$$

- $\lim\limits_{x\to\infty} \frac{x^5+x^2+2}{x^3+1} = \lim\limits_{x\to\infty} \frac{\frac{x^5+x^2+2}{x^3}}{\frac{x^3+1}{x^3}}$

$$= \lim_{x\to\infty} \frac{x^2+\frac{1}{x}+\frac{2}{x^3}}{1+\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\infty+0+0}{1+0} = +\infty$$

- $\lim\limits_{x\to\infty} \frac{-4x^3+7x}{2x^2-3x-10} = \lim\limits_{x\to\infty} \frac{\frac{-4x^3+7x}{x^2}}{\frac{2x^2-3x-10}{x^2}}$

$$= \lim_{x\to\infty} \frac{-4x+\frac{7}{x}}{2-\frac{3}{x}-\frac{10}{x^2}} = \frac{-\infty+0}{2-0-0} = -\infty$$

Problems (3.1):

1. Evaluate the following limits:

(a) $\lim\limits_{x\to 2} \frac{x^2}{x^3-9}$

(b) $\lim\limits_{x\to -3} \frac{\sqrt{2x+22}-4}{x+3}$

(c) $\lim\limits_{x\to 4} \frac{1}{x^2-16}$

(d) $\lim\limits_{x\to 0^+} \frac{1}{3x}$

(e) $\lim_{x\to 0^+} \frac{1}{x}$

(g) $\lim\limits_{x\to 3^+} \frac{1}{x-3}$

(h) $\lim\limits_{x\to -\infty} \frac{10x^5+x^4+31}{x^6}$

(i) $\lim\limits_{x\to\infty} \left(\frac{-x}{7x+4}+\frac{5x+2}{2x^3-1}\right)$

(j) $\lim\limits_{x\to -2^-} \frac{x^2-2}{x-2}$

(f) $\lim_{x\to-\infty} \frac{9-x-x^3}{3+2x+x^2}$

(k) $lim_{x\to\infty} \frac{2x+3}{5x+7}$

2. Let $f(x) = \begin{cases} \frac{x-2}{x-1} & \text{if } x \le 0 \\ \frac{1}{x^2} & \text{if } x > 0 \end{cases}$

Find:

(a) $f(0)$

(b) $lim_{x\to+\infty} f(x)$

(c) $lim_{x\to-\infty} f(x)$

(d) $lim_{x\to0^+} f(x)$

(e) $lim_{x\to0^-} f(x)$

(f) Does the Limit exists at $x = 0$?

3. Let $g(x) = \begin{cases} 3-x & \text{if } x < 2 \\ 6 & \text{if } x = 2 \\ \frac{x}{2} & \text{if } x > 2 \end{cases}$

Find:

(a) $g(2)$

(c) $g(-1)$

(b) $g(3)$

(d) $lim_{x\to2^+} g(x)$

(e) $lim_{x\to2^-} g(x)$

(f) Does the Limit exists at x =2?

Continuity (Continuous Function)

We say f is a continuous function at the point x_0 if there is no interrupt at x_0, and f is a contimous function at the interval x_0 if there is no any interrupt in this interval.

Definition: f is a continuous function at $x_0 \Leftrightarrow$

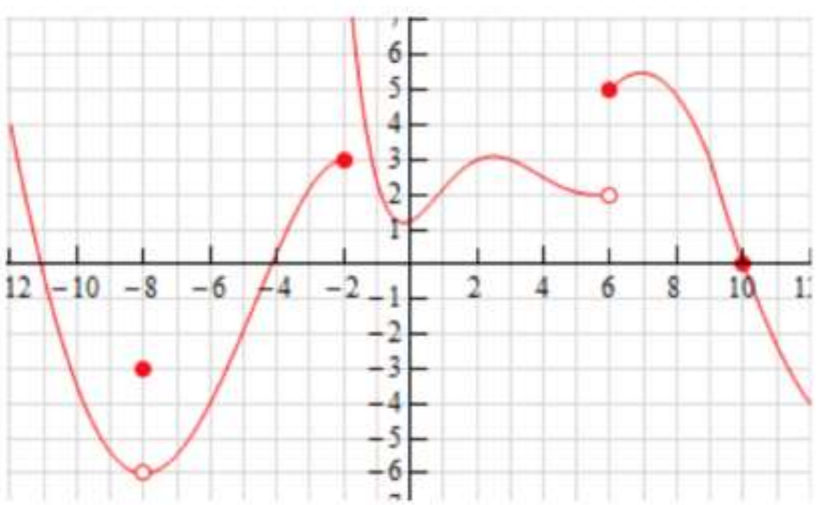
1. $f(x_0)$ exists
2. $\lim_{x \to x_0} f(x)$ exists (i.e., $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = L$)
3. $\lim_{x \to x_0} f(x) = f(x_0)$.

Definitions:

- If $\lim_{x \to x_0^+} f(x) = f(x_0)$, then $f(x)$ is continuous from the right at x_0
- If $\lim_{x \to x_0^-} f(x) = f(x_0)$, then $f(x)$ is continuous from the left at x_0
- $f(x)$ is continuous at $x_0 \Leftrightarrow f(x)$ is continuous from the right and the left.

Example 1: Let $f(x)$ be defined as shown below. Check if $f(x)$ is continuous at x_0 where

$$x_0 = 10, 6, -2, -8.$$



- $f(10) = 0$

- $\lim\limits_{x\to 10^+} f(x) = 0$

  $\lim\limits_{x\to 10^-} f(x) = 0$

  $\because \lim\limits_{x\to 10^+} f(x) = \lim\limits_{x\to 10^-} f(x) = 0 \Longrightarrow \lim\limits_{x\to 10} f(x) = 0$ "Exists"

- $\because \lim\limits_{x\to 10} f(x) = f(10) \Longrightarrow f(x)$ is continuous at 10

Note $f(x)$ is contimous from the right and from the left at $x_0 = 10$

- $f(6) = 5$

- $\lim_{x\to 6^+} f(x) = 5$

  $\lim\limits_{x\to 6^-} f(x) = 2$

  $\because \lim\limits_{x\to 6^+} f(x) \neq \lim\limits_{x\to 6^-} f(x) \Longrightarrow \lim\limits_{x\to 6} f(x)$ "Does Not Exists"

  $\because \lim_{x\to 6} f(x)$ "Does Not Exists" $\Longrightarrow f(x)$ is discontinuous at 6

Note $f(x)$ is continuous from the right at $x_0 = 6$

- $f(-2) = 3$

- $\lim\limits_{x\to -2^+} f(x) = +\infty$

  $\lim\limits_{x\to -2^-} f(x) = 3$

  $\because \lim\limits_{x\to -2^+} f(x) \neq \lim\limits_{x\to -2^-} f(x) \Longrightarrow \lim\limits_{x\to -2} f(x)$ "Does Not Exists"

- $\because \lim\limits_{x\to -2} f(x)$ "Does Not Exists" $\Longrightarrow f(x)$ is discontinuous at -2

Note $f(x)$ is continuous from the left at $x_0 = -2$

- $f(-8) = -3$

- $\lim\limits_{x\to -8^+} f(x) = -6$

  $\lim\limits_{x\to -8^-} f(x) = -6$

$$\because \lim_{x\to -8^+} f(x) = \lim_{x\to -8^-} f(x) = -6 \Longrightarrow \lim_{x\to -8} f(x) = -6 \qquad \text{"Exists"}$$

Note $f(x)$ is Not continuous neither from the right nor from the left at $x_0 = -8$

Example 2: Let $f(x) = \begin{cases} x+1 & \text{if } x \geq 0 \\ 1 & \text{if } x < 0 \end{cases}$

Is $f(x)$ continuous at $x_0 = 0$?

Solution:

- $f(0) = 0 + 1 = 1$
- $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x+1) = 0 + 1 = 1$

  $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (1) = 1$

  $\because \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 1 \Longrightarrow \lim_{x\to 0} f(x) = 1$ "Exists"
- $\because \lim_{x\to 0} f(x) = f(0) \Longrightarrow f(x)$ is continuous at 0

Example 3: Let $f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$

Is $f(x)$ continuous at $x_0 = 2$?

Solution:

- $f(2) = 3$
- $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x^2-4}{x-2} = \lim_{x\to 2^+} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2^+} (x+2) = 4$

  $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^2-4}{x-2} = \lim_{x\to 2^-} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2^-} (x+2) = 4$

  $\because \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = 4 \Longrightarrow \lim_{x\to 2} f(x) = 4$ "Exists"

  $\because \lim_{x\to 2} f(x) \neq f(2) \Longrightarrow f(x)$ is discontinuous at 2

Example 4: Let $f(x) = \begin{cases} \frac{x^3-1}{x-1} & \text{if } x \neq 1 \\ K & \text{if } x = 1 \end{cases}$ be a continuous function at $x_0 = 1$, Find the value of K ?

Solution:

$\because f(x)$ is continuous function at $x_0 = 1 \Longrightarrow f(1) = \lim\limits_{x\to 1} f(x)$

$$\therefore \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1} f(x)$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x^3-1}{x-1} = \lim_{x\to 1^+} \frac{(x-1)(x^2+x+1)}{x-1} = \lim_{x\to 1^+} (x^2+x+1) = 3$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^3-1}{x-1} = \lim_{x\to 1^-} \frac{(x-1)(x^2+x+1)}{x-1} = \lim_{x\to 1^-} (x^2+x+1) = 3$$

Hence, $\lim\limits_{x\to 1} f(x) = 3 = f(1) = \text{k} \Longrightarrow K = 3$

Example 5: Let $f(x) = \begin{cases} x^2 - 2 & \text{if } x \leq 2 \\ \text{C } x + 3 & \text{if } x > 2 \end{cases}$

be a continuous function at $x_0 = 2$, Find the value of C ?

Solution:

$\because f(x)$ is continuous function at $x_0 = 2 \Longrightarrow \lim\limits_{x\to 2} f(x)$ exists

$$\therefore \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2} f(x)$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 - 2 = 2^2 - 2 = 2$$

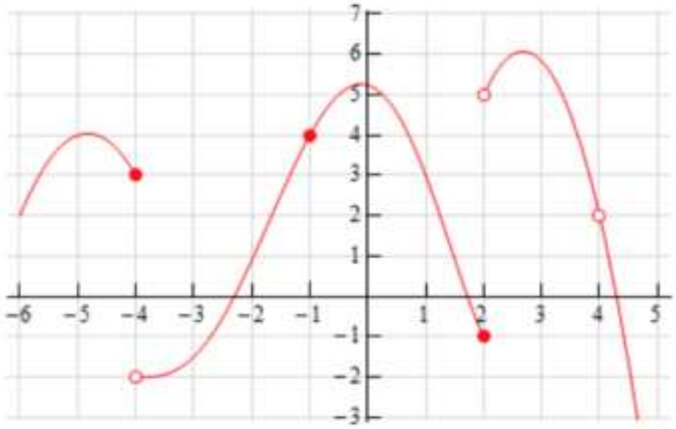
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} Cx + 3 = 2C + 3$$

$$\because \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) \Longrightarrow 2 = 2C + 3 \Longrightarrow 2C = -1 \Longrightarrow$$

$$C = \frac{-1}{2}$$

Problems (3.2):

1. Let $f(x)$ be defined as shown below. Check if $f(x)$ is continuous at x_0, where $x_0 = -4, -1, 2, 4$. If it is not continuous, is it right (or left) continuous at x_0 ? why?



2. Let $f(x) = \begin{cases} \frac{x^3+3x}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

   , is $f(x)$ continuous function at $x_0 = 0$? If $f(x)$ is discontimuous, redefine $f(x)$ to be continuous at $x_0 = 0$?

3. Let $f(x) = \begin{cases} ax + 3 & \text{if } x \geq 1 \\ 3x^2 + 1 & \text{if } x < 1 \end{cases}$

   , be a continuous function at $x_0 = 1$, find the value of a ?

4. Let $f(x) = \begin{cases} 2x + M & \text{if } x \leq -1 \\ x^2 + N & \text{if } x > -1 \end{cases}$

, be a continuous function at $x_0 = -1$ and $f(2) = 7$, find the values of M and N?