

جامعة بغداد



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قسم الرياضيات / المرحلة الاولى

مادة التفاضل والتكامل

الفصل السادس

الدوال الاسية و اللوغارتمية

Logarithm and Exponential Functions

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CHAPTER 6:

Logarithm and Exponential Fun.

The Natural Logarithm Function:

We denote it by "ln"

$$y = \ln(x); x > 0$$

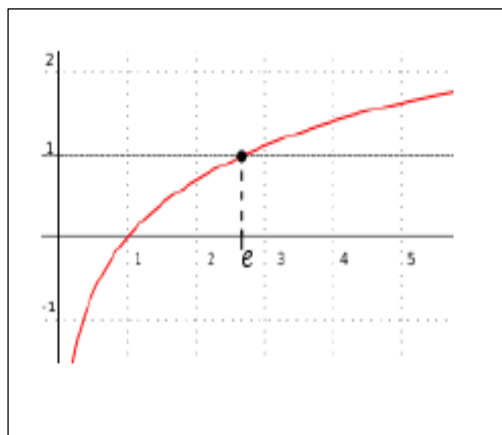
$$\ln: (0, \infty) \rightarrow \mathbb{R}$$

$$\ln e = 1$$

$$\ln 1 = 0$$

$$\ln 2 = 0.69$$

$$\ln 10 = 2.3$$



Properties of ln:

Let $x > 0$ and $y > 0$, then:

$$1 \quad \ln(xy) = \ln x + \ln y$$

$$2 \quad \ln\left(\frac{x}{y}\right) = \ln x - \ln y$$

$$3 \quad \ln(x^a) = a \ln x$$

Examples: Evaluate the following if you know that ($\ln 2 = 0.69$) :

$$1 \quad \ln 16 = \ln 2^4 = 4 \ln 2 = 4 \cdot (0.69) = 2.76$$

$$2 \quad \ln 8 = \ln 2^3 = 3 \cdot \ln 2 = 3 \cdot (0.69) = 2.07$$

$$3 \quad \ln \frac{1}{2} = \ln 1 - \ln 2 = 0 - (0.69) = -0.69$$

$$4 \quad \ln \sqrt{2} = \ln 2^{\frac{1}{2}} = \frac{1}{2} \cdot \ln 2 = \frac{1}{2} \cdot (0.69) = 0.345$$

The Derivative of ln

$$\frac{d}{dx} \ln u = \frac{1}{u} \cdot \frac{du}{dx}$$

Examples: Find y' for the following functions:

1 $y = \ln(x^2 + 2x)$

$$\Rightarrow y' = \frac{2x+2}{x^2+2x}$$

2 $y = \ln(\sin x * x^2)$

$$\Rightarrow y' = \frac{1}{\sin x * x^2} \cdot [\sin x * 2x + x^2 \cos x]$$

3 $y = \ln(\tan x + \sec x)$

$$\Rightarrow y' = \frac{\sec^2 x + \sec x \cdot \tan x}{\tan x + \sec x}$$

4 $y = (\ln x)^3$

$$\Rightarrow y' = 3(\ln x)^2 \cdot \frac{1}{x}$$

5 $y = (\ln \tan x)^5 \cdot \cos x^2$

$$\Rightarrow y' = (\ln \tan x)^5 \cdot (-\sin x^2 \cdot 2x) + \cos x^2 \cdot 5(\ln \tan x)^4 \frac{1}{\tan x} \cdot \sec^2 x$$

6 $y = \ln(\cos x)$

$$\Rightarrow y' = \frac{-\sin}{\cos x} = -\tan x$$

7 $y = \ln(\ln x)$

$$\Rightarrow y' = \frac{\frac{1}{x}}{\ln x}$$

8 $y = \ln x - \frac{1}{2} \ln(1 + x^2) - \frac{\tan^{-1} x}{x}$

$$\Rightarrow y' = \frac{1}{x} - \frac{1}{2} \cdot \frac{2x}{1+x^2} - \frac{x \cdot \frac{1}{1+x^2} - \tan^{-1} x}{x^2}$$

9 $y = x[\sin(\ln x) + \cos(\ln x)]$

$$\Rightarrow y' = x \left[\cos(\ln x) \cdot \frac{1}{x} - \sin(\ln x) \cdot \frac{1}{x} \right] + [\sin(\ln x) + \cos(\ln x)] \cdot 1$$

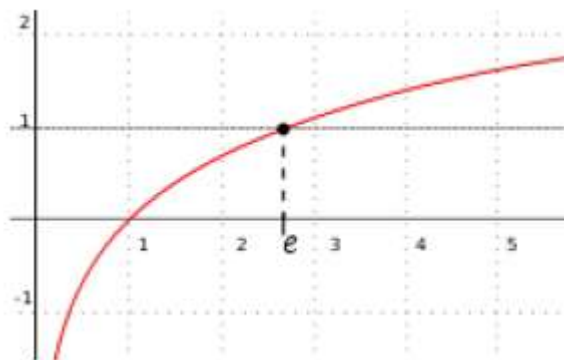
10 $y = x \sec^{-1} x - \ln(x + \sqrt{x^2 - 1}); x > 1$

$$\Rightarrow y' = x \cdot \frac{1}{|x|\sqrt{x^2 - 1}} + \sec^{-1} x \cdot 1 - \frac{1 + \frac{1}{2}(x^2 - 1)^{-\frac{1}{2}} \cdot 2x}{x + \sqrt{x^2 - 1}}$$

The Limit of $\ln$

$$\lim_{x \rightarrow \infty} \ln x = +\infty$$

$$\lim_{x \rightarrow 0^+} \ln x = -\infty$$



Examples: Proof the following:

$$1 \quad \lim_{x \rightarrow 1} (x - \ln x) \stackrel{?}{=} 1$$

$$\text{Proof: } \lim_{x \rightarrow 1} (x - \ln x) = 1 - \ln 1 = 1 - 0 = 1$$

$$2. \lim_{x \rightarrow 1} \cos(\ln x) \stackrel{?}{=} 1$$

$$\text{Proof: } \lim_{x \rightarrow 1} \cos(\ln x) = \cos(\ln 1) = \cos(0) = 1$$

$$3. \lim_{x \rightarrow 1} \ln x^{(x+1)} \stackrel{?}{=} 0$$

$$\begin{aligned} \text{Proof: } \lim_{x \rightarrow 1} \ln x^{(x+1)} &= \lim_{x \rightarrow 1} (x+1) \ln x \\ &= \lim_{x \rightarrow 1} (x+1) \lim_{x \rightarrow 1} \ln x = (1+1) \cdot \ln 1 = 2 \cdot 0 = 0 \end{aligned}$$

$$4 \quad \lim_{x \rightarrow 0} \frac{\ln(x+1)}{x} \stackrel{?}{=} 1$$

$$\text{Proof: } \lim_{x \rightarrow 0} \frac{\ln(x+1)}{x} \stackrel{L/R}{=} \lim_{x \rightarrow 0} \frac{\frac{1}{x+1}}{\frac{1}{1}} = \lim_{x \rightarrow 0} \frac{1}{x+1} = \frac{1}{0+1} = \frac{1}{1} = 1$$

$$5. \lim_{x \rightarrow 1} \ln(2-x)^{2\cos x} \stackrel{?}{=} 0$$

$$\begin{aligned} \text{Proof: } \lim_{x \rightarrow 1} \ln(2-x)^{2\cos x} &= \lim_{x \rightarrow 1} [2\cos x \cdot \ln(2-x)] \\ &= 2 \lim_{x \rightarrow 1} \cos x \cdot \lim_{x \rightarrow 1} \ln(2-x) \\ &= 2 \cdot \cos(1) \cdot \ln(2-1) = 2 \cdot \cos(1) \cdot 0 = 0 \end{aligned}$$

Problems (6.1): Find y' of the following functions:

$$1 \quad y = \ln(x\sqrt{x^2+1})$$

$$2 \quad y = -5\ln(3x\sqrt{x+2})$$

$$3 \quad y = t \ln t - \cos t^2$$

$$4 \quad y = x^3 \ln x - \frac{x}{\ln x}$$

$$5 \quad y = \frac{1}{2} \ln \frac{1+w}{1-w}$$

$$6 \quad y = -3 \ln \frac{\sin w}{1+w^3}$$

$$7 \quad y = \ln \frac{\sec x}{2+3x}$$

$$8 \quad y = \ln(t^2 + 4) - t \tan^{-1} \frac{t}{2}$$

$$9 \quad y = z(\ln z)^3$$

$$10 \quad y = \csc^3 x \ln x^3 + \tan^{-1} \ln x$$

The Exponential Function:

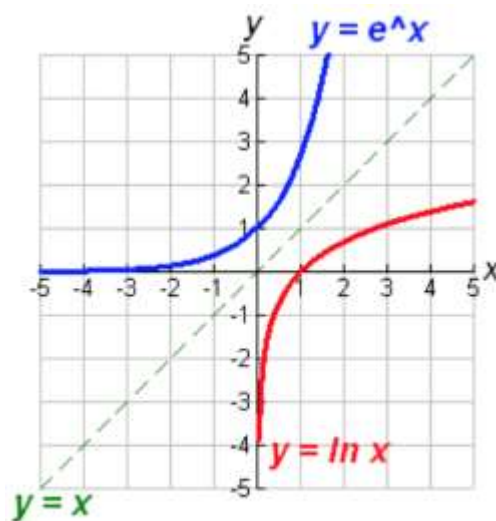
We denote it by " e^x " or " $\exp(x)$ "

$$y = e^x$$

$$e^x: \mathbb{R} \rightarrow (0, \infty)$$

$$e^x = \ln^{-1}(x)$$

$$e^0 = 1, \quad e^2 = 7.29$$



Properties of e^x :

$$1 \quad e^0 = 1$$

$$2 \quad e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$$

$$3 \quad \frac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$$

$$4 \quad (e^x)^r = e^{rx}, \forall r \in \mathbb{R}$$

$$5 \quad e^{-x} = \frac{1}{e^x}$$

$$6 \quad e^{\ln x} = x = \ln e^x$$

Examples: Simplify the following:

1 $\ln(e^{-x^2}) = -x^2$

2 $\ln\left(e^{\frac{1}{x}}\right) = \frac{1}{x}$

3 $e^{\ln \frac{1}{x}} = \frac{1}{x}$

4 $e^{2\ln x} = e^{\ln x^2} = x^2$

5 $\exp(\ln x - 2\ln y) = \exp(\ln x - \ln y^2) = \exp\left(\ln \frac{x}{y^2}\right) = \frac{x}{y^2}$

6 $e^{x+\ln x} = e^x \cdot e^{\ln x} = e^x \cdot x = x \cdot e^x$

The Derivative of e^x

$$\frac{d}{dx} e^u = e^u \cdot \frac{du}{dx}$$

Examples: Find y' for the following functions:

1 $y = e^{\tan^{-1} x}$

$$\Rightarrow y' = e^{\tan^{-1} x} \cdot \frac{1}{1+x^2}$$

2 $y = \exp(3) \cdot \exp(\sin 3\theta)$

$$\Rightarrow y' = \exp(3) \cdot \exp(\sin 3\theta) \cdot \cos 3\theta \cdot 3$$

3 $y = e^{x^3}$

$$\Rightarrow y' = e^{x^3} \cdot 3x^2$$

4 $y = \ln \frac{e^x}{1+e^x}$

$$\Rightarrow y' = \frac{1}{\frac{e^x}{1+e^x}} \cdot \frac{(1+e^x)e^x - e^x e^x}{(1+e^x)^2}$$

5 $y = \frac{1}{2}(e^{\sin x} - e^{-2x})$

$$\Rightarrow y' = \frac{1}{2}(e^{\sin x} \cdot \cos x - e^{-2x} \cdot (-2))$$

6 $y = \sec^{-1}(e^{2x})$

$$\Rightarrow y' = \frac{1}{|e^{2x}| \sqrt{(e^{2x})^2 - 1}} \cdot e^{2x} \cdot 2$$

7 $y = e^{\sin 2x}$

$$\Rightarrow y' = e^{\sin 2x} \cdot \cos 2x \cdot 2$$

$$8 \quad y = e^{(\ln x)^3}$$

$$\Rightarrow y' = e^{(\ln x)^3} \cdot 3(\ln x)^2 \cdot \frac{1}{x}$$

$$9 \quad y = e^{\cos(x^2)}$$

$$\Rightarrow y' = e^{\cos(x^2)} \cdot -\sin(x^2) \cdot 2x$$

$$10 \quad y = e^{(3x - e^{-x^2})}$$

$$\Rightarrow y' = e^{(3x - e^{-x^2})} \cdot (3 - e^{-x^2} \cdot (-2x))$$

Problems (6.2): Find y' of the following functions:

$$1 \quad y = x^2 \cdot e^{5x^2}$$

$$2 \quad y = \exp(\pi) \cdot x + \exp(\cos \pi x)$$

$$3 \quad y = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

$$4 \quad y = e^{\sin^{-1} x} \cdot \sqrt{x}$$

$$5 \quad y = (9t^2 - 6t + 2)\exp(3t)$$

$$6 \quad y = \exp(\tan t) \cdot \ln t^5$$

$$7 \quad y = w^7 \cdot e^{-\sqrt{w}}$$

$$8 \quad y = e^{(\sqrt[3]{x} + \tan x^2)}$$

$$9 \quad y = \exp(\tan^{-1}(t^3))$$

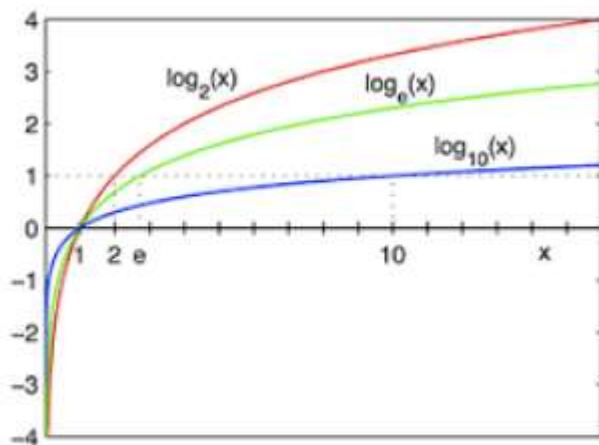
$$10 \quad y = \ln(e^{-\sin \theta} - e^{-\theta^3 + 5})$$

The General Logarithm Function:

We denote it by " $\log_a x$ "

$$y = \log_a x = \frac{\ln x}{\ln a}; x > 0, a > 0, a \neq 1$$

$$\log_a x: (0, \infty) \rightarrow \mathbb{R}$$



Properties of $\log_a x$:

$$1 \quad \log_a(x \cdot y) = \log_a x + \log_a y$$

$$2 \quad \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$3 \quad \log_a x^y = y \log_a x$$

$$4 \quad \log_a a = 1$$

$$5 \quad \log_a 1 = 0 \Leftrightarrow \ln 1 = 0$$

Examples: Simplify the following:

$$1 \quad \log_2 16 = \frac{\ln 16}{\ln 2} = \frac{\ln 2^4}{\ln 2} = \frac{4 \ln 2}{\ln 2} = 4$$

$$2 \quad \log_{\frac{1}{7}} 49 = \frac{\ln 49}{\ln \frac{1}{7}} = \frac{\ln 7^2}{-\ln 7} = \frac{2 \ln 7}{-\ln 7} = -2$$

$$3 \quad \log_{10} 10 = 1$$

$$4 \quad \log_{10} 100 = \log_{10} 10^2 = 2 \log_{10} 10 = 2 \cdot 1 = 2$$

$$5 \quad \log_{10} \frac{1}{1000} = \log_{10} 10^{-3} = -3 \log_{10} 10 = -3 \cdot 1 = -3$$

The Derivative of $\log_a x$

$$\frac{d}{dx} \log_a u = \frac{1}{\ln a} \cdot \frac{1}{u} \cdot \frac{du}{dx}, \text{ where } u > 0, a > 0, a \neq 1$$

Examples:

Find y' for the following functions:

$$1 \quad y = \log_2(x^2 + 3x) \Rightarrow y' = \frac{1}{(x^2+3x)} \cdot \frac{1}{\ln 2} \cdot (2x + 3)$$

$$2 \quad y = \log_3 x^5 \cdot \cos x^2$$

$$\Rightarrow y' = \log_3 x^5 \cdot (-\sin x^2 \cdot 2x) + \cos x^2 \cdot \frac{1}{x^5} \cdot \frac{1}{\ln 3} \cdot 5x^4$$

$$3 \quad y = \log_7(\tan x + \sin x)$$

$$\Rightarrow y' = \frac{1}{(\tan x + \sin x)} \cdot \frac{1}{\ln 7} \cdot (\sec^2 x + \cos x)$$

$$4 \quad y = \ln x \cdot \log_{10} x \Rightarrow y' = \ln x \cdot \frac{1}{x} \cdot \frac{1}{\ln 10} \cdot 1 + \log_{10} x \cdot \frac{1}{x}$$

$$5 \quad y = \log_a \sin^{-1} x + \frac{x}{e^x}, \text{ where } a \text{ is a constant.}$$

$$\Rightarrow y' = \frac{1}{\sin^{-1} x} \cdot \frac{1}{\ln a} \cdot \frac{1}{\sqrt{1-x^2}} + \frac{e^x \cdot 1 - x \cdot e^x \cdot 1}{e^{2x}}$$

Problems (6.3): Find y' of the following functions:

$$1 \quad y = x \log_5 x - \cos x^2$$

$$2 \quad y = e^{-4t} \log_2(1-t) + \log_a(3t^2), \text{ where } a \text{ is a constant.}$$

$$3 \quad y = \log_4(\cos 3w) \sec \sqrt{w}$$

$$4 \quad y = \log_{11}(\tan(3\pi t))$$

$$5 \quad y = \sqrt[3]{1 + \log_3(z^5)}$$

$$6 \quad y = \log_n(\cos^{-1} \theta) + \sin \theta \cdot \log_3 t^2, \text{ where } n \text{ is a constant.}$$

$$7 \quad y = (\log_6(5z))^3$$

$$8 \quad y = \sin(\log_2 \theta) + \sec \theta^2$$

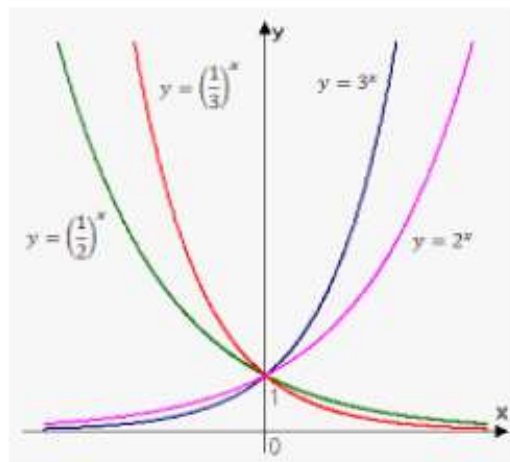
$$9 \quad y = \log_b(we^{w^2}) + \sin^{-1} w, \text{ where } b \text{ is a constant.}$$

$$10 \quad y = \cot(\log_7 t^2) \cdot \frac{1}{t^3}$$

The General Exponential Function:

We denote it by " a^x "

$$y = a^x = e^{x \ln a}; a > 0$$
$$a^x: \mathbb{R} \rightarrow (0, \infty)$$



Properties of a^x :

- 1 $a^1 = a$, where $a > 0$
- 2 $a^0 = 1$
- 3 $a^u \cdot a^v = a^{u+v}$
- 4 $\left(a^{\frac{m}{n}}\right)^n = a^m$
- 5 $(a \cdot b)^u = a^u \cdot b^u$, where $a > 0$ and $b > 0$

The Derivative of a^x

$$\because a^u = e^{u \ln a}$$

$$\xrightarrow{\frac{d}{dx}} \frac{d}{dx} a^u = \frac{d}{dx} (e^{u \ln a})$$

$$\rightarrow \frac{d}{dx} a^u = a^u \cdot \ln a \cdot \frac{du}{dx}$$

Examples: Find y' for the following functions:

$$1 \quad y = 2^{x^2 + \sec x} \Rightarrow y' = 2^{x^2 + \sec x} \cdot \ln 2 \cdot (2x + \sec x \tan x)$$

$$2 \quad y = 4^{\sin^{-1} x} \Rightarrow y' = 4^{\sin^{-1} x} \cdot \ln 4 \cdot \frac{1}{\sqrt{1-x^2}}$$

$$3 \quad y = x^\pi \cdot \pi^x \Rightarrow y' = x^\pi \cdot \pi^x \cdot \ln \pi \cdot 1 + \pi^x \cdot \pi \cdot x^{\pi-1} \cdot 1$$

Example: Find the value of x , where $3^x = 2^{x+1}$

Solution: $3^x = 2^{x+1}$

$$\stackrel{\ln}{\Rightarrow} \ln 3^x = \ln 2^{x+1}$$

$$\Rightarrow x \ln 3 = (x + 1) \ln 2$$

$$\Rightarrow x \ln 3 = x \ln 2 + \ln 2$$

$$\Rightarrow x \ln 3 - x \ln 2 = \ln 2$$

$$\Rightarrow x (\ln 3 - \ln 2) = \ln 2$$

$$\Rightarrow x \left(\ln \frac{3}{2} \right) = \ln 2 \Rightarrow x = \frac{\ln 2}{\ln \frac{3}{2}}$$

Problems (6.4): Find y' of the following functions:

1 $y = 5^{x^2+x-1}$

2 $y = 6^{\sin \omega + \ln w + 3}$

3 $y = 2^{\sec \sqrt{t}}$

4 $y = e^{5x} 3^{\tan x}$

5 $y = \ln \frac{x^4}{1+x^3} + 7^{\frac{x^2}{2}}$

6 $y = 2^{-t^2} \cos t^3$

7 $y = \pi^{\cos x} e^{\sqrt{x}} - 5^{-4x^3}$

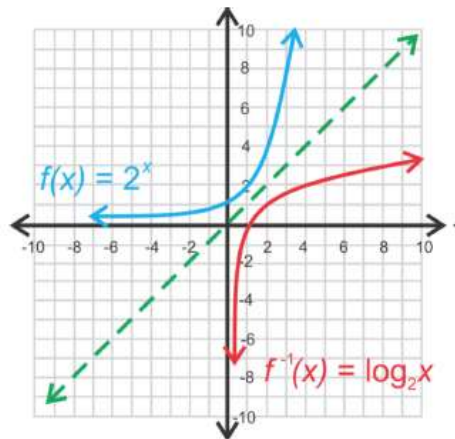
8 $y = \ln(1 + e^{2x}) 5x^2$

9 $y = -4 \ln w + \frac{4}{w} - 2^{\sin(\sqrt{w})}$

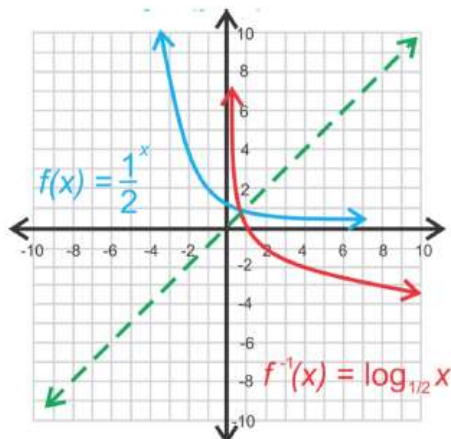
10 $y = 8^{\cos \pi \theta} \ln \sqrt{\theta} + 3^{\sec(5\theta)}$

The Relation between a^x and $\log_a x$

$$\log_a y = x \Leftrightarrow y = a^x; a > 0 \text{ \& } a \neq 0$$



$$a > 1$$



$$0 < a < 1$$

Logarithmic Differentiation

If in a function where the base and the exponent are variables, we should change the form before we take the derivative.

Examples: Find y' for the following:

1. $y = x^{x^2}$

$$\xrightarrow{\ln} \ln y = \ln x^{x^2}$$

$$\rightarrow \ln y = x^2 \ln x$$

$$\xrightarrow{\frac{d}{dx}} \frac{y'}{y} = x^2 \frac{1}{x} + \ln x \cdot 2x$$

$$\rightarrow y' = y * \left(x^2 \frac{1}{x} + \ln x \cdot 2x \right)$$

$$\xrightarrow{y=x^{x^2}} y' = x^{x^2} * \left(x^2 \frac{1}{x} + \ln x \cdot 2x \right)$$

$$2. y = (x^2 + 1)^{\ln x}$$

$$\xrightarrow{\ln} \ln y = \ln(x^2 + 1)^{\ln x}$$

$$\rightarrow \ln y = \ln x \ln(x^2 + 1)$$

$$\xrightarrow{\frac{d}{dx}} \frac{y'}{y} = \ln x \cdot \frac{2x}{x^2 + 1} + \ln(x^2 + 1) \cdot \frac{1}{x}$$

$$\rightarrow y' = y * \left(\ln x \cdot \frac{2x}{x^2 + 1} + \ln(x^2 + 1) \cdot \frac{1}{x} \right)$$

$$\rightarrow y' = (x^2 + 1)^{\ln x} * \left(\ln x \cdot \frac{2x}{x^2 + 1} + \ln(x^2 + 1) \cdot \frac{1}{x} \right)$$

$$3 \quad y = (\sin x)^{\tan x}$$

$$\xrightarrow{\ln} \ln y = \ln(\sin x)^{\tan x}$$

$$\rightarrow \ln y = \tan x \cdot \ln(\sin x)$$

$$\xrightarrow{\frac{d}{dx}} \frac{y'}{y} = \tan x \cdot \frac{\cos x}{\sin x} + \ln(\sin x) \cdot \sec^2 x$$

$$\rightarrow y' = y * \left(\tan x \frac{\cos x}{\sin x} + \ln(\sin x) \cdot \sec^2 x \right)$$

$$\rightarrow y' = (\sin x)^{\tan x} * \left(\tan x \cdot \frac{\cos x}{\sin x} + \ln(\sin x) \cdot \sec^2 x \right)$$

$$\rightarrow y' = (\sin x)^{\tan x} * \left(\frac{\sin x}{\cos x} \cdot \frac{\cos x}{\sin x} + \ln(\sin x) \cdot \sec^2 x \right)$$

$$\rightarrow y' = (\sin x)^{\tan x} * (1 + \sec^2 x \cdot \ln(\sin x))$$

$$4. y = \sqrt[5]{\frac{(x+2)^3(x+1)^2}{(x-4)^2(x+3)}}$$

$$\rightarrow y = \left(\frac{(x+2)^3(x+1)^2}{(x-4)^2(x+3)} \right)^{\frac{1}{5}}$$

$$\xrightarrow{\ln} \ln y = \frac{1}{5} * \ln \left(\frac{(x+2)^3(x+1)^2}{(x-4)^2(x+3)} \right)$$

$$\rightarrow \ln y = \frac{1}{5} * \ln((x+2)^3(x+1)^2) - \ln((x-4)^2(x+3))$$

$$\begin{aligned} \rightarrow \ln y &= \frac{1}{5} * (\ln(x+2)^3 + \ln(x+1)^2) - (\ln(x-4)^2 + \ln(x+3)) \\ \rightarrow \ln y &= \frac{1}{5} * (3\ln(x+2) + 2\ln(x+1) - 2\ln(x-4) - \ln(x+3)) \\ \frac{d}{dx} \frac{y'}{y} &= \frac{1}{5} \left(\frac{3}{x+2} + \frac{2}{x+1} - \frac{2}{x-4} - \frac{1}{x+3} \right) \\ \rightarrow y' &= y * \frac{1}{5} \left(\frac{3}{x+2} + \frac{2}{x+1} - \frac{2}{x-4} - \frac{1}{x+3} \right) \\ \rightarrow y' &= \sqrt[5]{\frac{(x+2)^3(x+1)^2}{(x-4)^2(x+3)}} * \frac{1}{5} \left(\frac{3}{x+2} + \frac{2}{x+1} - \frac{2}{x-4} - \frac{1}{x+3} \right) \end{aligned}$$

Problems (6.5):

1 Find y' for the following:

(a) $y = x^x$

(b) $y = e^{\sqrt{z^2+e^z}}$

(c) $y = x^{\sqrt{5}+\ln x}$

(d) $y = n^{\ln n}$

(e) $y = (\tan t)^{3\sin^{-1} t}$

2 Using L'R. proof the following:

(a) $\lim_{x \rightarrow \infty} \frac{3x^2+x+4}{5x^2+8x} = \frac{3}{5}$

(b) $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x - 1}{\cos x} = 0$

(c) $\lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0$

(d) $\lim_{x \rightarrow \infty} \frac{\ln\left(1+\frac{1}{x}\right)}{x^{-1}} = 1$

(e) $\lim_{x \rightarrow 1} \frac{x \ln x - x + 1}{(x-1) \ln x} = 1$

(f) $\lim_{x \rightarrow 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{x^3} = \frac{1}{6}$

(g) $\lim_{x \rightarrow -\infty} \frac{3x-2}{e^{x^2}} = 0$