SEPARATION AXIOMS

Definition: T_0-Space.

A Topological space (X,τ) is called a T_0-space if for every distinct Points $x \neq y$ in X, there exists an open set containing only one of them, but not contain the other.

i.e. (X,τ) is T_0-Space $\Longleftrightarrow \begin{cases} \forall x \neq y \text{ in } X, \exists U \in \tau \\ \text{s.t. } (x \in U \wedge y \notin U) \\ \text{or } (x \notin U \wedge y \in U) \end{cases}$

So, (X,τ) is not T_0-Space $\Longleftrightarrow \begin{cases} \exists x \neq y \text{ in } X \text{ s.t.} \\ \forall U \in \tau \text{ then} \\ (x, y \in U) \text{ or } (x, y \notin U) \end{cases}$

Example: Let $X = \{1,2,3\}$ and $\tau = \{X, \phi, \{1\}, \{2\}, \{1,2\}\}$

$1 \neq 2$ in X, $\exists U = \{1\} \in \tau$; $1 \in U \wedge 2 \notin U$

$1 \neq 3$ in X, $\exists U = \{1\} \in \tau$; $1 \in U \wedge 3 \notin U$

$2 \neq 3$ in X, $\exists U = \{2\} \in \tau$; $2 \in U \wedge 3 \notin U$

$\therefore (X,\tau)$ is T_0-Space.

Example: Let $X = \{1,2\}$ and $\tau = \{X, \phi\}$.

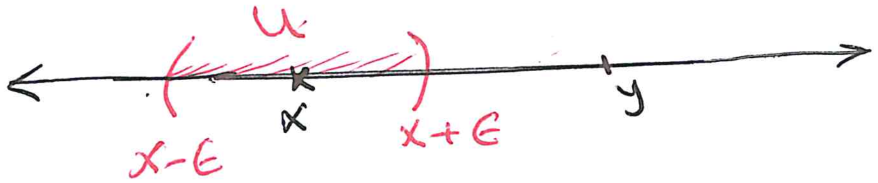
Then $1 \neq 2$ in X but $\nexists U \in \tau$; $(1 \in U \wedge 2 \notin U)$ or $(1 \notin U \wedge 2 \in U)$

$\therefore (X,\tau)$ is not T_0-Space.

Example: The usual topological space $(\mathbb{R}, \tau_u)$ is T_0- space.

Since $\forall x \neq y$ in $\mathbb{R}$. Let $\epsilon = \frac{|x-y|}{3}$, then

$U = (x-\epsilon, x+\epsilon) \in \tau_u$; $x \in U \wedge y \notin U$.



Example: The space $(\mathbb{N}, \tau_{cof})$ is T_0-space,

Since, $\forall x \neq y$ in $\mathbb{N}$, then Let $U = \mathbb{N} - \{y\}$.

$U \in \tau_{cof}$, $x \in U \wedge y \notin U$.

Example: The space $(\mathbb{N}, \tau)$;

$$\tau = \{A_n \subseteq \mathbb{N} : A_n = \{1, 2, 3, \ldots, n\} ; n \in \mathbb{N}\} \cup \{\mathbb{N}, \phi\}$$

$\forall x \neq y$ in $\mathbb{N}$; then $x < y$ or $y < x$.

If $x < y$, Let $U = \{1, 2, 3, \ldots, x\} = A_x \in \tau$.

$x \in U \wedge y \notin U$.

If $y < x$; Let $U = \{1, 2, 3, \ldots, y\} = A_y \in \tau$.

$y \in U \wedge x \notin U$.

Hence, $(\mathbb{N}, \tau)$ is T_0- space.

<u>Example</u>: The space $(\mathbb{N}, \tau)$;

$$\tau = \{A_n \subseteq \mathbb{N} ; A_n = \{n, n+1, n+2, \dots\}\} \cup \{\phi\}$$

$\forall n \neq m$ in $\mathbb{N}$, $n < m$ or $m < n$

If $n < m$, then Let $U = A_m = \{m, m+1, m+2, \dots\} \in \tau$.

$m \in U \wedge n \notin U$.

If $m < n$; then let $U = A_n = \{n, n+1, n+2, \dots\} \in \tau$

$n \in U \wedge m \notin U$.

Hence $(\mathbb{N}, \tau)$ is T_0-space.

<u>Example</u>: If X has more than one elements. then

i) (X, D) is T_0-space. Since $\forall x \neq y$ in X, then $\exists U = \{x\} \in D$; $x \in U \wedge y \notin U$.

ii) (X, I) is not T_0-space. Since $\forall x \neq y$ in X $\nexists U \in I$; $(x \in U \wedge y \notin U)$ or $(y \in U \wedge x \notin U)$

Theorem: (X,τ) is T_0-Space iff $\overline{\{x\}} \neq \overline{\{y\}}$ $\forall\, x \neq y$ in X.

$\Rightarrow$: Suppose X is T_0-Space and $x \neq y$ in X.

To prove $\overline{\{x\}} \neq \overline{\{y\}}$

Since X is T_0-Space and $x \neq y$, then

$\exists\, U \in \tau ; (x \in U \wedge y \notin U)$ or $(x \notin U \wedge y \in U)$.

$\Rightarrow (x \in U \wedge U \cap \{y\} = \phi)$ or $(y \in U \wedge U \cap \{x\} = \phi)$

$\Rightarrow x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$

$\Rightarrow \overline{\{x\}} \neq \overline{\{y\}}$

$\Leftarrow$: Suppose $\overline{\{x\}} \neq \overline{\{y\}}$. To prove X is T_0-Space.

Since $\overline{\{x\}} \neq \overline{\{y\}} \Rightarrow \exists\, a ; (a \in \overline{\{x\}} \wedge a \notin \overline{\{y\}})$ or $(a \in \overline{\{y\}} \wedge a \notin \overline{\{x\}})$

Suppose; $a \in \overline{\{x\}} \wedge a \notin \overline{\{y\}}$

$\Rightarrow (\forall\, U \in \tau ;\ a \in U$ then $U \cap \{x\} \neq \phi) \wedge$
$(\exists\, V \in \tau ;\ a \in V \wedge V \cap \{y\} = \phi)$.

$\Rightarrow (\forall\, U \in \tau ; a \in U$ then $x \in U) \wedge$
$(\exists\, V \in \tau ;\ a \in V$ and $y \notin V)$.

$\Rightarrow \exists\, V \in \tau ;\ a \in V$, $x \in V$ and $y \notin V$.

$\therefore X$ is T_0-space.

Similarly; if $a \in \overline{\{y\}}$ and $a \notin \overline{\{x\}}$.

Theorem: The Property of being a T_0-Space is a hereditary Property.

Proof: Let (X, τ) be a T_0-Space, and (W, τ_W) be a subspace of (X, τ). To Prove (W, τ_W) is T_0-space.

Let $x \neq y$ in W. Since $W \subseteq X$, then $x \neq y$ in X. Since X is T_0-space, then $\exists U \in \tau$; $(x \in U \wedge y \notin U)$ or $(y \in U \wedge x \notin U)$.

$\Rightarrow$ let $V = U \cap W \in \tau_W$.

Then $(x \in V \wedge y \notin V)$ or $(y \in V \wedge x \notin V)$.

Hence, (W, τ_W) is a T_0-space.

Question: Prove or disprove; The continuous image of a T_0-Space is T_0-space.

Disprove: Let $X = \{1, 2, 3\}$, then the function $f: (X, D) \longrightarrow (X, I)$ is continuou function and (X, D) is T_0-space, but (X, I) is not a T_0-space.

Theorem: A T_0-space is a topological property.

Let $(X,\tau) \cong (Y,\tau')$ and X is a T_0-space.
To prove (Y,τ') is a T_0-space.
Since, $X \cong Y$, then $\exists\ f: X \longrightarrow Y$ is continuous, 1-1, onto and f^{-1} continuous, or f is open.
Let $y_1 \neq y_2$ in Y. Since f is onto, then $\exists\ x_1, x_2 \in X$; $f(x_1) = y_1$ and $f(x_2) = y_2$.
Since f is function, then $x_1 \neq x_2$ in X.
Since X is T_0-space, then $\exists\ U \in \tau$; $(x_1 \in U \wedge x_2 \notin U)$ or $(x_2 \in U \wedge x_1 \notin U)$.
Since f is open function, then $f(U)$ open in Y, such that, $(f(x_1) \in f(U) \wedge f(x_2) \notin f(U))$ or $(f(x_2) \in f(U) \wedge f(x_1) \notin f(U))$. (since f is 1-1).
thus, $f(U) \in \tau'$ such that $(y_1 \in f(U) \wedge y_2 \notin f(U))$ or $(y_2 \in f(U) \wedge y_1 \notin f(U))$
$\therefore (Y,\tau')$ is T_0-space.
Similarly; If Y is T_0-space, then X is T_0-space.

Theorem: Let (X,τ) and (Y,τ') are two topological spaces. Then $X\times Y$ is T_0-space if and only if both X and Y are T_0-spaces.

Proof: $\Rightarrow$ Let $X\times Y$ is a T_0-space. To prove X and Y are T_0-spaces.

Let $x_1 \neq x_2$ in X and $y_1 \neq y_2$ in Y.

Then $(x_1,y_1) \neq (x_2,y_2)$ in $X\times Y$.

Since $X\times Y$ is T_0-space, then $\exists U\times V \in \tau_{X\times Y}$

such that $((x_1,y_1)\in U\times V \wedge (x_2,y_2)\notin U\times V)$ or

$((x_1,y_1)\notin U\times V \wedge (x_2,y_2)\in U\times V)$

$\Rightarrow \exists U\in\tau;\ (x_1\in U \wedge x_2\notin U)$ or $(x_1\notin U\wedge x_2\in U)$

and $\exists V\in\tau';\ (y_1\in V\wedge y_2\notin V)$ or $(y_1\notin V\wedge y_2\in V)$

$\Rightarrow (X,\tau)$ and (Y,τ') are T_0-spaces.

$\Leftarrow$ Let both (X,τ) and (Y,τ') are T_0-spaces, to prove $X\times Y$ is T_0-space.

Let $(x_1,y_1)\neq(x_2,y_2)$ in $X\times Y$.

$\Rightarrow x_1\neq x_2$ in X or $y_1\neq y_2$ in Y.

$\Rightarrow [\exists U\in\tau;(x_1\in U\wedge x_2\notin U)$ or $(x_1\notin U\wedge x_2\in U)]$ or

$[\exists V\in\tau';(y_1\in V\wedge y_2\notin V)$ or $(y_1\notin V\wedge y_2\in V)]$

$\Rightarrow \exists U\times V\in\tau_{X\times Y};\ ((x_1,y_1)\in U\times V\wedge(x_2,y_2)\notin U\times V)$

or $((x_1,y_1)\notin U\times V\wedge(x_2,y_2)\in U\times V)$.

$\therefore X\times Y$ is a T_0-space.

Definition: A topological space (X, τ) is called a T_1-space if for every distinct points $x \neq y$ in X, there exists an open sets U and V such that $(x \in U \wedge y \notin U)$ and $(x \notin V \wedge y \in V)$

Example: Let $X = \{1, 2\}$ and $\tau = D = \{X, \phi, \underset{U}{\{1\}}, \underset{V}{\{2\}}\}$.
It's clear that (X, τ) is a T_1-space.

Remark Every T_1-space is T_0-space, not conversly. For example:
Let $X = \{1, 2\}$ and $\tau = \{X, \phi, \overset{U}{\{1\}}\}$.
It's clear that (X, τ) is a T_0-space, but its not T_1-space, since $\nexists V \in \tau \ni (2 \in V \wedge 1 \notin V)$.

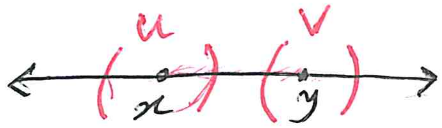
Theorem A space (X, τ) is T_1-space if and only if $\{x\}$ is closed $\forall x \in X$.

جداً مهم

Example: Show that $(\mathbb{R}, \tau_u)$ is T_1-space.

Answer: Since $\forall x \in \mathbb{R}$, then $\{x\}^c = (-\infty, x) \cup (x, \infty)$ which is open set. Hence $\{x\}$ is closed for all $x \in \mathbb{R}$. So $(\mathbb{R}, \tau_u)$ is a T_1-space.

Method 2: Let $x \neq y$ in $(\mathbb{R}, \tau_u)$



Let $r = |y - x|$

Let $\epsilon = \frac{r}{3}$

Let $U = (x-\epsilon, x+\epsilon)$ and $V = (y-\epsilon, y+\epsilon)$.

It is clear that U and V are open sets in $(\mathbb{R}, \tau_u)$

So $(x \in U \wedge y \notin U)$ and $(x \notin V \wedge y \in V)$.

Hence $(\mathbb{R}, \tau_u)$ is T_1-space.

Example: Show that $(\mathbb{N}, \tau_{cof})$ is a T_1-space.

Answer: Since $\forall x \in \mathbb{N}$, then $\{x\}^c = \mathbb{N} - \{x\}$ which is an open set, then $\{x\}$ is closed set. Hence, $(\mathbb{N}, \tau_{cof})$ is T_1-space

ملاحظة: $\mathbb{N}-\{x\}$ مجموعة مفتوحة في $(\mathbb{N}, \tau_{cof})$ لأن متممتها $\{x\}$ مجموعة منتهية.

Method 2: Let $U = \mathbb{N} - \{x\}$ and $V = \mathbb{N} - \{y\}$ for $x \neq y$ in $(\mathbb{N}, \tau_{cof})$.

Since, $U^c = \{x\}$ which is finite set, then U is open set and $(y \in U \wedge x \notin U)$.

So, $V^c = \{y\}$ is finite set, then V is open and $(x \in V \wedge y \notin V)$.

Hence, $(\mathbb{N}, \tau_{cof})$ is T_1-space.

Example Show that (X,D) is T_1-space.

Answer: Since $\forall x \in X$, then $X-\{x\} \in D$ $= P(X)$, So $X-\{x\}$ is open set, implies $\{x\}$ is closed set. Hence (X,D) is T_1-space.

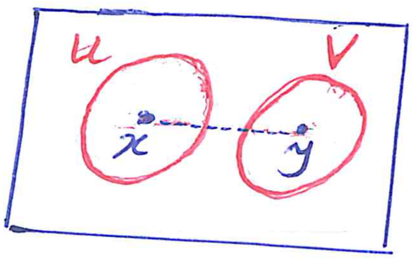
Method 2: $\forall x \neq y$ in (X,D)

Let $U=\{x\}$ and $V=\{y\}$.

It is clear that U and V are open sets, $(x \in U \wedge y \notin U)$ and $(y \in V \wedge x \notin V)$.

Hence, (X,τ) is T_1-space.

Theorem Every metric space is a T_1-space.



Proof: Let (X,d) be a metric space, $x \neq y$ in X.

Let $r = d(x,y)$

Let $\epsilon = \frac{r}{3}$, $U = B_\epsilon(x)$ and $V = B_\epsilon(y)$.

It is clear that U and V are open sets with $(x \in U \wedge y \notin U)$ and $(y \in V \wedge x \notin V)$.

Hence, (X,d) is T_1-space.

Proposition Every T_1-space, has a property that each finite subset is closed.

or: If (X, τ) is a T_1-space, then every finite subset of X is closed.

Proof:- Let (X, τ) be a T_1-space, and let $B = \{x_1, x_2, \dots, x_n\}$ be a finite subset of X. To prove B is closed set.

Since (X, τ) is T_1-space, then $\{x\}$ is closed set $\forall x \in X$. So $\{x_1\} \cup \{x_2\} \cup \dots \cup \{x_n\}$ is closed. Hence B is closed set.

Corollary: If X is finite set, then (X, τ) is T_1-space if and only if $\tau = D$.

Proof: $\Rightarrow$: Suppose (X, τ) is a T_1-space.
To prove $(X, \tau) = (X, D)$; (i.e. $\tau = D$)
Must prove $\{x\} \in \tau$ $\forall x \in X$.
$\forall x \in X$, then $\{x\}^c = X - \{x\}$ is finite set. So $\{x\}^c$ is closed set, by propo. since X is T_1-space. Hence $\{x\}$ is open.
Therefore, $\tau = D$.

$\Leftarrow$ Let $\tau = D$, T.P. (X, τ) is T_1-space.
$\forall x \neq y$ in X, let $U = \{x\}$ and $V = \{y\}$.
$U, V \in \tau$ and $(x \in U \wedge y \notin U)$ and $(x \notin V \wedge y \in V)$.
Hence X is T_1-space.

Theorem: A space (X,τ) is T_1-space iff $\{x\}$ is closed set $\forall x \in X$. جداً مهم

Corollary:

i. (X,τ) is T_1-space iff $\overline{\{x\}} = \{x\}$ $\forall x \in X$.

ii. (X,τ) is T_1-space iff $\{x\} = \bigcap\{F : F \in \mathcal{F} \wedge x \in F\}$ $\forall x \in X$.

iii. (X,τ) is T_1-space iff $\{x\}^b = \phi$ $\forall x \in X$.

iv. (X,τ) is T_1-space iff $\{x\}^b \subseteq \{x\}$ $\forall x \in X$.

v. (X,τ) is T_1-space iff $\{x\}' \subseteq \{x\}$ $\forall x \in X$.

vi. (X,τ) is T_1-space iff $\{x\}' = \phi$ $\forall x \in X$.

Theorem A T_1-space is a hereditary property.

Proof: Let (X,τ) be a T_1-space, and $W \subseteq X$.

T.P. (W,τ_W) is T_1-space.

Let $x \neq y$ in $W \Rightarrow x \neq y$ in X

$\Rightarrow \exists U, V \in \tau \ni (x \in U \wedge y \notin U)$ and $(x \notin V \wedge y \in V)$ Since X is T_1-sp.

Now, Let $U_1 = U \cap W$ and $V_1 = V \cap W$.

It is clear that $U_1, V_1 \in \tau_W$, and $(x \in U_1 \wedge y \notin U_1)$ and $(x \notin V_1$ and $y \in V_1)$.

Hence (W,τ_W) is T_1-space.

Theorem A T_1-space is a topological property.

Prove: Let $(X,\tau) \cong (Y,\tau')$ and X is T_1-sp.
T.P. (Y,τ') is T_1-space.

Since $X \cong Y$, then $\exists f:(X,\tau) \to (Y,\tau')$;
f is 1-1, onto, cont^s and open.
(or f^{-1} cont^s)

Let $y_1 \neq y_2$ in Y. Since f is onto, then
$\exists x_1, x_2 \in X \ni f(x_1) = y_1 \wedge f(x_2) = y_2$.
Since f is 1-1, then $x_1 \neq x_2$. (Since $y_1 \neq y_2$)
Since, (X,τ) is T_1-space, then $\exists U, V \in \tau$
s.t. $(x_1 \in U \wedge x_2 \notin U)$ and $(x_1 \notin V \wedge x_2 \in V)$.
Since f is open, then $f(U), f(V)$ are open sets in (Y,τ') and $(f(x_1) \in f(U) \wedge f(x_2) \notin f(U))$ and $(f(x_2) \in f(V) \wedge f(x_1) \notin f(V))$.
$\therefore$ (Y,τ') is T_1-space

Similarly; If (Y,τ') is T_1-space, then (X,τ) is T_1-space.

Theorem: If (X,τ) and (Y,τ') are two topological spaces. Then, both X and Y are T_1-spaces iff $(X \times Y, \tau_{X \times Y})$ is a T_1-space

Definition: A topological space (X,τ) is called a T_2-space if for every two distinct points $x \neq y$ in X, there exist two disjoint open sets U and V satisfies, $x \in U \wedge y \in V$.

i.e. (X,τ) is T_2-space $\Leftrightarrow$ $\forall x \neq y$ in X
$\exists U, V \in \tau$; $U \cap V = \phi$, $x \in U$ and $y \in V$

A T_2-space is called some times, Hausdorff space.

Example Consider the topological space (X,τ) where $X = \{1,2,3\}$ and $\tau = D$.
It is clear that (X,τ) is a T_2-space, since for every $x \neq y$ in X, we can take $U = \{x\}$ and $V = \{y\}$, then $U, V \in \tau = D$, $U \cap V = \{x\} \cap \{y\} = \phi$, $x \in \{x\} = U$ and $y \in \{y\} = V$.

Remark Every T_2-space is T_1-space. But the convers is not true, in general. For example. The space (N, τ_{cof}) is T_1-space but not T_2-space, since $1 \neq 2$ in N but $\nexists U, V \in \tau_{cof}$ such that $U \cap V = \phi$, $1 \in U$ and $2 \in V$.

Remark: $T_2\text{-space} \underset{\not\Leftarrow}{\Longrightarrow} T_1\text{-space} \underset{\not\Leftarrow}{\Longrightarrow} T_0\text{-space}$

Example: The space (X,τ), where $X=\{1,2\}$ and $\tau=\{X,\phi,\{1\}\}$ is a T_0-space which is not T_1-space and not T_2-space.

Example: When X has more than one element, then

i. (X,I) is not T_0-space, not T_1-space and not T_2-space.

ii. (X,D) is T_0-space, T_1-space and T_2-space.
$\forall x,y\in X$, $\{x\}=U$, $\{y\}=V$

Example Show that (X,d), the metric space is T_2-space.
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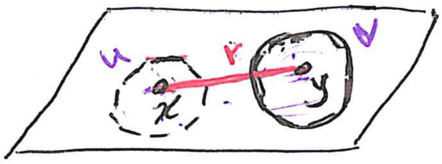


Solution: $\forall x\neq y$ in X

Let $r=d(x,y)$

Let $\epsilon=\frac{r}{3}$, take $U=B_\epsilon(x)$ and $V=B_\epsilon(y)$.

It is clear that $U\cap V=\phi$, $x\in U$ and $y\in V$

So $U \wedge V$ are open sets. Hence (X,d) is T_2-sp.

Example The usual topology $(\mathbb{R},\tau_u)$ is T_2-space.

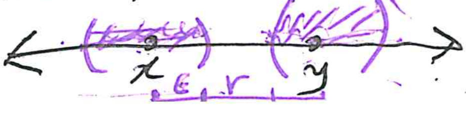
Solution: $\forall x\neq y$ in $\mathbb{R}$, let $r=|y-x|$.

then we can take $\epsilon=\frac{r}{3}$, such that

$U=(x-\epsilon,x+\epsilon)$, $V=(y-\epsilon,y+\epsilon)$. $U,V\in\tau_u$

Clearly; $x\in U$, $y\in V$ and $U\cap V=\phi$.

$\therefore (\mathbb{R},\tau_u)$ is T_2-space



Theorem: A space (X,τ) is T_2-space iff the diagonal $\Delta = \{(x,x) : x \in X\}$ is closed subset of $X \times X$.

Theorem: A T_2-space is a hereditary Property.

Proof: Let (X,τ) be a T_2-space.
Let (W,τ_W) be a subspace of (X,τ).
Must prove (W,τ_W) is a T_2-space.
Let $x \neq y$ in $W \Rightarrow x \neq y$ in X $(W \subseteq X)$

Since X is a T_2-space $\Rightarrow \exists U, V \in \tau$;
$U \cap V = \phi$, $x \in U \wedge y \in V$.

Let $U_1 = U \cap W$ and $V_1 = V \cap W$
then $U_1, V_1 \in \tau_W$

* $U_1 \cap V_1 = (U \cap W) \cap (V \cap W) = (U \cap V) \cap W$
$= \phi \cap W = \phi$

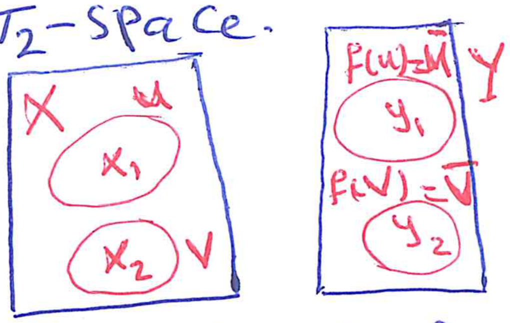
* $x \in U \wedge x \in W \Rightarrow x \in U \cap W = U_1$

* $y \in V \wedge y \in W \Rightarrow y \in V \cap W = V_1$

Hence, (W,τ_W) is a T_2-space.

Theorem A T_2-space is a topological property.

Proof: Let $(X,\tau) \cong (Y,\tau')$ and (X,τ) is T_2-space.
To prove (Y,τ') is T_2-space.



Let $y_1 \neq y_2$ in Y.
Since $X \cong Y$
$\therefore \exists f:(X,\tau) \to (Y,\tau')$;
f is 1-1, onto, continuous and open function.
* $y_1 \neq y_2$ in Y and f is onto, $\exists x_1, x_2 \in X$;
$f(x_1) = y_1$ and $f(x_2) = y_2$.
Since f is 1-1 and $y_1 \neq y_2 \Rightarrow x_1 \neq x_2$.
$x_1 \neq x_2$ in X and X is T_2-space $\Rightarrow \exists U, V \in \tau$
such that $x_1 \in U \wedge x_2 \in V \wedge U \cap V = \phi$.
Since f is open function $\Rightarrow f(U), f(V) \in \tau'$.
$y_1 = f(x_1) \in f(U)$ and $y_2 = f(x_2) \in f(V)$.

Must prove $f(U) \cap f(V) = \phi$.

Suppose $\exists y \in f(U) \cap f(V) \Rightarrow$
$y \in f(U) \wedge y \in f(V)$
$\Rightarrow \exists a \in U; y = f(a)$ and $\exists b \in V; y = f(b)$
$\therefore y = f(a) = f(b)$ and f is 1-1 $\Rightarrow a = b$
$\Rightarrow U \cap V \neq \phi$ Contradiction.
$\therefore f(U) \cap f(V) = \phi$ and Y is T_2-space.

Similarly; If Y is T_2-space, then X is T_2-space.

Theorem Let (X, τ) and (Y, τ') be two topological spaces, then $X \times Y$ is T_2-space if and only if both X and Y are T_2-space.

Example: Show which of the following spaces is T_2-space.

① $(\mathbb{R}, \tau_u) \times (\mathbb{N}, \tau_{cof})$.

$(\mathbb{R}, \tau_u)$ is T_2-space, but $(\mathbb{N}, \tau_{cof})$ is not T_2-space. Then $(\mathbb{R}, \tau_u) \times (\mathbb{N}, \tau_{cof})$ is not T_2-space.

② $(X, D) \times (X, I)$ where X has more than one element.

(X, D) is T_2-space, but (X, I) is not T_2-space. Then $(X, D) \times (X, I)$ is not T_2-space.

③ $(\mathbb{R}, \tau_u) \times (X, D)$

Both $(\mathbb{R}, \tau_u)$ and (X, D) are T_2-spaces. Then $(\mathbb{R}, \tau_u) \times (X, D)$ is T_2-space.

④ $(\mathbb{N}, D) \times (\mathbb{N}, \tau_{cof})$

$(\mathbb{N}, D)$ is T_2-space, but $(\mathbb{N}, \tau_{cof})$ is not T_2-space. Then $(\mathbb{N}, D) \times (\mathbb{N}, \tau_{cof})$ is not T_2-space.

Definition: A sequence S is a function from $\mathbb{N}$ into any nonempty set X.
i.e. $S: \mathbb{N} \to X$; $S(n) \in X$ $\forall n \in \mathbb{N}$, and denoted by $(S_n)_{n\in\mathbb{N}}$.

Definition Let (X, τ) be a topological space. and $(S_n)_{n\in\mathbb{N}}$ be a sequence in X:
The sequence $(S_n)_{n\in\mathbb{N}}$ is convergence to x_0 in X (denoted by $S_n \to x_0$) if the following condition is hold:

$\forall U \in \tau; x_0 \in U, \exists K \in \mathbb{N}; x_n \in U$ $\forall n \geq K$.

then x_0 is called limite Point for (S_n).

Example: Let $X = \{1, 2\}$ and $\tau = I = \{X, \phi\}$.
Let $S: \mathbb{N} \to X$; $S(n) = 1$ $\forall n \in \mathbb{N}$ be a sequence in X. Then $S_n \to 1$ and $S_n \to 2$.

* To Prove $S_n \to 1$:
the Unique open set Contain 1 is X.
and $\exists K = 1 \in \mathbb{N}$; $S(n) = 1 \in X$ $\forall n \geq K$.

$S(1) = 1 \in X$, $S(2) = 1 \in X$, $S(3) \in X$, $S(4) = 1 \in X$
$S(5) = 1 \in X$, --- $\therefore S(n) \in X$ $\forall n \geq 1$

* To Prove $S_n \to 2$:
The Unige open set Contain 2 is X, and
$\exists K = 1$; $S(n) = 1 \in X$ $\forall n \geq K$.

Example: Let $X = \{1, 2, 3\}$ and
$\tau = \{X, \phi, \{1\}, \{2\}, \{1, 2\}\}$

Let $S : \mathbb{N} \to X$; $S(n) = 3 \quad \forall n \in \mathbb{N}$.

* Then, $S(n)$ is disconvergence to 1.
Since $\exists\, U = \{1\} \in \tau$; $\nexists\, K \in \mathbb{N}$; $S(n) = 3 \in U$ $\forall n \geq K$.

denoted by $S_n \not\to 1$

* So $S_n \not\to 2$: Since $\exists\, U = \{2\} \in \tau$; $\nexists\, K \in \mathbb{N}$ such that $S(n) = 3 \in U$ $\forall n \geq K$.

* $S_n \to 3$: the unique open set contain 3 is $U = X$, and $\exists\, K = 1 \in \mathbb{N}$;
$S(n) \in U \quad \forall n \geq K$

$S(n) \in U = X \quad \forall n \geq K = 1$

$S(1) = 3 \in X$

$S(2) = 3 \in X$

$S(3) = 3 \in X$

$S(4) = 3 \in X$

$\vdots$

Example: Let $X = \{1,2,3\}$ and

$\tau = \{X, \phi, \{1\}, \{2\}, \{1,2\}\}$

A sequence $S : \mathbb{N} \to X$; $S(n) = \begin{cases} 1 & \text{if } n \text{ is odd.} \\ 2 & \text{if } n \text{ is even.} \end{cases}$

then, $S(n) \not\to 1$.

Since $\exists\, U = \{1\} \in \tau$; $1 \in U$ and $\nexists\, K \in \mathbb{N}$;

$S(n) \in U \quad \forall n \geq K$.

If $S(K) \in U \Rightarrow S(K) = 1$ and K is odd.

So $K+1$ is even and $S(K+1) = 2 \notin U = \{1\}$.

Also, $S_n \not\to 2$.

Since $\exists\, U = \{2\} \in \tau$; $\nexists\, K \in \mathbb{N}$; $S(n) \in U$

$\forall n \geq K$.

In Other hand, $S_n \to 3$, Since the Unique open set Contain 3 is X and $\exists\, K = 1 \in \mathbb{N}$ such that $S(n) \in X \quad \forall n \geq K$.

Example: Let $X = \{1,2,3\}$ and

$\tau = \{X, \phi, \{1,2\}, \{3\}\}$

Let $S : \mathbb{N} \to X$; $S(n) = \begin{cases} 1 & \text{if } n \text{ odd} \\ 2 & \text{if } n \text{ even} \end{cases}$

Then $S_n \to 1$ and $S_n \to 2$ but $S_n \not\to 3$.

Show that.

<u>Example</u>: Give an example to Convergence sequence in topological spaces has infinite number of deference limit points.

Let $X = \mathbb{R}$, $\tau = I = \{X, \phi\}$.

$S : \mathbb{N} \to X$; $S(n) = \begin{cases} \sqrt{2} & n \text{ is odd.} \\ 0 & \text{if } n \text{ is even.} \end{cases}$

Then, $S_n \to r \quad \forall r \in \mathbb{R}$. Since the unique open set contain r is X, and $\exists K = 1 \in \mathbb{N}$; $S(n) \in X \quad \forall n \geq K$.

<u>Example</u>: Give an example to a sequence in $\mathbb{R}$ which has no limit point:

Let $X = \mathbb{R}$ and $\tau = D = \mathbb{P}(X)$.

$S : \mathbb{N} \to X$; $S(n) = \begin{cases} 1 & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$

$\forall r \in \mathbb{R}$, $\exists U = \{r\} \in \tau$; $r \in U$ and $\nexists K \in \mathbb{N}$; $S(n) \in U \quad \forall n \geq K$.

$\therefore \quad S(n) \not\to r \quad \forall r \in \mathbb{R}$

In other example we can show that a sequence $S : \mathbb{N} \to \mathbb{R}$; $S(n) = \begin{cases} \sqrt{2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$ in $(\mathbb{R}, \tau_u)$, has no limit point:

Example Give an example to Convergence sequence in topological space has five deference limit Points.

Answer. Let $X=\{1,2,3,4,5\}$ and $\tau=\{X,\phi\}$.

Define $S:\mathbb{N}\longrightarrow X$; $S(n)=1$ $\forall n\in\mathbb{N}$.

Then $S_n\longrightarrow x$ $\forall x\in\{1,2,3,4,5\}$.

Remark: Let X has more than one Points. and $S:\mathbb{N}\longrightarrow X$ be a sequence.

1) If $X=I=\{X,\phi\}$, then $S_n\longrightarrow x$ $\forall x\in X$.

2) If $X=D$, then $S_n\not\longrightarrow x$ $\forall x\in X$.

***Example**: Let X be a non-empty set, $a\in X$. and let $S:\mathbb{N}\longrightarrow X$; $S(n)=a$ $\forall n\in\mathbb{N}$ be a sequence in X. Then:

① If $\tau=I$, then $S_n\longrightarrow x$ $\forall x\in X$

② If $\tau=D$, then $S_n\not\longrightarrow x$ $\forall x\in X$.

③ If $\tau=\{U\subseteq X: a\in U\}\cup\{\phi\}$, then $S_n\longrightarrow x$ $\forall$ $x\in X$.

Since, $\forall x\in A$, $\forall U\in\tau$; $x\in U$ implies $a\in U$

So, $S(n)=a\in U$ $\forall$ $n\in\mathbb{N}$.

$\therefore \exists K=1\in\mathbb{N}$; $S(n)\in U$ $\forall n\geqslant K$

$S(n)=a\in U$

Theorem: If (X, τ) is a T_2-Space, then every Convergence sequence in X has a unique limit Point:

Proof: Let $(S_n)_{n \in \mathbb{N}}$ be a convergence sequence in X.

Suppose $S_n \to x$ and $S_n \to y$; $x \neq y$ in X.

Since X is T_2-Space, then $\exists\, U, V \in \tau$; $U \cap V = \phi$, $x \in U$ and $y \in V$.

$\left.\begin{array}{l} S_n \to x \\ x \in U \end{array}\right\} \therefore \exists K_1 \in \mathbb{N}$; $S_n \in U \;\; \forall n \geq K_1$.

$\left.\begin{array}{l} S_n \to y \\ y \in V \end{array}\right\} \therefore \exists K_2 \in \mathbb{N}$; $S_n \in V \;\; \forall n \geq K_2$.

let $K = \max\{K_1, K_2\}$

$\therefore S_n \in U \cap V \;\; \forall n \geq K$. which is a contraduction

$\therefore (S_n)_{n \in \mathbb{N}}$ has a unique limit Point.

Theorem: If $f:(X,\tau)\to(Y,\tau')$ is Continuous function and $(S_n)_{n\in\mathbb{N}}$ is Convergence sequence in X such that $S_n\to x$, then $f(S_n)\to f(x)$.

Proof: To Prove $f(S_n)\to f(x)$ in Y.

Let $V\in\tau'$; $f(x)\in V$.

Since f is continuous, then $f^{-1}(V)\in\tau$

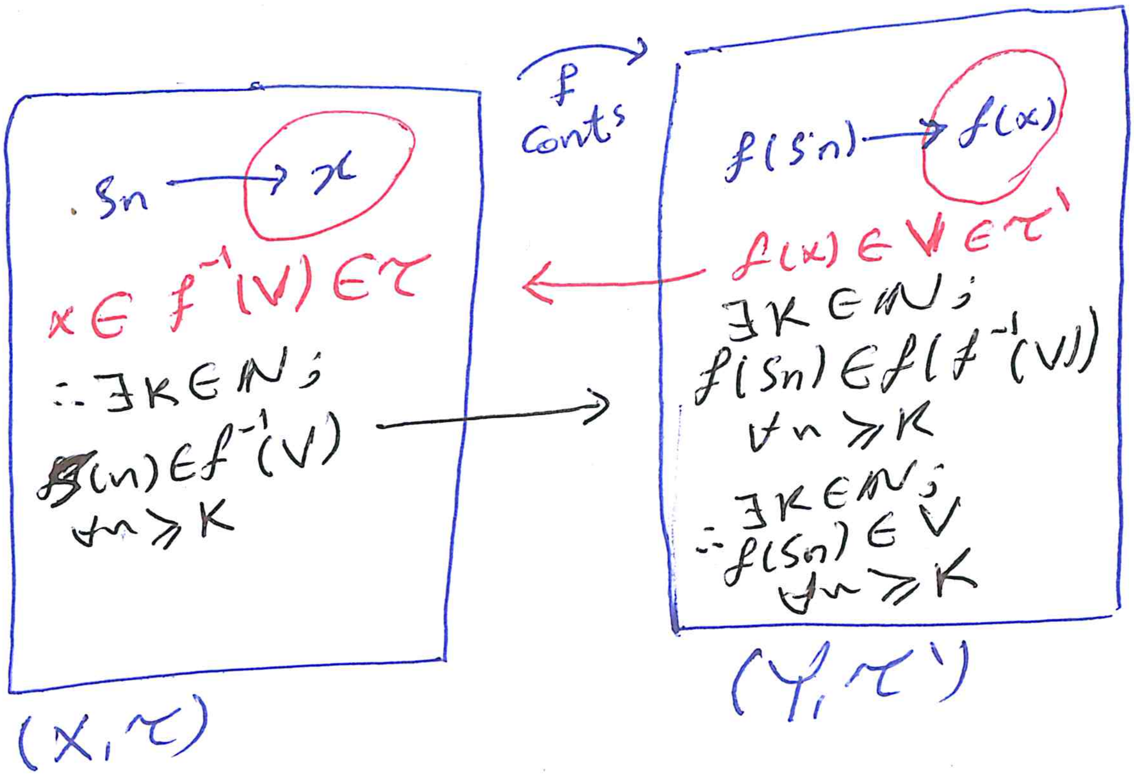
and $x\in f^{-1}(V)\in\tau$.

Since $S_n\to x$, $\therefore \exists K\in\mathbb{N}$; $S_n\in f^{-1}(V)$ $\forall n\geq K$.

$\Rightarrow f(S_n)\in f(f^{-1}(V))$ $\forall n\geq K$

$\therefore \exists K\in\mathbb{N}$; $f(S_n)\in V$ $\forall n\geq K$.

$\therefore f(S_n)\to f(x)$



Theorem: Every compact set in a T_2-space is closed.

Proof: Let (X, τ) be a T_2-space, A be a compact subset of X. To prove A is closed.
Must prove $X - A$ is open.

Let $x \in X - A = A^c$, then $x \neq a \ \forall a \in A$.

Since, (X, τ) is T_2-space and $x \neq a \ \forall a \in A$, then $\exists U_a, V_a \in \tau$; $a \in U_a$, $x \in V_a$ and $U_a \cap V_a = \phi$.

Therefore, $\{U_a ; a \in A\}$ is an open cover for A, which is compact set, Implies, $\exists a_1, a_2, \cdots, a_n \in A$, s.t. $A \subseteq \bigcup_{i=1}^{n} U_{a_i}$.

In other hand, let $V = V_{a_1} \cap V_{a_2} \cap \cdots \cap V_{a_n} = \bigcap_{i=1}^{n} V_{a_i}$.

thus, $x \in V \in \tau$.

$\forall i = 1, 2, \cdots, n$; $x \in V \subseteq V_{a_i}$ and $V_{a_i} \cap U_{a_i} = \phi$

then $V \cap U_{a_i} = \phi \quad \forall i = 1, 2, \cdots, n$

Therefore, $V \cap (\bigcup_{i=1}^{n} U_{a_i}) = \phi$.

Since $A \subseteq \bigcup_{i=1}^{n} U_{a_i}$, then $V \cap A = \phi$.

Hence, $x \in V \in \tau$ and $V \subseteq A^c$, A^c is open set and A is closed.

Theorem: If $f:(X,\tau)\to(Y,\tau')$ is continuous, bijective function and X is compact space, Y is T_2-space, then f is homeomorphism.

Proof: It is enough to prove f is closed function.
Let F be a closed subset of X, Since X is compact, then F is compact set (the closed subset of a compact space is compact).
Since f is continuous function, then $f(F)$ is compact set in Y (the continuous image of a compact set is compact).
Since Y is T_2-space, then $f(F)$ is closed in Y (The compact set in a T_2-space is closed).
Hence f is closed function. So f is homeomorphism.

$$\underset{\text{Compact}}{f:(X,\tau)} \longrightarrow \underset{T_2\text{-Space}}{(Y,\tau)}$$

f cont's, bij

F closed in X

F compact $\longrightarrow$ $f(F)$ compact in Y

Theorem: If A is closed set and B is compact set in T_2-space, then $A \cap B$ is compact.

Proof: Since B is compact in a T_2-space X, then B is closed.
$\Rightarrow A \cap B$ is closed (Since, both A and B are closed sets).

Since B is compact and $A \cap B$ closed subset of B, Then $A \cap B$ is compact.
(closed subset of a compact space is compact).

Corollary: If A and B are compact set in T_2-space (X, τ), then $A \cap B$ is compact.

Proof: Since X is T_2-space, A and B are compact subsets of X, then A and B are closed sets.
$\Longrightarrow A \cap B$ is closed
$\therefore A \cap B$ is closed subset of a compact space A. $\longrightarrow A \cap B$ is compact.

طريقة ثانية البرهان:

Proof: Since X is T_2-space and A is compact subset of X, then A is closed.
Since A is closed and B is compact subsets of a compact space X, then $A \cap B$ is compact.

Theorem: If X is a compact T_2-space, $A \subseteq X$, then A is closed $\Leftrightarrow A$ is compact.

Remark:

1) If $f:(\mathbb{N},\tau_{cof}) \longrightarrow (X,\tau)$ is continuous, Onto function, then (X,τ) is compact.

Since $(\mathbb{N},\tau_{cof})$ is compact, and the continuous image of a compact space is compact, then $f(\mathbb{N})$ is compact. And since f is onto, then $f(\mathbb{N}) = X$. Thus, X is compact.

2) In $(\mathbb{R},\tau_u)$, let $A \subseteq \mathbb{R}$, then A is closed $\Longleftrightarrow$ A is compact. } True or false

Answer: false: Since $\mathbb{N}$ is closed subset of $(\mathbb{R},\tau_u)$, but $\mathbb{N}$ is not compact.

$\mathbb{N}^c = (-\infty,1) \cup (1,2) \cup (2,3) \cup (3,4) \cup \dots \cup (n,n+1) \cup \dots$

$C = \{(n-1,n+1) : n \in \mathbb{N}\}$ open cover for $\mathbb{N}$

3) Answer True or False: In $(\mathbb{R},\tau_u)$ there is a converge sequence that has more than one limite Point.

Answer: False; since $(\mathbb{R},\tau_u)$ is T_2-space, and in a T_2-space every converge sequence has a unique limit Point.

Definition: Regular space

A topological space (X,τ) is called regular space if for every closed set $F \subseteq X$, and every element $x \notin F$, there exists two disjoint open sets U and V such that $x \in U$ and $F \subseteq V$.

i.e.

$$(X,\tau) \text{ is regular space} \Longleftrightarrow \begin{cases} \forall F \subseteq X;\ F \text{ is closed} \\ \forall x \notin F,\ \exists U, V \in \tau; \\ x \in U \wedge F \subseteq V \\ \text{and } U \cap V = \phi. \end{cases}$$

Regular space is denoted by R-space.

Example (1): Let $X = \{1,2,3,4\}$ and

$\tau = \{X, \phi, \{1,2\}, \{3,4\}\}$

then (X,τ) is R-space.

Since, the family of closed sets in (X,τ) is

$\mathcal{F} = \{X, \phi, \{3,4\}, \{1,2\}\}$

* $F = \{3,4\} \wedge x = 1$, $1 \notin F$.
  $\exists U = \{1,2\}$, $V = \{3,4\}$; $1 \in U \wedge F \subseteq V; U \cap V = \phi$
* $F = \{3,4\} \wedge x = 2$, $2 \notin F$.
  $\exists U = \{1,2\}$, $V = \{3,4\}$; $1 \in U \wedge F \subseteq V \wedge U \cap V = \phi$.
* $F = \{1,2\}$, $x = 3$, $3 \notin F$.
  $\exists U = \{3,4\}$, $V = \{1,2\}$; $3 \in U \wedge F \subseteq V \wedge U \cap V = \phi$.
* $F = \{1,2\}$, $x = 4$, $4 \notin F$.
  $\exists U = \{3,4\}$, $V = \{1,2\}$; $4 \in U \wedge F \subseteq V \wedge U \cap V = \phi$.

Example (2): Let $X=\{1,2,3\}$ and
$\tau=\{X,\phi,\{1\}\}$
then (X,τ) is not R-space.
Since the family of closed sets in (X,τ) is
$\mathcal{F}=\{X,\phi,\{2,3\}\}$.
Let $F=\{2,3\}$, $x=1$, $x\notin F$,
but $\nexists\, U,V\in\tau;\ 1\in U \wedge F\subseteq V \wedge U\cap V=\phi$.

Remark:

1) The regular space need not to be T_i-space $(i=0,1,2)$
See Example (1); (X,τ) is R-space which is not T_0-space, not T_1-space, not T_2-space.

2) $(\mathbb{N},\tau_{cof})$ is not R-space, but it is T_0-space and T_1-space.
Let $F=\{2\}$ closed set, $x=1\notin F$.
$\nexists\, U,V\in\tau_{cof};\ x\in U$, $F\subseteq V$ and $U\cap V=\phi$.
Since $\forall\, U,V\in\tau_{cof} \Rightarrow U\cap V\neq\phi;\ U\neq\phi\neq V$.

3) $(\mathbb{R},\tau_u)$ is R-space.
Let F be a closed subset of R, $x\notin F$.
Let $\epsilon=\min\{d(x,y);\ y\in F\}$
Let $N_y=(y-\frac{\epsilon}{3},\ y+\frac{\epsilon}{3})$ $\forall y\in F$.
Let $V=\bigcup_{y\in F} N_y$ and $U=(x-\frac{\epsilon}{3},\ x+\frac{\epsilon}{3})$.
Thus; $U,V\in\tau_u$, $x\in U \wedge F\subseteq V$ and
$U\cap V=\phi$.
Hence, $(\mathbb{R},\tau_u)$ is R-space.

Remark: In a topological space (X,τ), if $\tau = \mathcal{F}$, then X is R-space.

Proof: $\forall F$ closed subset of X and $\forall x \notin F$,
$\Rightarrow x \in F^c = X - F \in \tau$ (since F is closed).
in other hand $\mathcal{F} = \tau \Rightarrow F$ is open set.
Now, F and F^c are open sets, such that
$x \in F^c \wedge F \subseteq F \wedge F \cap F^c = \phi$.

Example: Let $X = \mathbb{N}$ and
$\tau = \{A_n : A_n = \{1, 2, \cdots, n\}; n \in \mathbb{N}\} \cup \{\mathbb{N}, \phi\}$
Is (X,τ) R-space?

Answer: No, Let $F = \{2, 3, 4, 5, \cdots\}$ be a closed set in (X,τ) and $1 \notin F$.
There is no two open sets $U, V \in \tau$;
$1 \in U \wedge F \subseteq V$ and $U \cap V = \phi$.
* $\mathbb{N}$ is the unique open set contain F.

Example: Let $X = \mathbb{N}$ and
$\tau = \{A_n : A_n = \{n, n+1, n+2, \cdots\}; n \in \mathbb{N}\} \cup \{\phi\}$
Is (X,τ) R-space?

Answer: No, Let $F = \{1\}$ be a closed set and $x = 2 \notin F$. $\nexists U, V \in \tau$; $x \in U \wedge F \subseteq V$ and $U \cap V = \phi$.
* $\mathbb{N}$. is the unique open set contain F.

Theorem: The space (X,τ) is R-space iff $\forall x \in X$, $\forall W \in \tau$; $x \in W$ then $\exists U \in \tau$ such that $x \in U \subseteq \overline{U} \subseteq W$

Theorem: The Property of being a R-space is a hereditary Property.

Proof: Let (X,τ) be R-space, and (W,τ_W) be a subspace of (X,τ).

To prove (W,τ_W) is R-space.

Let $x \in W$ and B closed set in (W,τ_W); $x \notin B$.

Since $W \subseteq X$, then $x \in X$.

B is closed set in (W,τ_W), then $\exists F$ closed set in X; $B = F \cap W$, and $x \notin F$.

Now, Since (X,τ) is R-space, F closed subset of X and $x \notin F$, then $\exists U, V \in \tau$; $x \in U \wedge F \subseteq V$ and $U \cap V = \phi$.

Let $U' = U \cap W$ and $V' = V \cap W$.

Then $U', V' \in \tau_W$

$x \in U \wedge x \in W \Rightarrow x \in U'$

$B = F \cap W \subseteq V \cap W = V' \Rightarrow B \subseteq V'$

$U' \cap V' = (U \cap W) \cap (V \cap W) = (U \cap V) \cap W$.

$= \phi \cap W = \phi$

Theorem: The Property of being a R- space is a topological Property.

Proof: Let $(X, \tau) \cong (Y, \tau')$ and let (X, τ) be a R-space. To Prove (Y, τ') is R-space.

Since $X \cong Y$, then $\exists f: X \longrightarrow Y$ is continuous, 1-1, onto and open function.

Let F be a closed subset of Y and $y \in Y$; $y \notin F$.

Since f is onto, then $\exists x \in X$; $y = f(x)$

f is Continuous, then $f^{-1}(F)$ is closed set in X.

f is 1-1, then $x \notin f^{-1}(F)$.

$f^{-1}(F)$ closed in X, $x \notin f^{-1}(F)$ and X is R-space, then $\exists U, V \in \tau$; $x \in U$, $f^{-1}(F) \subseteq V \wedge U \cap V = \phi$.

Since f is open function, then $f(U), f(V)$ open in Y.

Since f is 1-1 and onto, then

$$f(x) = y \in f(U) \text{ and } f(f^{-1}(F)) = F \subseteq f(V).$$

$$U \cap V = \phi \Rightarrow f(U) \cap f(V) = f(U \cap V) = f(\phi) = \phi.$$

Therefore, (Y, τ') is R-space.

In similar Way, if Y is R-space, then X is R-space

Remark: Prove or disprove, the Continuous image of a R-space is R-space.

Answer: Disprove, for example

$$f: (\mathbb{N}, D) \longrightarrow (\mathbb{N}, \tau_{cof}) \; ; \; f(x) = x \; . \; \forall x \in \mathbb{N}.$$

f is Continuous and $(\mathbb{N}, D)$ is R-space but $(\mathbb{N}, \tau_{cof})$ is not R-space.

Theorem: Let (X,τ) and (Y,τ') be two topological spaces, then the product space $X \times Y$ is R-space iff both X and Y are R-space.

i.e.

$X \times Y$ is R-space $\Longleftrightarrow$ X and Y are R-space.

Example:

① $(\mathbb{N},D) \times (\mathbb{R},\tau_{cof})$ is not R-space, since $(\mathbb{N},D)$ is R-space but $(\mathbb{R},\tau_{cof})$ is not R-space.

② $(X,D) \times (X,I)$ is R-space, since both (X,D) and (X,I) are R-spaces.
To prove (X,I) is R-space, is clearly, since $\nexists F \subsetneqq X$ closed subset of X; $x \notin F$.

③ $(\mathbb{R},\tau_u) \times (\mathbb{R},I)$ is R-space, since both $(\mathbb{R},\tau_u)$ and $(\mathbb{R},I)$ are R-spaces.

④ $(\mathbb{N},\tau_{cof}) \times (\mathbb{N},D)$ is not R-space since $(\mathbb{N},D)$ is R-space, but $(\mathbb{N},\tau_{cof})$ is not R-space.

⑤ $(\mathbb{N},I) \times (\mathbb{N},\tau_{cof})$ is not R-space. Since $(\mathbb{N},I)$ is R-space. but $(\mathbb{N},\tau_{cof})$ is not R-space.

Definition: T_3-space

A space (X,τ) is called T_3-space if it is T_1-space and regular-space

i.e. X is T_3-sp. $\Leftrightarrow$ X is T_1-sp. and R-sp.

Example: If X has more than one elements, then:

(i) (X,D) is T_3-sp., since it is T_1-sp. and R-sp.

(ii) (X,I) is not T_1-sp. $\Rightarrow$ (X,I) is not T_3-sp.

Example

$(\mathbb{N},\tau_{cof})$ is T_1-space, but not R-space

then $(\mathbb{N},\tau_{cof})$ is not T_3-space.

Example Let $X=\{1,2,3\}$, $\tau=\{X,\phi,\{1\},\{2,3\}\}$

Then (X,τ) is R-space, but not T_1-space,

So (X,τ) is not T_3-space.

Example: $(\mathbb{R},\tau_u)$ is T_1-space and R-space.

So $(\mathbb{R},\tau_u)$ is T_3-space.

Theorem Every metric space is T_3-space.

Proof: Since, Every m.sp. is T_1-sp. and R-sp. then it is T_3-sp.

<u>Theorem:</u> The T_3-space is a hereditary Property

Proof: Since T_1-space and R-space are hereditary properities, then T_3- is hereditary Property.

<u>Theorem:</u> The T_3-space is a topological Property.

Proof: Since T_1-space and R-space are topological properties, then T_3-space is a topological Property.

<u>Theorem</u> If (X,τ) is T_3-space, then X is T_2-SP.

<u>Proof:</u> Let (X,τ) be a T_3-space (T_1-sp. and R-sp).

To Prove (X,τ) is T_2-space.

let $x \neq y$ in X, Since X is T_1-space, then $\{x\}, \{y\}$ are closed sets. So $x \notin \{y\}$ (Since $x \neq y$).

Now, $x \notin \{y\}$ and $\{y\}$ is closed set and (X,τ) is R-sp., then $\exists U, V \in \tau \ni$
$x \in U \wedge \{y\} \subseteq V$ and $U \cap V = \phi$.

Hence, $\exists U, V \in \tau \ni x \in U, y \in V$ and $U \cap V = \phi$

So, (X,τ) is T_2-space

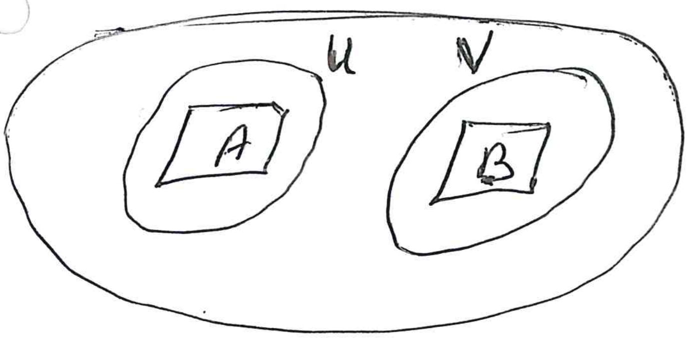
<u>Remark;</u> $T_3\text{-sp.} \underset{\not\Leftarrow}{\Rightarrow} T_2\text{-sp.} \underset{\not\Leftarrow}{\Rightarrow} T_1\text{-sp.} \underset{\not\Leftarrow}{\Rightarrow} T_0\text{-sp.}$

$\Downarrow$
R-sp.

Definition: Normal Space (N-space)

A space (X,τ) is called normal space iff for every two disjoint closed sets A and B there are two disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

i.e.: (X,τ) is N-sp. $\Leftrightarrow$ $\forall A, B$ closed subsets of X; $A \cap B = \phi$ then $\exists U, V \in \tau$; $A \subseteq U, B \subseteq V$, $U \cap V = \phi$



Example Let $X = \{1,2,3\}$ and $\tau = \{X, \phi, \{1\}, \{2\}, \{1,2\}\}$

To show that (X,τ) is N-space,

$\mathcal{F} = \{X, \phi, \{2,3\}, \{1,3\}, \{3\}\}$

Then there are no two closed, disjoint, subsets of X, (i.e. $\nexists A, B \in \mathcal{F}$; $A \cap B = \phi$), Hence, (X,τ) is N-space.

($F \Rightarrow F$ is true statement)

Remark: The above example is N-space, which is not T_1-space (since $1 \neq 3$; $\nexists U \in \tau \ni 3 \in U \wedge 1 \notin U$)

So, it is not R-sp. ($\{2,3\}$ closed set and $1 \notin \{2,3\}$ $\nexists U, V \in \tau$; $1 \in U \wedge \{2,3\} \subset V$, $U \cap V = \phi$)

① N-space $\not\Rightarrow$ T_0-sp.

Ex:- $(\mathbb{N}, I)$ is N-sp.

Since $\nexists A, B$ closed subsets of $\mathbb{N}$ $\ni$ $A \cap B = \phi$, $A \neq \phi \neq B$

But $(\mathbb{N}, I)$ is not T_0-sp.

Since, $1 \neq 2$ in $\mathbb{N}$ but $\nexists U \in \tau_I$; $(1 \in U \wedge 2 \notin U)$ or $(1 \notin U \wedge 2 \in U)$

T_0-sp. $\not\Rightarrow$ N-space

Ex:- $(\mathbb{N}, \tau_{cof})$ is T_0-sp.

But it is not N-sp.

$\{1\}, \{2\}$ are closed sets $\{1\} \cap \{2\} = \phi$ but $\nexists U, V \in \tau_{cof}$ such that $\{1\} \subseteq U \wedge \{2\} \subseteq V$. $\wedge$ $U \cap V = \phi$

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② N-space $\not\Rightarrow$ T_1-space

Example: $(\mathbb{N}, I)$: In above we see that $(\mathbb{N}, I)$ is N-space.

But $(\mathbb{N}, I)$ is not T_1-sp.

Since $1 \neq 2$ in $\mathbb{N}$ but $\nexists U, V \in I$; $(1 \in U \wedge 2 \notin U) \wedge (1 \notin V \wedge 2 \in V)$

T_1-space $\not\Rightarrow$ N-space

for example:

$(\mathbb{N}, \tau_{cof})$ is T_1-space which is not N-space

Example: Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$

(X, τ) is N-sp. but not T_0-sp., not T_1-sp. not T_2-sp., not R-space and not T_3-sp.

Remark: the Property of being a N-Space is not a hereditary Property.

For example: Let $X=\{1,2,3,4,5\}$ and

$$\tau=\{X,\phi,\{1,3,5\},\{1,4,5\},\{1,5\},\{1,3,4,5\}\}$$

It is clear that (X,τ) is N-space, Since

$$\mathcal{F}=\{X,\phi,\{2,4\},\{2,3\},\{2,3,4\},\{2\}\}$$

there are no two disjoint closed sets.

Now, Let $W=\{3,4,5\}\subseteq X$.

$$\tau_W=\{W,\phi,\{3,5\},\{4,5\},\{5\}\}$$

$$\mathcal{F}_W=\{W,\phi,\{4\},\{3\},\{3,4\}\}$$

$\{3\},\{4\}$ are disjoint closed subsets in (W,τ_W) but there are no two disjoint open sets U and V in τ_W ; $\{3\}\subseteq U$ $\wedge$ $\{4\}\subseteq V$.

Theorem A closed Subspace of N-Space is N-space.

Proof: Let (X,τ) be an N-SP. and W closed subset of X. T.P. (W,τ_W) is N-SP.

Let A,B disjoint closed subsets of W

$\Rightarrow$ A,B disjoint closed subsets of X and X is N-SP.

$\Rightarrow \exists\, U,V\in\tau$; $A\subseteq U$, $B\subseteq V$ and $U\cap V=\phi$

$\Rightarrow U\cap W,\ V\cap W\in\tau_W$; $A\subseteq U\cap W$, $B\subseteq V\cap W$ and $(U\cap W)\cap(V\cap W)=(U\cap V)\cap W=\phi\cap W=\phi$ $\therefore (W,\tau_W)$ is N-SP.

<u>Theorem:</u> The Property of being a N-space is a topological space.

Proof:

Let $(X,\tau) \cong (Y,\tau')$ with (X,τ) is N-sp.

T.P. (Y,τ') is N-sp.

Since $(X,\tau) \cong (Y,\tau')$ then there exist $f:(X,\tau) \longrightarrow (Y,\tau')$; f cont', 1-1, onto, open

Let A, B are disjoint closed subsets of Y

Since f is onto, then $f^{-1}(A), f^{-1}(B)$ are in X;

$f^{-1}(A) \neq \phi \neq f^{-1}(B)$

Since f is 1-1 and $A \cap B = \phi \Rightarrow f^{-1}(A) \cap f^{-1}(B) = \phi$

Since f is cont^s and A, B closed in $Y \Rightarrow$

$f^{-1}(A), f^{-1}(B)$ closed in X

Since (X,τ) is N-space $\Rightarrow \exists U, V \in \tau$;

$f^{-1}(A) \subseteq U$, $f^{-1}(B) \subseteq V$ and $U \cap V = \phi$

Since f is open function $\Rightarrow f(U), f(V)$ open in Y

Since f is 1-1 $\wedge$ $U \cap V = \phi \Rightarrow f(U) \cap f(V) = \phi$

$f(f^{-1}(A)) \subseteq f(U)$ and $f(f^{-1}(B)) \subseteq f(V)$

$\Rightarrow A \subseteq f(U)$, $B \subseteq f(V)$, $f(U) \cap f(V) = \phi$

$\therefore (Y,\tau')$ is N-sp.

Similarly if (Y,τ') is N-sp. $\Rightarrow (X,\tau)$ is N-space

Definition: T_4-Space

A topological space (X,τ) is T_4-space iff it is T_1-space and N-space.

i.e. T_4-sp. $\Leftrightarrow$ N-sp. and T_1-sp.

Example: If X be a non-empty set, which has more than one element, then

(i) (X,D) is N-sp. and T_1-sp. $\Rightarrow$ (X,D) is T_4-sp.

(ii) (X,I) is N-sp. but not T_1-sp. $\Rightarrow$ (X,I) is not T_4-sp.

Example:

(N,τ_{cof}) is T_1-sp. but not N-sp. $\Rightarrow$ (N,τ_{cof}) is not T_4-sp.

Theorem: Every T_4-space is T_3-space.

Proof: Let (X,τ) be a T_4-sp. (N-sp. and T_1-sp.)

T.P. (X,τ) is T_3-sp. (i.e T.P. X is R-sp.)

Let A be a closed subset of X; $x \notin A$.

Since (X,τ) is T_1-sp. $\Rightarrow$ $\{x\}$ is closed set

Since $x \notin A \Rightarrow \{x\} \cap A = \phi$

Since (X,τ) is N-sp. $\Rightarrow \exists\, U,V \in \tau$; $U \cap V = \phi$ and $\{x\} \subseteq U$, $A \subseteq V$

$\Rightarrow \exists\, U,V \in \tau$; $x \in U$, $A \subseteq V$ and $U \cap V = \phi$

$\Rightarrow$ (X,τ) is R-space.

and Since X is T_4-sp. $\Rightarrow$ X is T_1-sp.

$\Rightarrow (X,\tau)$ is T_3-space.

Corollary: Every T_4-space is R-space.

Proof: Since every T_4-space is T_3-space and a T_3-space is T_1-sp. and R-sp. then a T_4-sp. is R-sp.

i.e. Let (X,τ) be a T_4-sp.

$\Rightarrow$ (X,τ) is a T_3-sp.

$\Rightarrow$ (X,τ) is T_1-sp. and R-sp.

$\Rightarrow$ (X,τ) is R-sp.

Remark:

$$T_4\text{-sp} \Rightarrow T_3\text{-sp.} \Rightarrow T_2\text{-sp} \Rightarrow T_1\text{-sp.} \Rightarrow T_0\text{-sp.}$$
$$\not\Leftarrow \qquad \not\Leftarrow \qquad \not\Leftarrow \qquad \not\Leftarrow$$

Remark: The property of being a T_4-space is not a hereditary property.
Since N-sp. is not hereditary property

Theorem: A closed subspace of T_4-space is T_4-space.

Proof: Let (X, τ) be a T_4-space (N-sp. and T_1-sp.)
Let W be a closed subset of X.
then (W, τ_W) is N-space (since the closed subspace of N-sp. is N-sp.).

and (W, τ_W) is T_1-space (since T_1-sp. is a hereditary property).

Hence, (W, τ_W) is T_4-space

Theorem: The property of being a T_4-space is a topological property.

Proof: Since N-space and T_1-space are topological property, then T_4-space is topological property.

Theorem: Every compact T_2-space is R-space

Corollary: Every compact T_2-space is T_3-space

Theorem: Every compact T_2-space is N-space.

Coro Every compact T_2-sp. is T_4-space

$$\underbrace{T_4 \underset{\not\Leftarrow}{\Rightarrow} T_3 \underset{\not\Leftarrow}{\Rightarrow} T_2}\ \underset{\Leftarrow}{\Rightarrow} T_1 \underset{\not\Leftarrow}{\Rightarrow} T_0$$

$\left\{\begin{array}{l}\text{If } (X,\tau) \text{ is compact space} \\ \text{then } T_4 \equiv T_3 \equiv T_2\end{array}\right.$