CHAPTER FIVE
CONNECTED SPACES

Definition 5.1:
A topological space (X,τ) is called "disconnected space", If there are two disjoint nonempty open sets A and B such that $A \cup B = X$.

i.e.
The space (X,τ) is disconnected $\Longleftrightarrow$ $\exists A \neq \phi \neq B;$ $A, B \in \tau$, $A \cap B = \phi$ and $A \cup B = X$.

The space (X,τ) is **Connected** if it is not disconnected.

Example 5.2: Let $X = \{1,2,3\}$ and $\tau = \{X, \phi, \{1\}, \{2,3\}\}$. Then (X,τ) is a disconnected space, Since, Let $A = \{1\}$, $B = \{2,3\}$, $A, B \in \tau$, $A \neq \phi$ and $B \neq \phi$, $A \cap B = \phi$ and $A \cup B = X$

Example 5.3: Let $X = \{1,2,3\}$ and $\tau = \{X, \phi, \{1\}, \{2\}, \{1,2\}\}$. Then (X,τ) is Connected space. Since, $\nexists A, B \in \tau$; $A \neq \phi \neq B$, $A \cap B = \phi$ and $A \cup B = X$.

Remark 5.4:

① (X,D) is disconnected space, when X has more than one element. Since for $\phi \neq A \subsetneq X$, then $A, A^c \in D$, $A \neq \phi \neq A^c$, $A \cup A^c = X$.

② (X,I) is Connected space.

③ If $\tau = \mathcal{F}$ and $\tau \neq I$, then (X,τ) is disconnected space.

Since $\tau \neq I \Rightarrow \exists A \in \tau$; $\phi \neq A \subsetneq X$.

Since $\tau = \mathcal{F} \Rightarrow A, A^c \in \tau$ and so $\phi \neq A^c \neq X$.

Now, $A, A^c \in \tau$; $A \neq \phi \neq A^c$, $A \cap A^c = \phi$ and $X = A \cup A^c$.

Example 5.5: The space (X, τ_{cof}) is Connected space, where X is an infinite set, since for any two nonempty open sets A and B then $A \cap B \neq \phi$.

$(\mathbb{N}, \tau_{cof})$, $(\mathbb{R}, \tau_{cof})$, (E, τ_{cof}), (O, τ_{cof}) are Connected spaces.

Example 5.6. The usual space $(\mathbb{R}, \tau_u)$ is Connected, Since $\mathbb{R}$ cannot be written as a Union of two disjoint nonempty open sets.

For any $x \in \mathbb{R}$, then $(-\infty, x) \cup (x, \infty) = \mathbb{R} - \{x\} \neq \mathbb{R}$.

Example 5.7:

1) The space $(\mathbb{N}, \tau)$ is connected, where
$\tau = \{A_n \subseteq \mathbb{N} : A_n = \{1, 2, 3, \ldots, n\}\} \cup \{\mathbb{N}, \phi\}$.
Since $A_n \cap A_m \neq \phi$, $\forall n, m \in \mathbb{N}$.

2) The space $(\mathbb{N}, \tau)$ is connected, where
$\tau = \{A_n \subseteq \mathbb{N} : A_n = \{n, n+1, n+2, \ldots\}\} \cup \{\phi\}$.
Since $A_n \cap A_m \neq \phi$, $\forall n, m \in \mathbb{N}$.

Example 5.8: Let $X = \{1, 2, 3, 4\}$. Define two topologies on X, τ and τ'; such that (X, τ) is connected and (X, τ') is disconnected.

i) $\tau = \{X, \phi, \{1\}\}$, then (X, τ) is connected.

2) $\tau' = \{X, \phi, \{1,2\}, \{3,4\}\}$, then (X, τ') is disconnected.

Proposition 5.9: (X, τ) is connected space iff X cannot be written as a union of two nonempty disjoint closed sets.

Proof: $\Rightarrow$ Suppose (X, τ) is connected space.
If X can be written as a union of two disjoint, nonempty closed sets A and B, then
$A \neq \phi$, $B \neq \phi$ and $A \cap B = \phi$, $A \cup B = X$,
implies $A^c \neq \phi \neq B^c$ and $A^c \cap B^c = \phi$, $A^c \cup B^c = X$.
A, B are closed sets, then A^c & B^c are open sets.
Implies, X is disconnected, which is a contradiction.

$\Leftarrow$: Suppose X cannot be written as a union of two nonempty disjoint closed sets. To Prove X is Connected.

If X is disconnected space, then $\exists A, B \in \tau$; $A \neq \phi \neq B$, $A \cap B = \phi$ and $A \cup B = X$.

Implies A^c, B^c are closed sets, so that $A^c = B$ and $B^c = A$.

Hence, A^c, B^c are closed sets;

$A^c = B \neq \phi$ and $B^c = A \neq \phi$

$A^c \cap B^c = B \cap A = \phi$

$A^c \cup B^c = B \cup A = X$.

which is a contraduction.

Then X is Connected.

$\Rightarrow$: Suppose X is Connected.

If $X = A \cup B$; $A \neq \phi \neq B$, $A \cap B = \phi$, A, B closed sets

$\Rightarrow X^c = (A \cup B)^c$; $A^c \neq \phi \neq B^c$, $(A \cap B)^c = \phi^c$, A^c, B^c open sets

Thus $A^c = B \wedge B^c = A$

$\Rightarrow \phi = A^c \cap B^c$; $A^c \neq \phi \neq B^c$, $A^c \cup B^c = X$, A^c, B^c open

Let $A^c = U$, $B^c = V$

$\Rightarrow U \cap V = \phi$, $U \neq \phi \neq V$, $U \cup V = X$, $U, V \in \tau$

$\therefore X$ disconnected $\divideontimes$ Contraduction

$\Leftarrow$ Suppose X cannot be written as a union of two nonempty disjoint closed set.

If X is disconnected

Then $\exists U, V \in \tau$; $U \neq \phi \neq V$, $U \cap V = \phi$ and $U \cup V = X$

Thus; $U^c = V$, $V^c = U$

$\Rightarrow \exists U^c, V^c$ closedsets, $U^c \neq \phi \neq V^c$, $(U \cap V)^c = \phi^c$ and $(U \cup V)^c = X^c$

$\Rightarrow \exists U^c, V^c$ closed sets; $U^c \neq \phi \neq V^c$, $U^c \cup V^c = X$ and $U^c \cap V^c = \phi$.

let $U^c = A$, $V^c = B$. that is a contraduction.

Theorem: (X,τ) is Connected Space iff the only subsets of the Space X which are open and closed are X and ϕ.

Proof: $\Rightarrow$: Suppose X is Connected.

If $\exists\, A \subsetneqq X$, $A \neq \phi$, A is open and closed.

$\Rightarrow A^c \subsetneqq X$, $A^c \neq \phi$, A^c is open and closed.

$\Rightarrow A, A^c$ are open sets; $A \cap A^c = \phi$, $A^c \neq \phi \neq A$, and $X = A \cup A^c$.

thus, X is disconnected, which is a contradiction.

$\Leftarrow$: Suppose the only open and closed subsets of X are X and ϕ.

If X is disconnected.

Then $\exists\, A, B \subseteq X$; $A \neq \phi \neq B$, $A \cap B = \phi$, $A, B \in \tau$ and $X = A \cup B$.

Thus, $A = B^c$, and since B is open then B^c is closed, so A is closed.

$\therefore$ A is open and closed.

which is a contradiction.

$\therefore$ X is Connected

Theorem: (X,τ) is connected space iff every continuous function from X to $(\{1,2\},D)$ is constant function.

Proof: Let $f:(X,\tau)\longrightarrow(\{1,2\},D)$ be continuous function, and (X,τ) is connected space.

To prove f is constant function.

If not, then $\exists a\in X; f(a)=1$ and $\exists b\in X; f(b)=2$.

Let $A=\{a\in X: f(a)=1\}$

$B=\{b\in X: f(b)=2\}$

Then $A\neq\phi$ and $B\neq\phi$.

And $A\cap B=\phi$ (If $A\cap B\neq\phi$, then $\exists x\in A\cap B$ $\therefore f(x)=1$ and $f(x)=2$ contraduction).

$\{1\}\in D$, and f continuous function, then $f^{-1}(\{1\})=A\in\tau$.

So, $\{2\}\in D$, then $f^{-1}(\{2\})=B\in\tau$.

$X=A\cup B$, (If not, $\exists x\in X; f(x)\neq1$ and $f(x)\neq2$, Thus f is not function).

Then X is disconnected.

That is a contraduction. So f is constant.

$\Leftarrow$: Suppose that every continuous function from (X,τ) into $(\{1,2\},D)$ is constant, to prove X is connected.

If X is disconnected, then $\exists A,B\subseteq X; A,B\in\tau$, $A\neq\phi\neq B$, $A\cap B=\phi$ and $X=A\cup B$.

Define, $f:(X,\tau)\to(\{1,2\},D)$; $f(x)=\begin{cases}1 & \forall x\in A\\ 2 & \forall x\in B.\end{cases}$

Then f is continuous, not constant function. Contraduction. Then X connected.

Question: Prove or disprove,

1. If (X, τ) is connected space, and (W, τ_W) is a subspace of X, then W is connected.

Answer: Disprove, for example

Let $X = \{1, 2, 3, 4\}$

$\tau = \{X, \phi, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$

Then, (X, τ) is connected space.

Let $W = \{2, 3\}$, then $\tau_W = D$, So (W, τ_W) is disconnected.

2. If (W, τ_W) is connected subspace of (X, τ) then (X, τ) is connected.

Answer: Disprove.

Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}, \{2, 3\}\}$

And let $W = \{2, 3\}$, then $\tau_W = I$, It is cleare that (X, τ) is disconnected but (W, I) is connected.

Theorem: The Continuous image of Connected space is Connected.

Proof: Let (X,τ) be a Connected space and $f:(X,\tau)\to(Y,\tau')$ is a Continuous, onto, mapping. To prove (Y,τ') is Connected.

If not, then $\exists A,B\subseteq Y$; $A\neq\phi\neq B$, $A\cap B=\phi$, $A,B\in\tau'$ and $A\cup B=Y$.

* Since f is onto, then $f^{-1}(A)\neq\phi\neq f^{-1}(B)$.
* Since f is Continuous, then $f^{-1}(A), f^{-1}(B)\in\tau$.
* Since f is a function and $A\cap B=\phi$, then $f^{-1}(A)\cap f^{-1}(B)=\phi$.
* $Y=A\cup B \Rightarrow f^{-1}(Y)=f^{-1}(A\cup B)$

$\Rightarrow f^{-1}(Y)=f^{-1}(A)\cup f^{-1}(B)$, $A\cap B=\phi$.

$\Rightarrow f^{-1}(Y)=X$ since f is onto

$\therefore$ $X=f^{-1}(A)\cup f^{-1}(B)$.

that is a contradiction.

$\therefore$ (Y,τ') is Connected.

Question: Prove or disprove; If $f:(X,\tau)\to(Y,\tau')$ is Continuous, onto function and Y is Connected, then X is Connected.

Answer: Disprove;

$f:(N,D)\longrightarrow(N,I)$; $f(x)=x$ $\forall x\in N$.

(N,I) is Connected, and f is onto, then (N,I) is Continuous function, but (N,D) is disconnected

Theorem: Connectedness is topological Property.