Chapter Three: Compact Spaces

Definition Let (X,τ) be a topological space, $\{A_\alpha\}_{\alpha\in\Lambda}$ be a family of subsets of X. If $X\subseteq\bigcup_{\alpha\in\Lambda}A_\alpha$ then the family $\{A_\alpha\}_{\alpha\in\Lambda}$ is called cover of X.

If Λ is finite set, then the cover $\{A_\alpha\}_{\alpha\in\Lambda}$ is called finite cover of X.

If $A_\alpha\in\tau$, $\forall\alpha\in\Lambda$, then the cover $\{A_\alpha\}_{\alpha\in\Lambda}$ is called open cover of X.

If $A_\alpha\in\mathcal{F}$, $\forall\alpha\in\Lambda$ (i.e. A_α is closed set), then $\{A_\alpha\}_{\alpha\in\Lambda}$ is called closed cover of X.

Example: Let $X=\{1,2,3\}$, $\tau=\{X,\phi,\{1\},\{2\},\{1,2\}\}$

then $C_1=\{\{1\},\{2\},\{3\}\}$ is a cover of X.

$C_2=\{X,\{1\},\{2\}\}$ is an open cover of X.

$C_3=\{\{2,3\},\{1,3\}\}$ is a closed cover of X

Example In $(\mathbb{R}, \tau_u)$,

$C_1 = \{E, O, \mathbb{Q}, Irr\}$ is a finite cover of $\mathbb{R}$.

$C_2 = \{\mathbb{Q}, Irr\}$ is a finite cover of $\mathbb{R}$.

$C_3 = \{\{x\} : x \in \mathbb{R}\}$ is an infinite cover of $\mathbb{R}$.

$C_4 = \{(a, b) : a, b \in \mathbb{R}\}$ is an open cover of $\mathbb{R}$.

$C_5 = \{[a, b] : a, b \in \mathbb{R}\}$ is a closed cover of $\mathbb{R}$.

Definition: Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a cover of X and $\{B_i\}_{i \in \Lambda}$ be a subfamily of $\{A_\alpha\}_{\alpha \in \Lambda}$ and $X \subseteq \bigcup_{i \in \Lambda} B_i$, then $\{B_i\}_{i \in \Lambda}$ is called subcover of $\{A_\alpha\}_{\alpha \in \Lambda}$.

Example: In the above example, C_2 is a subcover from C_1

Example Let $X = \{1, 2, 3\}$

$\tau = \{X, \phi, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}\}$

Let $C_1 = \{\{1\}, \{2\}, \{1, 2\}, \{1, 3\}\}$. open cover of X.

$C_2 = \{\{1, 2\}, \{1, 3\}\}$.

Then C_2 is a finite subcover from C_1.

Definition: A space (X, τ) is called **compact** iff every open cover of X has a finite subcover.

X is compact $\Leftrightarrow$ $\left[\forall C = \{U_\alpha\}_{\alpha \in \Lambda},\ U_\alpha \in \tau, \forall \alpha\ \ X = \bigcup_{\alpha \in \Lambda} U_\alpha\right], \left[\exists \alpha_1, \alpha_2, \dots, \alpha_n \text{ s.t. } X = \bigcup_{i=1}^{n} U_{\alpha_i}\right]$

X is not compact $\Leftrightarrow$ $\exists C = \{U_\alpha\}_{\alpha \in \Lambda}$ an open cover of X ; $\nexists \alpha_1, \alpha_2, \dots, \alpha_n$; $X = \bigcup_{i=1}^{n} U_{\alpha_i}$

Example: In a space $(\mathbb{N}, \tau)$;

$$\tau = \{\mathbb{N}, \phi, E^+, O^+\}$$

then $(\mathbb{N}, \tau)$ is compact, since, since every open cover $C \subseteq \tau$, has a finite subcover. (Since τ is finite set).

Example: In a space $(\mathbb{N}, \tau)$; $\tau = D_{\mathbb{N}} = P(\mathbb{N})$. the discreat topology on $\mathbb{N}$.

then $(\mathbb{N}, \tau)$ is not compact, since the open cover $\{\{n\} : n \in \mathbb{N}\}$ has no finite subcover.

Remark: ① If τ is finite set, then (X, τ) compact

② If X is finite set, then (X, τ) is compact

Example: Consider the space $(\mathbb{N}, \tau)$, where,

$\tau = \{\mathbb{N}\} \cup \mathbb{P}(O^+)$.

$(\mathbb{N}, \tau)$ is compact, since every open cover of $\mathbb{N}$ must covers E^+. So, there is no open set except $\mathbb{N}$ covers E^+, (i.e. The unique open set contains E is $\mathbb{N}$). This means, every open cover of $\mathbb{N}$ must contain the open set $\mathbb{N}$. Hence $\{\mathbb{N}\}$ is a finite subcover.

Example: Consider the space $(\mathbb{N}, \tau)$, where

$\tau = \{A_n = \{n, n+1, n+2, \ldots\} : n \in \mathbb{N}\} \cup \{\phi\}$.

$A_1 = \{1, 2, 3, \ldots\} = \mathbb{N}$

$A_2 = \{2, 3, 4, \ldots\}$

$A_3 = \{3, 4, 5, \ldots\}$

$\vdots$

$(\mathbb{N}, \tau)$ is compact, since every open cover of $\mathbb{N}$ must covers 1. the unique open set covers 1 is $\mathbb{N} = A_1$. Implies, every open cover of $\mathbb{N}$ must contains the open set $\mathbb{N}$ A_1 to covers 1. Hence $\{A_1\}$ is a finite subcover for C, where C is an open cover of $\mathbb{N}$.

Example: Consider the space $(\mathbb{N}, \tau)$; where
$\tau = \{A_n = \{1, 2, \dots, n\} : n \in \mathbb{N}\} \cup \{\mathbb{N}, \phi\}$.
$(\mathbb{N}, \tau)$ is not compact, since $\{A_n : n \in \mathbb{N}\}$ is an open cover of $\mathbb{N}$ which has no finite subcover.

Example: Consider the space $(\mathbb{R}, \tau)$, where
$\tau = \{\mathbb{R}, \phi, \mathbb{Q}, Irr\}$.
Since τ is finite set, then every open cover of $\mathbb{R}$ has a finite subcover. $(\mathbb{R}, \tau)$ is compact space.

Example: Consider the space $(\mathbb{R}, \tau_u)$.
the usual topological space.
$(\mathbb{R}, \tau_u)$ is not compact, since $\{(-n, n) : n \in \mathbb{N}\}$ is an open cover of $\mathbb{R}$ which has no finite subcover.

Example: Show that the space $(\mathbb{N}, \tau_{cof})$ is compact.

Solution: Let $C = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of $\mathbb{N}$. ($U_\alpha \in \tau_{cof} \Rightarrow \mathbb{N} - U_\alpha$ is finite set)
Take, $U_{\alpha_0} \in C$. then $\mathbb{N} - U_{\alpha_0}$ is finite set.
Let $\mathbb{N} - U_{\alpha_0} = \{x_1, x_2, x_3, \dots, x_n\}$
Since C covers $\mathbb{N}$, then C covers the set $\{x_1, x_2, x_3, \dots, x_n\}$.
Let U_{α_1} contains x_1, U_{α_2} contains x_2, ..., U_{α_n} contains x_n. Therefore, $\{U_{\alpha_0}, U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}\}$ is a finite subcover for $\mathbb{N}$. Hence $\mathbb{N}$ is compact.

Compact Subspace

Definition (Subspace)

Let (X, τ) be a topological space, and W be a subset of X, then

$\tau_W = \{U \cap W : U \in \tau\}$ is a topology on W, and (W, τ_W) is called a subspace of (X, τ)

Example

Let (X, τ) be a topological space, where $X = \{1, 2, 3, 4\}$

$\tau = \{X, \phi, \{3,4\}, \{1,3,4\}, \{2,3,4\}\}$

Let $W = \{1, 2\}$

$\tau_W = \{W, \phi, \{1\}, \{2\}\}$

then τ_W is a topology on W and (W, τ_W) is a topological space, which is a subspace of (X, τ).

Definition

Let (X, τ) be a topological space, and W be a subset of X. Then W is called compact set if every open cover from X cover W has a finite subcover.

W is compact $\Leftrightarrow$ $\left[\forall \{U_\alpha\}_{\alpha \in \Lambda} ; U_\alpha \in \tau \text{ and } W \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha \Rightarrow \exists \alpha_1, \alpha_2, \cdots, \alpha_n ; W = \bigcup_{i=1}^{n} U_{\alpha_i} \right]$

Note: If W is compact subset of (X, τ), then (W, τ_W) is called compact subspace of (X, τ).

Example Let (X, τ) be a topological space, where $X = \mathbb{N}$, the set of natural numbers and $\tau = \mathbb{P}(\mathbb{N}) = D$

Let $W = \{1, 2, 3\}$

It is clear that every open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ from τ cover W has finite subcover U_{α_1} covers 1, U_{α_2} covers 2 and U_{α_3} cover 3

So $W \subset U_{\alpha_1} \cup U_{\alpha_2} \cup U_{\alpha_3}$.

Example Let $X = \mathbb{N} \cup \{0\}$

$$\tau = \mathbb{P}(\mathbb{N}) \cup \{H \subseteq X : 0 \in H \text{ and } H^c \text{ is finite}\}$$

Let $W = \{0, 2, 4, 6, 8, \dots\} = \{0\} \cup E^+$

Prove or disprove W is compact ?!

It is clear that W is compact, Since every open cover $\{U_\alpha\}_{\alpha \in \Delta}$ from τ cover W, has at least one set U_{α_0} cover zero, So $U_{\alpha_0}^c$ is finite.

Let $U_{\alpha_0}^c = \{x_1, x_2, x_3, \dots, x_n\}$

then for each x_i, there exist U_{α_i} such that $x_i \in U_{\alpha_i}$, $i = 1, 2, \dots, n$.

So, $U_{\alpha_0}, U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}$ cover W.

Hence, W is compact.

مثال

In $(\mathbb{N}, \tau)$, $\tau = \{\mathbb{N}, \phi, E\}$
then E is compact but not closed & not bounded

ملاحظة: نظرية Heine-Borel خاصة بالفضاءات المترية فقط. والمثال اعلاه يوضح انها غير متحققة في الفضاءات غير المترية

Example: Let $X = \mathbb{N} \cup \{0, -1\}$
$\tau = P(\mathbb{N}) \cup \{H \subseteq X : -1, 0$

Heine-Borel Theorem

A subset A of a space $(\mathbb{R}, \tau_u)$ is compact iff A is closed and bounded.

حيث ان هذه النظرية متحققة في الفضاءات المترية فقط . وكون الفضاء $(\mathbb{R}, \tau_u)$ هو فضاء متري فهو يحقق نظرية Heine-Borel .

المثال التالي يوضح ان نظرية Heine-Borel غير متحققة في الفضاءات غير المترية

In $(\mathbb{R}, \tau)$; $\tau = \{\mathbb{R}, \phi, \mathbb{N}\}$, then $\mathbb{N}$ is compact subset of $\mathbb{R}$ which is not closed and not bounded.

Heine-Borel Theorem

A subset A of a space $(\mathbb{R}, \tau_u)$ is compact iff A is closed and bounded.

A is compact $\Leftrightarrow$ A is closed $\wedge$ bounded

Example: In $(\mathbb{R}, \tau_u)$ show that any set from the following sets is compact by using Heine-Borel theorem.

$A = (2, 3)$: not closed, bounded $\Rightarrow$ not compact

$B = [5, 7]$: closed and bounded $\Rightarrow$ compact

$C = [-2, 1)$: not closed, bounded $\Rightarrow$ not compact

$D = \mathbb{N}$: closed, not bounded $\Rightarrow$ not compact

$E = \{2, 3, 4\}$: closed and bounded $\Rightarrow$ compact

$F = \mathbb{Q}^{Irr}$: not closed, not bounded $\Rightarrow$ not compact

Note: In $(\mathbb{R}, \tau_u)$ Every closed interval is compact.

Definition (hereditary Property)

The Property "P" is called a hereditary Property of a space X, if every subspace W of X has the Property "P".

Q: prove or disprove; Compactness is a hereditary Property

Remark: Compactness is not hereditary Property. For example.

In the space $(\mathbb{N}, \tau)$, wher

$\tau = \{\mathbb{N}\} \cup \mathbb{P}(E^+)$

It is clear that $\mathbb{N}$ is compact space, Since every open cover of $\mathbb{N}$ must contain the set $U = \mathbb{N}$, So $\{U\} = \{\mathbb{N}\}$ is a finite subcover.

In othe hand, $W = E^+ \subseteq \mathbb{N}$, which is not compact. Since $\{\{n\} : n \in E^+\}$ $\{\{2n\} : n \in \mathbb{N}\}$ is an open cover of E^+ which has no finite subcover.

Prove or disprove the following Statements
If A and B are Compact sets in a space (X, τ), then:
i. $A \cup B$ is Compact.?!
ii. $A \cap B$ is Compact.?!

i. $A \cup B$ is Compact.

Proof:- Let $\{U_\alpha\}_{\alpha \in \Lambda}$ is an open Cover for $A \cup B$. Then:

$C = \{U_\alpha\}_{\alpha \in \Lambda}$ is an open cover of A and $\{U_\alpha\}_{\alpha \in \Lambda}$ is an open cover of B.

Since A is compact then there is a finite subcover $\{U_{A_1}, U_{A_2}, \dots, U_{A_n}\}$ of $C = \{U_\alpha\}_{\alpha \in \Lambda}$ such that $A \subseteq \bigcup_{i=1}^{n} U_{A_i}$.

So, B is compact, then there is a finite subcover $\{U_{B_1}, U_{B_2}, \dots, U_{B_m}\}$ of $\{U_\alpha\}_{\alpha \in \Lambda}$. such that $B \subseteq \bigcup_{i=1}^{m} U_{B_i}$.

So $A \cup B \subseteq \bigcup_{i=1}^{n+m} U_{\alpha_i}$ which is a finite subcover of $A \cup B$.

ii. $A \cap B$ is not compact.

Let (X, τ) be a topological space where

$X = \mathbb{N} \cup \{-1, 0\}$

$\tau = \mathbb{P}(\mathbb{N}) \cup \{H \subseteq X : -1, 0 \in H \wedge H^c \text{ is finite}\}$

Let $A = \mathbb{N} \cup \{0\}$, and $B = \mathbb{N} \cup \{-1\}$.

* It is clear that A is compact, since every open cover of A must contain a set U_α such that $0 \in U_{\alpha_0}$. So $U_{\alpha_0}^c$ is finite set. Let $U_\alpha^c = \{x_1, x_2, \dots, x_n\}$. Then for every element x_i, $\exists U_{\alpha_i}$ such that $x_i \in U_{\alpha_i}$

Hence $A \subseteq U_{\alpha_0} \cup U_{\alpha_1} \cup U_{\alpha_2} \cup \dots \cup U_{\alpha_n}$

and $\{U_{\alpha_0}, U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}\}$ is a finite subcover of A.

* In similar way, B is compact.
* But $A \cap B = \mathbb{N}$ which is not compact. Since the cover $\{\{n\} : n \in \mathbb{N}\}$ of $A \cap B$, has no finite subcover.

Remark:

1. If τ is finite, then (X, τ) is compact for any set X.

   $(\mathbb{R}, I)$, $(\mathbb{N}, I)$, $(\mathbb{N}, \tau)$ where
   $$\tau = \{\mathbb{N}, \phi, E^+, O^+\}$$
   $(\mathbb{R}, \tau)$ where $\tau = \{\mathbb{R}, \phi, Q, Irr\}$

2. If X is finite, then (X, τ) is compact for any τ.

   $X = \{1, 2, 3\}$, $\tau_1 = \{X, \phi, \{1\}\}$
   $$\tau_2 = \{X, \phi, \{1\}, \{2\}, \{1,2\}\}$$
   $$\tau_3 = \{X, \phi, \{1\}, \{1,2\}, \{1,3\}\}$$
   $$\tau_4 = \mathbb{P}(X) = D$$
   then (X, τ_i) is compact

We have seen it previously, compactness is not a hereditary property. In the following theorem, we will put a condition on the subset of a compact space to be compact.

Theorem: A closed subset of a compact space is compact.

Proof: Let (X,τ) be a compact space, and B be a closed subset of X.

To prove, B is compact.

Let $\{U_\alpha : \alpha \in \Lambda\}$ be an open cover of B.

i.e. $U_\alpha \in \tau \ \forall \alpha \in \Lambda$; $B \subseteq \cup\{U_\alpha : \alpha \in \Lambda\}$.

Since B is closed set, then B^c is open.

$$X = B \cup B^c \subseteq \cup\{U_\alpha : \alpha \in \Lambda\} \cup B^c.$$

Thus, $\{U_\alpha : \alpha \in \Lambda\} \cup \{B^c\}$ is an open cover of X, which is compact space.

Then $\exists\, \alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda$;

$$X = \cup\{U_{\alpha_i} : i = 1, 2, \ldots, n\} \cup B^c.$$

Since $B \subseteq X$ and $B \cap B^c = \phi$, then

$$B \subseteq \cup\{U_{\alpha_i} : i = 1, 2, \ldots, n\}.$$

Hence, $\{U_{\alpha_i} : i = 1, 2, \ldots, n\}$ is a finite subcover for B.

Therefore, B is compact.

Theorem: The continuous image of a compact space is compact.

Proof: Let (X,τ) be a compact space, and $f:(X,\tau)\to(Y,\tau')$ be a continuous function.

To prove, $f(X)$ is compact.

Let $\{V_\alpha : \alpha\in\Lambda\}$ be an open cover for $f(X)$.

Then, $V_\alpha\in\tau'$ $\forall\alpha\in\Lambda$ and $f(X)\subseteq\bigcup\{V_\alpha:\alpha\in\Lambda\}$.

Since f is continuous function, then $f^{-1}(V_\alpha)\in\tau$ $\forall\alpha\in\Lambda$ and $f^{-1}(f(X))\subseteq f^{-1}(\bigcup\{V_\alpha:\alpha\in\Lambda\})$.

Implies; $X\subseteq\bigcup\{f^{-1}(V_\alpha):\alpha\in\Lambda\}$.

Thus; $\{f^{-1}(V_\alpha):\alpha\in\Lambda\}$ is an open cover for X.

Since X is compact, then $\exists\,\alpha_1,\alpha_2,\dots,\alpha_n\in\Lambda$ such that $X\subseteq\bigcup\{f^{-1}(V_{\alpha_i}):i=1,2,\dots,n\}$

Implies; $f(X)\subseteq f(\bigcup\{f^{-1}(V_{\alpha_i}):i=1,2,\dots,n\})$

$\Rightarrow f(X)\subseteq\bigcup\{f(f^{-1}(V_{\alpha_i})):i=1,2,\dots,n\}$

$\Rightarrow f(X)\subseteq\bigcup\{V_{\alpha_i}:i=1,2,\dots,n\}$

Hence, $\{V_{\alpha_i}:i=1,2,\dots,n\}$ is a finite subcover of $f(X)$.

Therefore, $f(X)$ is compact.

Corollary: If the Product space $X \times Y$ is compact, then X and Y are compact spaces.

Proof: The Projection function $P_X : X \times Y \to X$; $P_X((x,y)) = x$ is continuous and onto.

and $X \times Y$ is compact, then $P_X(X \times Y)$ is compact.

Since $P_X(X \times Y) = X$, then X is compact.

In similar way, Y is compact.

the Projection function $P_Y : X \times Y \to Y$; $P_Y((x,y)) = y$ is continuous and onto function.

And $X \times Y$ is compact, then $P_Y(X \times Y)$ is compact.

$P_Y(X \times Y) = Y$, then Y is compact.

Corollary: If (X, τ) is compact and $f : (X, \tau) \to (Y, \tau')$ is continuous and onto function, then (Y, τ') is compact.

Proof: Since f is onto, then $f(X) = Y$.

Since f is continuous, and X is compact, then $f(X)$ is compact.

So Y is compact.

Note: The condition, that f is onto, in the above Corollary is necessity.

Q: Prove or disprove; If $f:(X,\tau)\to(Y,\tau')$ is a continuous function and (X,τ) is compact space, then (Y,τ') is compact.

Answer: Disprove: Let $\mathbb{N}$ be the set of all natural numbers.

$\tau=\{A_n : A_n=\{n,n+1,n+2,\dots\}; n\in\mathbb{N}\}\cup\{\phi\}$.

$\tau'=D$ is the discreat topology on $\mathbb{N}$.

It's clear that $(\mathbb{N},\tau)$ is compact space.

Let $f:(\mathbb{N},\tau)\to(\mathbb{N},D)$; $f(x)=1$ $\forall x\in\mathbb{N}$.

Since f is a constant function, then f is continuous function.

$f(\mathbb{N})=\{1\}$ which is compact subset of $(\mathbb{N},D)$. (Since $\{1\}$ is finite set).

But $(\mathbb{N},D)$ is not compact space.

Since $\{\{n\}: n\in\mathbb{N}\}$ is an open cover of $(\mathbb{N},D)$ which has no finite subcover.

$A_1=\{1,2,3,\dots\}=\mathbb{N}$

$A_2=\{2,3,4,\dots\}$

$A_3=\{3,4,5,\dots\}$

$A_4=\{4,5,6,\dots\}$

$C=$ open cover for $\mathbb{N}$

$\therefore A_1\in C$

$\{A_1\}$ finite subcover for C.

Tichonov theorem:

If (X, τ) and (Y, τ') are compact spaces, then $X \times Y$, is compact space.

Example: Is the following product spaces are compact.

① $(\mathbb{R}, \tau_u) \times (\mathbb{N}, \tau_{cof})$?

$(\mathbb{N}, \tau_{cof})$ is compact, but $(\mathbb{R}, \tau_u)$ is not compact.

$\therefore (\mathbb{R}, \tau_u) \times (\mathbb{N}, \tau_{cof})$ is not compact.

② $(\mathbb{R}, \tau_{cof}) \times (\mathbb{N}, \tau_{cof})$?

Since both, $(\mathbb{R}, \tau_{cof})$ and $(\mathbb{N}, \tau_{cof})$ are compact spaces, then the product space $(\mathbb{R}, \tau_{cof}) \times (\mathbb{N}, \tau_{cof})$ is compact.

③ $(\mathbb{N}, \tau_{cof}) \times (\mathbb{N}, \tau_{cof})$?

Since, $(\mathbb{N}, \tau_{cof})$ is compact, then the product space $(\mathbb{N}, \tau_{cof}) \times (\mathbb{N}, \tau_{cof})$ is compact.

④ $(\mathbb{N}, \tau_{cof}) \times (\mathbb{R}, I)$?

Since, $(\mathbb{N}, \tau_{cof})$ and $(\mathbb{R}, I)$ are compact spaces. then $(\mathbb{N}, \tau_{cof}) \times (\mathbb{R}, I)$ is compact.

(5) $(\mathbb{R}, D) \times (\mathbb{R}, I)$?

Since $(\mathbb{R}, D)$ is not compact, then the Product space $(\mathbb{R}, D) \times (\mathbb{R}, I)$ is not compact.

(6) If $X = \{1, 2, 3\}$, then Is $(X, D) \times (\mathbb{R}, I)$ compact. ?

Since X is finite set, then (X, D) is compact space. So, $(\mathbb{R}, I)$ is compact. Then, the Product space $(X, D) \times (\mathbb{R}, I)$ is compact.

(7) What is the condition on the spaces (X, D) and (Y, I) to be the Product space $(X, D) \times (Y, I)$ is compact. ?!

Answer: Must be X finite set.

X is finite set $\Rightarrow$ (X, D) is compact space, and (Y, I) is compact space, since $I = \{Y, \phi\}$. Implies; $(X, D) \times (Y, I)$ compact.

(8) Choose, topology τ on $\mathbb{R}$ and a set X to be $(\mathbb{R}, \boxed{\tau}) \times (\boxed{X}, D)$ is compact.

Answer: Let $\tau = I$ and X be a finite set then $(\mathbb{R}, I)$ and (X, D) are compact spaces. So, $(\mathbb{R}, I) \times (X, D)$ is compact.

Definition: Topological Property

A property P of a space (X, τ) is called a topological property iff every topological space (Y, τ') homeomorphic to (X, τ) also has the same property P.

Theorem: Compactness is a topological property.

Proof: Let $(X, \tau) \cong (Y, \tau')$.

$\therefore \exists f : (X, \tau) \to (Y, \tau')$; f is continuous, 1-1, onto and f^{-1} continuous. (f open)

* Suppose (X, τ) is compact.

$\therefore f(X)$ is compact (the continuous image of compact is compact)

Since f is onto, then $f(X) = Y$.

Hence (Y, τ') is compact.

* Suppose (Y, τ') is compact.

$\therefore f^{-1}(Y)$ is compact (f^{-1} is continuous and the continuous image of compact is compact).

Since $f : (X, \tau) \to (Y, \tau')$ is 1-1 and onto.

$\therefore f^{-1}(Y, \tau') \to (X, \tau)$ is 1-1 and onto.

$f^{-1}(Y) = X$ (f^{-1} is onto)

$\therefore (X, \tau)$ is compact.

Example:

① Is $(\mathbb{R}, I) \cong (\mathbb{R}, \tau_u)$?!

Answer: $(\mathbb{R}, I)$ is compact space and $(\mathbb{R}, \tau_u)$ is not compact space.

then $(\mathbb{R}, I) \not\cong (\mathbb{R}, \tau_u)$

② Is $(\mathbb{N}, \tau_{cof}) \cong (\mathbb{N}, D)$?!

Answer: $(\mathbb{N}, \tau_{cof})$ is compact space and $(\mathbb{N}, D)$ is not compact space.

then $(\mathbb{N}, \tau_{cof}) \not\cong (\mathbb{N}, D)$

③ If $(X, \tau) \cong (\mathbb{N}, \tau_{cof})$. Is (X, τ) compact?

Answer: Since $(\mathbb{N}, \tau_{cof})$ compact space and $(X, \tau) \cong (\mathbb{N}, \tau_{cof})$. then (X, τ) is compact.

④ If $(\mathbb{R}, I) \cong (Y, \tau)$. Is (Y, τ) compact ?!

Answer: Since $(\mathbb{R}, I)$ is compact space and $(\mathbb{R}, I) \cong (Y, \tau)$. Then (Y, τ) is compact.

⑤ If $(\mathbb{N}, D) \cong (X, \tau)$, is (X, τ) compact?

Answer: Since $(\mathbb{N}, D)$ is not compact, and $(\mathbb{N}, D) \cong (X, \tau)$, then (X, τ) is not compact.

Definition: Finite intersection Property (f.i.p.).

Let $\{A_\alpha : \alpha \in \Lambda\}$ be a family of sets. We call this family satisfy the finite intersection Property and denoted by (f.i.p.) if the intersection of any finite number of elements of this family is nonempty.

i.e.
$\{A_\alpha : \alpha \in \Lambda\}$ has f.i.p. $\Leftrightarrow \bigcap_{i=1}^{n} A_{\alpha_i} \neq \phi \quad \forall n \in \mathbb{N}$.

Example: The family $\{A_n : n \in \mathbb{N}\}$;

$A_n = (-\frac{1}{n}, \frac{1}{n})$

The intersection of any finite numbers of elements of this family is nonempty.
So $\{A_n : n \in \mathbb{N}\}$ has a (f.i.p.).

(Remark): Example Prove or disprove;
If $\{A_\alpha : \alpha \in \Lambda\}$ has a f.i.p., then $\bigcap \{A_\alpha : \alpha \in \Lambda\} \neq \phi$.

Answer: Disprove: Let $A_n = \{n, n+1, n+2, \dots\} \; \forall n \in \mathbb{N}$
then the family $\{A_n : n \in \mathbb{N}\}$ has a f.i.p.
But $\bigcap \{A_n : n \in \mathbb{N}\} = \phi$

$A_1 = \{1, 2, 3, \dots\} = \mathbb{N}$
$A_2 = \{2, 3, 4, \dots\}$
$A_3 = \{3, 4, 5, \dots\}$
$A_4 = \{4, 5, 6, \dots\}$
$A_5 = \{5, 6, 7, 8, \dots\}$

Theorem: A space (X,τ) is compact $\boxed{\text{iff}}$ every family of closed subsets of X has (f.i.p.) being intersection nonempty.

Proof: $\Rightarrow$: Suppose X is compact and $\{B_\alpha : \alpha \in \Lambda\}$ be a family of closed subsets of X, has a (f.i.p.), to prove $\bigcap\{B_\alpha : \alpha\in\Lambda\} \neq \phi$.

Suppose $\bigcap\{B_\alpha : \alpha \in \Lambda\} = \phi$

$$\Rightarrow \left(\bigcap\{B_\alpha : \alpha\in\Lambda\}\right)^c = \phi^c$$

$$\Rightarrow \bigcup\{B_\alpha^c : \alpha\in\Lambda\} = X$$

$\therefore$ $\{B_\alpha^c : \alpha\in\Lambda\}$ is an open cover for X,

Since X is compact, then $\exists\, \alpha_1, \alpha_2, \dots, \alpha_n \in \Lambda$ such that $X = \bigcup\{B_{\alpha_i}^c : i = 1, 2, \dots, n\}$

$$\Rightarrow X^c = \left(\bigcup\{B_{\alpha_i}^c : i = 1, 2, \dots, n\}\right)^c \qquad B^{c^c} = B$$

$$\Rightarrow \phi = \bigcap\{B_{\alpha_i} : i = 1, 2, \dots, n\}$$

which is a contraduction, since the family $\{B_\alpha : \alpha\in\Lambda\}$ has (f.i.p.).

$\therefore$ $\bigcap\{B_\alpha : \alpha\in\Lambda\} \neq \phi$.

$\Leftarrow$ Suppose every family of closed subsets of X has (f.i.p.) being intersection nonemty. To prove X is compact.

Suppose X is not compact.

$\therefore \exists \{U_\alpha : \alpha \in \Lambda\}$ an open cover for X which has no finite subcover.

i.e. $\boxed{X = \bigcup\{U_\alpha : \alpha \in \Lambda\}}$ and $\boxed{X \neq \bigcup\{U_{\alpha_i} : i = 1, 2, \ldots, n\} \ \forall n \in \mathbb{N}.}$

$\Rightarrow \boxed{X^c = \left(\bigcup\{U_\alpha : \alpha \in \Lambda\}\right)^c}$ and $\boxed{X^c \neq \left(\bigcup\{U_{\alpha_i} : i = 1, 2, \ldots, n\}\right)^c \ \forall n \in \mathbb{N}}$

$\Rightarrow \boxed{\phi = \bigcap\{U_\alpha^c : \alpha \in \Lambda\}}$ and $\boxed{\phi \neq \bigcap\{U_{\alpha_i}^c : i = 1, 2, \ldots, n\} \ \forall n \in \mathbb{N}}$

$\Rightarrow \{U_\alpha^c : \alpha \in \Lambda\}$ be a family of closed subsets of X, which has (f.i.p.), but $\bigcap\{U_\alpha^c : \alpha \in \Lambda\} = \phi$. That is a contraduction.

$\therefore X$ is compact.

Definition A space (X, τ) is called a Lindelöf space if every open cover of X has a countable subcover.

Remark: Prove or disprove the following statements.

i. Every compact space is Lindelöf space.

ii. Every Lindelöf space is compact space.

i- Proof: Let (X, τ) be a compact space.

T.P. X is a Lindelöf space.

Let $C = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover for X.

Since X is compact, then $\exists\, \alpha_1, \alpha_2, \dots, \alpha_n \in \Lambda$ such that $X = \bigcup_{i=1}^{n} U_{\alpha_i}$.

So, $\{U_{\alpha_i} : i = 1, 2, \dots, n\}$ is a countable subcover of C.

Hence, (X, τ) is Lindelöf space.

ii: disprove.

Consider the topological space $(\mathbb{N}, \tau)$, where $\tau = \{U_n : U_n = \{1, 2, 3, \dots, n\}, n \in \mathbb{N}\} \cup \{\mathbb{N}, \phi\}$

It is clear that $(\mathbb{N}, \tau)$ is not compact space. Since the open cover $C = \{U_n : n \in \mathbb{N}\}$ has no finite subcover.

In other hand; Every open cover of $\mathbb{N}$ has a countable subcover (Since $\mathbb{N}$ is countable set). Thus $(\mathbb{N}, \tau)$ is Lindelöf space.

Remark: Prove or disprove the following statements.

i- Every finite space is Lindelöf space.

ii- Every countable space is Lindelöf space.

Proof: i. Since every finite space is compact, and every compact space is Lindelöf.

ii- Since every open cover is countable subcover of itsself.

Remark: ① $(\mathbb{R}, \tau_u)$ is a Lindelöf space.

Proof: Let $C = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover for $\mathbb{R}$. Then U_α is a union of open intervals, $\forall \alpha \in \Lambda$. So, $\forall \alpha \in \Lambda \; \exists r \in \mathbb{Q} \; ; r \in U_\alpha$, (Since, $\mathbb{R}$ is dense set). Now, we will rename the sets U_α by $U_{\alpha r}$. Then $\{U_{\alpha r} : r \in \mathbb{Q}\}$ is a countable subcover of C, since $\mathbb{Q}$ is countable set. Hence, $(\mathbb{R}, \tau_u)$ is a Lindelöf space.

② $(\mathbb{N}, \tau)$, $(\mathbb{Z}, \tau)$ and $(\mathbb{Q}, \tau)$ are a Lindelöf spaces for any topology τ. Since $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are countable sets.

③ Prove or disprove, If (X, τ) is a Lindelöf space then X is countable.

disprove: Since $(\mathbb{R}, \tau_u)$ is a Lindelöf space, but $\mathbb{R}$ is uncountable.

④ The space (X, I); $I = \{X, \phi\}$, is a Lindelöf space. Since it is compact space. It's true for any set X.

Remark: A space (X, D) is a Lindelöf space if and only if X is Countable.

Note: ① The spaces $(\mathbb{R}, I)$, $(\mathbb{Q}, I)$, $(\mathbb{N}, I)$

(X, I)

and (Irr, I) are Lindelöf spaces. Since, they are Compact spaces.

② $(\mathbb{N}, \overset{\tau}{D})$, $(\mathbb{Z}, \overset{\tau}{D})$ and $(\mathbb{Q}, \overset{\tau}{D})$ are Lindelöf spaces. Since $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are Countable sets.

③ (X, τ_{cof}) is Lindelöf space, Since it is Compact.
$(\mathbb{N}, \tau_{cof})$, $(\mathbb{R}, \tau_{cof})$, $(\mathbb{Q}, \tau_{cof})$ and (Irr, τ_{cof}) are Lindelöf spaces.

④ Prove or disprove; The Lindelöfness is a hereditary Property.

disprove: The space $(\mathbb{R}, \tau_u)$ is a Lindelöf space.

$\boxed{\tau_W = ?!}$

let $W = Irr \subseteq \mathbb{R}$,
then (W, τ_W) is not Lindelöf space.

⑤ Prove or disprove, If (W, τ_W) is Lindelöf subspace of (X, τ), then (X, τ) is Lindelöf.

disprove: $(\mathbb{R}, D)$ is not Lindelöf.
let $W = \mathbb{N}$, then (W, τ_W) is lindelöf
$(W, \tau_W) = (\mathbb{N}, D)$.

Theorem: A closed subset of a Lindelöf space is Lindelöf

Theorem The continuous image of a Lindelöf space is Lindelöf.

Theorem: Lindelöfness is topological property.

Corollary: (X,τ) and (Y,τ') are lindelöf spaces if and only if $(X\times Y, \tau_{X\times Y})$ is Lindelöf.

The product space of two Lindelöf spaces is Lindelöf

What could be certain about the space X in the following statements:

① $X = \mathbb{N}$ as a subspace of $(\mathbb{R}, D)$.

Since $\mathbb{N}$ is countable set, then (X, τ_X) is Lindelöf space.

② $X = [0,1]$ as a subspace of $(\mathbb{R}, \tau_u)$.

Since X is closed and bounded $\Rightarrow$ X is compact.

Since $(\mathbb{R}, \tau_u)$ is Lindelöf space and X is closed subset of $(\mathbb{R}, \tau_u)$, then (X, τ_X) is Lindelöf space, [the closed subspace of Lindelöf space is Lindelöf]

③ $f : (\mathbb{R}, \tau_u) \longrightarrow (X, \tau)$ continuous onto function

Since $(\mathbb{R}, \tau_u)$ is Lindelöf space, then (X, τ) is Lindelöf space [the continuous image of a Lindelöf space is Lindelöf]

④ $f:(\mathbb{R},\tau_{cof})\longrightarrow(X,\tau)$ is continuous onto function

Since $(\mathbb{R},\tau_{cof})$ is compact space, then (X,τ) is compact [the continuous image of a compact space is compact].

So, since every compact space is Lindelöf then (X,τ) is Lindelöf space.

⑤ $(X,\tau)=(\mathbb{N},D)\times(\mathbb{N},\tau_{cof})$

Since both $(\mathbb{N},D)$ and $(\mathbb{N},\tau_{cof})$ are Lindelöf spaces, then (X,τ) is Lindelöf [Lindelöfness is topological property]

⑥ $(X,\tau)=(\mathbb{R},\tau_{cof})\times(\mathbb{N},D)$ [similar as (5)]

⑦ $(X,\tau)=(\mathbb{R},\tau_{cof})\times(\mathbb{N},I)$

Since both $(\mathbb{R},\tau_{cof})$ and $(\mathbb{N},I)$ are compact spaces, then (X,τ) is compact [compactness is topological property]

⑧ $(X,\tau)=(\mathbb{R},I)\times(\mathbb{N},\tau_{cof})$ [similar as (7)]

(9) If $(X,\tau)\times(\mathbb{R},I)$ is Lindelöf ?!

Since $(\mathbb{R},I)$ is Lindelöf space and so the Product space, then (X,τ) is Lindelöf space.

[Lindelöfness is topological space]

(10) If $(X,\tau)\times(\mathbb{R},\tau_{cof})$ is Compact ?

Since $(\mathbb{R},\tau_{cof})$ and the Product space are Compact spaces, then (X,τ) is Compact.

[Compactness is topological Property]

Theorem:

① The Continuous image of Compact space is compact.

② the Continuous image of Lindelöf space is Lindelöf.

Exercise:

① If $f:(X,I)\to(Y,\tau)$ is Continuous, onto function, what about (Y,τ)

Answer: Since (X,I) is Compact space, then $f(X)$ is compact [the Continuous image of a Compact space is compact]. And since f is onto, then $f(X)=Y$. Therefore, (Y,τ) is compact space.

So, since every compact space is Lindelöf, then both (Y,τ) and (X,I) are Lindelöf spaces.

* $f:(N,I)\to(Y,\tau)$, Continuous, Onto function. Then (Y,τ) is Compact and Lindelöf.

* $f:(\mathbb{R},I)\to(Y,\tau)$, Continuous, onto function. Then (Y,τ) is Compact and Lindelöf space.

② If $f:(X,\tau_{cof}) \longrightarrow (Y,\tau)$ continuous, onto function, what about (Y,τ) ?!

Since (X,τ_{cof}) is compact and f is continuous function, then $f(X)$ is compact. [the continuous image of compact space is compact.].
Since, f is onto, then $f(X)=Y$, So (Y,τ) is compact.
Every compact space is Lindelöf space, then both (X,τ_{cof}) and (Y,τ) are Lindelöf spaces.

* $f:(X,\tau_{cof}) \longrightarrow (Y,\tau)$ continuous, onto function, then (Y,τ) is compact and Lindelöf space.
* $f(\mathbb{R},\tau_{cof}) \rightarrow (Y,\tau)$ continuous, onto function, then (Y,τ) is compact and Lindelöf space.

(3) $f:(\mathbb{R},\tau_u)\longrightarrow(Y,\tau)$ is continous, onto function, what about (Y,τ) ?!

Since $(\mathbb{R},\tau_u)$ is Lindelöf space, then $f(\mathbb{R})$ is Lindelöf [the continous image of lindelöf space is Lindelöf].

Since f is onto, then $f(\mathbb{R})=Y$.

therefore, (Y,τ) is Lindelöf space.

(4) If $(X\times Y,\tau_{X\times Y})$ is Lindelöf space, what about (X,τ) and (Y,τ').?!

<u>Answer:</u> Since $P_X:(X\times Y,\tau_{X\times Y})\longrightarrow(X,\tau)$; $P_X((x,y))=x$ is continuous, onto function, then (X,τ) is Lindelöf [the continuous image of lindelöf is Lindelöf].

Similar: $P_Y:(X\times Y,\tau_{X\times Y})\rightarrow(Y,\tau')$; $P_Y((x,y))=y$ is continuous, onto function, then (Y,τ') is Lindelöf

Example: ① (X, I) is lindelöf space, since (X, I) is compact and every compact space is lindelöf.

② (X, D) is Lindelöf space $\Leftrightarrow$ X is countable set.

$\circledast$ $(\mathbb{R}, D)$ is not lindelöf, since $\mathbb{R}$ is uncountable

③ $(\mathbb{N}, D)$ is Lindelöf space, but not compact space.

④ $(\mathbb{R}, I)$, $(\mathbb{Q}, I)$, (Irr, I), $(\mathbb{Z}, I)$ and $(\mathbb{N}, I)$ are compact spaces and so, are Lindelöf spaces.

⑤ (X, τ_{cof}), $(\mathbb{R}, \tau_{cof})$, $(\mathbb{Q}, \tau_{cof})$, (Irr, τ_{cof}) and $(\mathbb{N}, \tau_{cof})$ are compact spaces, so the are Lindelöf spaces.

⑥ If X is finite set, then (X, τ) is compact space. Implies, (X, τ) is Lindelöf space.

Example:

If $f:(\mathbb{R},\tau_u)\times(\mathbb{N},\tau_{cof})\longrightarrow(X,\tau)$ is continuous, onto function, what about (X,τ)?!

Answer: Since $(\mathbb{R},\tau_u)$ and $(\mathbb{N},\tau_{cof})$ are Lindelöf spaces, then $(\mathbb{R},\tau_u)\times(\mathbb{N},\tau_{cof})$ is Lindelöf. And since f is continuous, onto function, then (X,τ) is Lindelöf space [the continuous image of Lindelöf space is Lindelöf].

If $f:(\mathbb{R},\tau_u)\times(\mathbb{R},I)\rightarrow(Y,\tau)$ is continuous, onto function, then what about (Y,τ).?

Since, both $(\mathbb{R},\tau_u)$ and $(\mathbb{R},I)$ are Lindelöf spaces, then $(\mathbb{R},\tau_u)\times(\mathbb{R},I)$ is Lindelöf space. And since the continuous image of Lindelöf space is Lindelöf, then (Y,τ) is Lindelöf.

(*) If $f:(\mathbb{N},\tau_{cof})\times(\mathbb{N},D)\to(X,\tau)$ is continuous, onto function, then what about (X,τ) ?!

Answer: Since $(\mathbb{N},\tau_{cof})$ and $(\mathbb{N},D)$ are Lindelöf spaces, then $(\mathbb{N},\tau_{cof})\times(\mathbb{N},D)$ is Lindelöf. And since the continuous image of a Lindelöf space is Lindelöf, then (X,τ) is Lindelöf.

(*) If $f:(\mathbb{N},D)\times(\mathbb{N},I)\to(X,\tau)$ is continuous, onto function, then (X,τ) is Lindelöf space.

(*) If $f:(\mathbb{R},D)\times(\mathbb{R},I)\to(X,\tau)$ is continuous, onto function, then (X,τ) is Lindelöf space. (Prove or disprove)

Disprove: Since $(\mathbb{R},D)$ not Lindelöf space where the open cover $\{\{x\}:x\in\mathbb{R}\}$ has no countable subcover.