Chapter two

Continuity and Derived Topological Spaces.

Definition: Let (X,τ) and (Y,τ') are two topological spaces and $f:(X,\tau)\to(Y,\tau')$ be a function. Then, the function f is called continuous if the inverse image of every open set in (Y,τ') is an open set in (X,τ).

i.e.

$$(f:(X,\tau)\to(Y,\tau') \text{ is continuous}) \Longleftrightarrow (f^{-1}(V)\in\tau \ \forall V\in\tau')$$

And f is discontinuous, if $\exists V\in\tau'$, such that $f^{-1}(V)\notin\tau$.

Example:

1) $f:(X,D)\to(Y,\tau')$ is continuous. Since, $\forall V\in\tau' \Rightarrow f^{-1}(V)\subseteq X \Rightarrow f^{-1}(V)\in D=P(X)$.

2) $f:(X,\tau)\to(Y,I)$ is continuous. Since,
If $V\in\tau'=I \Rightarrow V=Y$ or $V=\phi$
and $f^{-1}(Y)=X\in\tau$.
$f^{-1}(\phi)=\phi\in\tau$.

3) $f:(X,\tau)\to(X,\tau)\ \ni f(x)=x$ is continuous.
Since, $\forall V\in\tau \Rightarrow f^{-1}(V)=V\in\tau$.

4) $f:(X,\tau)\longrightarrow(Y,\tau')$; $f(x)=y_0$; $y_0\in Y$, $\forall x\in X$.
is Continuous. Since, if $V\in\tau'$
then; either $y_0\in V$ or $y_0\notin V$.
If $y_0\in V\Rightarrow f^{-1}(V)=X\in\tau$.
If $y_0\notin V\Rightarrow f^{-1}(V)=\phi\in\tau$.

5) $f:(X,D)\longrightarrow(Y,D)$ is continuous. check!!

Example: Let $X=\{1,2,3\}$, $\tau=\{X,\phi,\{1\},\{2\},\{1,2\}\}$
$Y=\{a,b\}$ and $\tau'=\{Y,\phi,\{a\}\}$.

① $f:(X,\tau)\longrightarrow(Y,\tau')$; $f(1)=f(2)=a$
$f(3)=b$

then, f is Continuous, Since
$Y\in\tau'\Rightarrow f^{-1}(Y)=X\in\tau$
$\phi\in\tau'\Rightarrow f^{-1}(\phi)=\phi\in\tau$
$\{a\}\in\tau'\Rightarrow f^{-1}(\{a\})=\{1,2\}\in\tau$.

② $g:(X,\tau)\longrightarrow(Y,\tau')$; $g(1)=b$
$g(2)=g(3)=a$

then, $\{a\}\in\tau'$ and $g^{-1}(\{a\})=\{2,3\}\notin\tau$.
So, g is discontinuous.

③ $h:(X,\tau)\longrightarrow(Y,\tau')$; $h(1)=h(2)=h(3)=b$.
$Y\in\tau'\longrightarrow h^{-1}(Y)=X\in\tau$.
$\phi\in\tau'\longrightarrow h^{-1}(\phi)=\phi\in\tau$.
$\{a\}\in\tau'\longrightarrow h^{-1}(\{a\})=\phi\in\tau$.
$\therefore$ h is Continuous.

Ex: $f:(\mathbb{R}, I) \longrightarrow (\mathbb{R}, \tau_u)$; $f(x) = x$.

f is discontinuous, Since $(0,1) \in \tau_u$ but,
$f^{-1}((0,1)) = (0,1) \notin I$

Ex: $f:(\mathbb{N}, I) \longrightarrow (\mathbb{N}, \tau_{cof})$; $f(x) = x$.
f is discontinuous, Since $V = \mathbb{N} - \{1\} \in \tau_{cof}$
but $f^{-1}(V) = V \notin I$

Ex: $f:(X, I) \longrightarrow (Y, I)$ is Continuous.
Since $Y \in I \longrightarrow f^{-1}(Y) = X \in I$.
$\phi \in I \longrightarrow f^{-1}(\phi) = \phi \in I$.

Ex: $f:(\mathbb{R}, \tau_u) \longrightarrow (\mathbb{R}, D)$; $f(x) = x$
f is discontinuous, Since $\{1\} \in D$ but
$f^{-1}(\{1\}) = \{1\} \notin \tau_u$.

Ex: $f:(X, D) \longrightarrow (Y, D)$ is Continuous.
since $\forall V \in D \Rightarrow f^{-1}(V) \subseteq X$
$\Rightarrow f^{-1}(V) \in D$.

Ex: $f:(\mathbb{N}, \tau_{cof}) \longrightarrow (\mathbb{N}, D)$; $f(x) = x$.
f is discontinuous. Since $\{1\} \in D$ but
$f^{-1}(\{1\}) = \{1\} \notin \tau_{cof}$.

Ex: $f:(\mathbb{R},\tau_u)\to(\mathbb{R},I)$ is continuous.

Since, If $V\in I$, then $V=\mathbb{R}$ or $V=\phi$.

So $f^{-1}(\mathbb{R})=\mathbb{R}\in\tau_u$.

and $f^{-1}(\phi)=\phi\in\tau_u$.

Similar: $f:(\mathbb{N},\tau_{cof})\to(\mathbb{N},I)$ is continuous.

Ex: $f:(\mathbb{R},D)\to(\mathbb{R},\tau_u)$; $f(x)=x$.

f is continuous; since $\forall\,(a,b)\in\tau_u$, then $f^{-1}((a,b))=(a,b)\in D$.

Similar: $f:(\mathbb{N},D)\to(\mathbb{N},\tau_{cof})$; $f(x)=x$.

f is continuous.

Theorem:- If $f:(X,\tau)\to(Y,\tau')$ and $g:(Y,\tau')\to(Z,\tau'')$ are continuous functions, then the composition $g\circ f:(X,\tau)\to(Z,\tau'')$ is continuous.

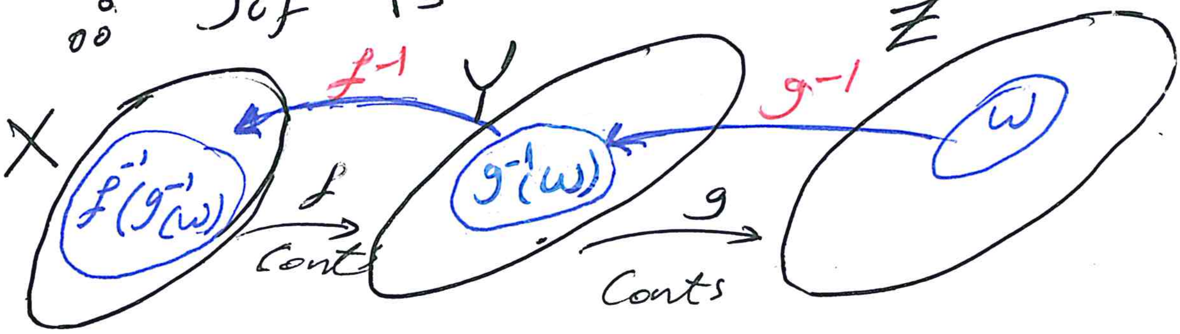
Proof: Let $W\in\tau'' \Rightarrow g^{-1}(W)\subseteq Y$ and $g^{-1}(W)\in\tau'$

$\to f^{-1}(g^{-1}(W))\subseteq X$ and $f^{-1}(g^{-1}(W))\in\tau$

$\Rightarrow (f^{-1}\circ g^{-1})(W)\in\tau$

$\Rightarrow (g\circ f)^{-1}(W)\in\tau$

$\therefore$ $g\circ f$ is continuous function.



Theorem: Let $f:(X,\tau) \to (Y,\tau')$ be a function. Then f is continuous iff satisfy one of the following Properties:

1) $f^{-1}(F) \in \mathcal{F} \ \forall F \in \mathcal{F}'$; $\mathcal{F}$ is a family of all closed sets in X and $\mathcal{F}'$ is a family of all closed sets in Y.

2) $f^{-1}(B) \in \tau \ \forall B \in \mathcal{B}'$; $\mathcal{B}'$ is a base for τ'.

3) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$; $B \subseteq Y$.

4) $f^{-1}(B^{\circ}) \subseteq (f^{-1}(B))^{\circ}$; $B \subseteq Y$.

Proof: 1) f is Continuous $\Leftrightarrow f^{-1}(F) \in \mathcal{F} \ \forall F \in \mathcal{F}'$.

$\Rightarrow$: Suppose f is Continuous.

Let $F \in \mathcal{F}'$ (T.P. $f^{-1}(F) \in \mathcal{F}$)

$F \in \mathcal{F}' \Rightarrow (Y-F) \in \tau'$

$\Rightarrow f^{-1}(Y-F) \in \tau$ since f is Cont^s.

$\Rightarrow f^{-1}(Y) - f^{-1}(F) \in \tau$.

$\Rightarrow X - f^{-1}(F) \in \tau$.

$\Rightarrow f^{-1}(F) \in \mathcal{F}$.

$\Leftarrow$: Suppose $f^{-1}(F) \in \mathcal{F} \ \forall F \in \mathcal{F}'$. T.P. f is Cont^s.

Let $V \in \tau' \Rightarrow Y - V \in \mathcal{F}'$.

$\Rightarrow f^{-1}(Y-V) \in \mathcal{F}$.

$\Rightarrow f^{-1}(Y) - f^{-1}(V) \in \mathcal{F} \Rightarrow X - f^{-1}(V) \in \mathcal{F}$

$\Rightarrow f^{-1}(V) \in \tau$. $\therefore f$ is Continuous function.

2) f is continuous $\Leftrightarrow f^{-1}(B) \in \tau \ \forall B \in \mathcal{B}'$;
$\mathcal{B}'$ is a base for τ'.

$\Rightarrow$: Suppose f is continuous function and let $B \in \mathcal{B}'$, where $\mathcal{B}'$ is a base for τ'.
To prove $f^{-1}(B) \in \tau$.
Since $B \in \mathcal{B}' \Rightarrow B \in \tau'$ (Since $\mathcal{B}' \subseteq \tau'$)
$\Rightarrow f^{-1}(B) \in \tau$ (Since f is cont's)

$\Leftarrow$: Suppose $f^{-1}(B) \in \tau \ \forall B \in \mathcal{B}'$. T.p. f is cont's.
Let $V \in \tau' \Rightarrow V = \bigcup_i B_i$; $B_i \in \mathcal{B}'$
$\Rightarrow f^{-1}(V) = f^{-1}(\bigcup_i B_i) = \bigcup_i (f^{-1}(B_i))$.
Since $B_i \in \mathcal{B}' \ \forall i$, then $f^{-1}(B_i) \in \tau \ \forall i$.
So, $\bigcup_i f^{-1}(B_i) \in \tau$.
$\Rightarrow f^{-1}(V) \in \tau \ \forall V \in \tau'$.
$\therefore f$ is continuous.

3) f is Continuous $\Leftrightarrow \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$; $B \subseteq Y$.

$\Rightarrow$: Suppose f is continuous function and $B \subseteq Y$.

$$B \subseteq \overline{B} \Rightarrow f^{-1}(B) \subseteq f^{-1}(\overline{B}).$$

Since $\overline{B}$ is closed set in Y and f is continuous, then (by 1) $f^{-1}(\overline{B})$ is closed set in X, So $f^{-1}(\overline{B}) \in \mathcal{F}'$.

Now, $\overline{f^{-1}(B)} = \bigcap \{F \in \mathcal{F}' : f^{-1}(B) \subseteq F\}$

$$\subseteq f^{-1}(\overline{B}).$$

$\Leftarrow$: Suppose $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) \; \forall B \subseteq Y$.

To prove f is continuous.

Let $F \in \mathcal{F}'$, to prove $f^{-1}(F) \in \mathcal{F}'$.

i.e. to prove $\overline{f^{-1}(F)} = f^{-1}(F)$.

clearly ; $f^{-1}(F) \subseteq \overline{f^{-1}(F)}$ ----(1). $(A \subseteq \overline{A})$

Since $F \in \mathcal{F}' \Rightarrow F = \overline{F}$

$$\left.\begin{array}{l} \Rightarrow f^{-1}(F) = f^{-1}(\overline{F}) \\ \text{By hypotheses, } \overline{f^{-1}(F)} \subseteq f^{-1}(\overline{F}) \end{array}\right\}$$

$$\Rightarrow \overline{f^{-1}(F)} \subseteq f^{-1}(F) \text{ ---- } (2).$$

From (1) and (2), we get $\overline{f^{-1}(F)} = f^{-1}(F)$.

$\therefore f^{-1}(F) \in \mathcal{F}'$ whenever $F \in \mathcal{F}'$.

So f is Continuous.

4) f is continuous $\Leftrightarrow f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ \ \forall B \subseteq Y$.

$\Rightarrow$: Suppose f is continuous and $B \subseteq Y$.

$B^\circ \subseteq B \Rightarrow f^{-1}(B^\circ) \subseteq f^{-1}(B)$ --- (1)

Since $B^\circ \in \tau'$ and f is continuous,

then $f^{-1}(B^\circ) \in \tau$ --- (2)

$(f^{-1}(B))^\circ = \bigcup\{U \in \tau : U \subseteq f^{-1}(B)\}$

Hence, $f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$.

$\Leftarrow$: Suppose $f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ \ \forall B \subseteq Y$.

To prove f is continuous.

Let $V \in \tau'$. To prove $f^{-1}(V) \in \tau$.

i.e. To prove $(f^{-1}(V))^\circ = f^{-1}(V)$. $(A \in \tau \Leftrightarrow A = A^\circ)$

Clearly; $(f^{-1}(V))^\circ \subseteq f^{-1}(V)$ --- (i). $(A^\circ \subseteq A)$

Now, Since $V \in \tau' \Rightarrow V^\circ = V \Rightarrow \boxed{f^{-1}(V^\circ) = f^{-1}(V)}$

Since $V \subseteq Y \Rightarrow \boxed{f^{-1}(V^\circ) \subseteq (f^{-1}(V))^\circ}$ (by hypotheses).

$\therefore f^{-1}(V) \subseteq (f^{-1}(V))^\circ$ --- (ii)

From (i) and (ii), then

$$(f^{-1}(V))^\circ = f^{-1}(V).$$

Hence, $f^{-1}(V) \in \tau$ and f is continuous.

Definition: Let $f:(X,\tau)\to(Y,\tau')$ be a function, then f is called:

1) Open function if the direct image for any open set in X is an open set in Y.

i.e. f is open function $\Leftrightarrow$

$$(\forall V\in\tau \Rightarrow f(V)\in\tau')$$

2) Closed function if the direct image of any closed set in X is a closed set in Y.

i.e. f is closed function $\Leftrightarrow$

$$(\forall F\in\mathcal{F} \Rightarrow f(F)\in\mathcal{F}')$$

Remark: Let $f:(X,\tau)\to(Y,\tau')$ be a function.

1) f is not open function if

$\exists V\in\tau$ but $f(V)\notin\tau'$

2) f is not closed function if

$\exists F\in\mathcal{F}$ but $f(F)\notin\mathcal{F}'$.

The three concepts, continuous function, open function and closed function are independent. As the following examples shows:

① Continuous, open, closed.

$f:(X,\tau) \longrightarrow (X,\tau)$; $f(x)=x \quad \forall x \in X$.

$\forall V \in \tau \Rightarrow f^{-1}(V)=V \in \tau$. $\therefore f$ is continuous.

$\forall V \in \tau \Rightarrow f(V)=V \in \tau$. $\therefore f$ is open.

$\forall F \in \mathcal{F} \Rightarrow f(F)=F \in \mathcal{F}$. $\therefore f$ is closed.

② Continuous, open, not closed.

$X=\{1,2,3\}$ $\quad \tau=\{X,\phi,\{1\}\}$

$Y=\{a,b,c\}$ $\quad \tau'=\{Y,\phi,\{a\}\}$

$f:(X,\tau) \longrightarrow (Y,\tau')$; $f(1)=f(2)=f(3)=a$

⊛ Continuous:

$Y \in \tau' \Rightarrow f^{-1}(Y)=X \in \tau$

$\phi \in \tau' \Rightarrow f^{-1}(\phi)=\phi \in \tau$

$\{a\} \in \tau' \Rightarrow f^{-1}(\{a\})=X \in \tau$

⊛ open:

$X \in \tau \Rightarrow f(X)=\{a\} \in \tau'$

$\phi \in \tau \Rightarrow f(\phi)=\phi \in \tau'$

$\{1\} \in \tau \Rightarrow f(\{1\})=\{a\} \in \tau'$

⊛ not closed.

$\{2,3\} \in \mathcal{F}$ but $f(\{2,3\})=\{a\} \notin \mathcal{F}'$

(3) Continuous, not open, closed

$X = \{1, 2, 3\}$ $\tau = \{X, \phi, \{1\}\}$

$Y = \{a, b, c\}$ $\tau' = \{Y, \phi, \{a, b\}\}$

$f(1) = f(2) = f(3) = c$; $f: (X, \tau) \longrightarrow (Y, \tau')$.

* Continuous:

$Y \in \tau' \Rightarrow f^{-1}(Y) = X \in \tau$.

$\phi \in \tau' \Rightarrow f^{-1}(\phi) = \phi \in \tau$.

$\{a, b\} \in \tau' \Rightarrow f^{-1}(\{a, b\}) = \phi \in \tau$.

* not open:

$\{1\} \in \tau$ but $f(\{1\}) = \{c\} \notin \tau'$.

* Closed.

$X \in \mathcal{F} \Rightarrow f(X) = \{c\} \in \mathcal{F}'$.

$\phi \in \mathcal{F} \Rightarrow f(\phi) = \phi \in \mathcal{F}'$.

$\{2, 3\} \in \mathcal{F} \Rightarrow f(\{2, 3\}) = \{c\} \in \mathcal{F}'$.

(4) Continuous, not open, not closed.

Let X has more than One element.

$f: (X, D) \longrightarrow (X, I)$; $f(x) = x$ $\forall x \in X$

It's clear that f is Continuous, not open and not closed function.

$\{x\} \in \tau = D$ but $f(\{x\}) = \{x\} \notin I$.

and $\{x\} \in \mathcal{F} = D$ but $f(\{x\}) = \{x\} \notin \mathcal{F}' = I$.

(5) discontinuous, open, closed.

Let X has more than element.

$f:(X,I) \longrightarrow (X,D)$; $f(x)=x$.

f is discontinuous, open and closed.

(6) discontinuous, Open, not closed.

$X=\{1,2,3\}$, $\tau=\{X,\phi,\{1\}\}$

$Y=\{a,b,c\}$ $\tau'=\{Y,\phi,\{a\},\{a,b\}\}$;

$f:(X,\tau)\longrightarrow(Y,\tau')$; $f(1)=f(2)=a$ and

$f(3)=b$.

(7) discontinuous, not open, closed

$X=\{1,2,3\}$, $\tau=\{X,\phi,\{1\}\}$

$Y=\{a,b,c\}$ $\tau'=\{Y,\phi,\{c\},\{b,c\}$

$f(1)=f(2)=a$

$f(3)=b$

(8) discontinuous, not open, not closed.

$X=\{1,2,3\}$ $\tau=\{X,\phi,\{1\},\{2\},\{1,2\}\}$

$Y=\{a,b,c\}$ $\tau'=\{Y,\phi,\{c\}\}$.

$f(1)=a$, $f(2)=b$, $f(3)=c$.

Remark: If $f:(X,\tau)\longrightarrow(Y,\tau')$ is bijective function, then
(f is open function $\Leftrightarrow$ f is closed function).

* bijective function means injective function and surjective function.

Remark: We can find a countinous function which is bijective but $f:(X,\tau)\to(Y,\tau)$ which is bijective but f^{-1} is not continuous.

For example:

$f:(\mathbb{R},D)\longrightarrow(\mathbb{R},I)$; $f(x)=x$

It is clear that f is continuous, bijective function, but $f^{-1}:(\mathbb{R},I)\to(\mathbb{R},D)$ is not continuous.

$\{1\}\in D$; $\{1\}$ open in $(\mathbb{R},D)$

but $f^{-1}(\{1\})=\{1\}\notin I$, not open in $(\mathbb{R},I)$

Definition: A function $f:(X,\tau)\to(Y,\tau')$ is called homeomorphism if its continuous, injective, surjective and f^{-1} continuous.*
(f open function)

Definition: The two spaces (X,τ) and (Y,τ') are called homeomorphic if there exists a homeomorphism function from (X,τ) to (Y,τ') and denoted by $(X,\tau)\cong(Y,\tau')$.

<u>Example:</u> Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}, \{1, 2\}\}$ and $Y = \{a, b, c\}$, $\tau' = \{Y, \phi, \{a\}, \{a, b\}\}$.

And let $f: (X, \tau) \to (Y, \tau')$;

$f(1) = a$, $f(2) = b$, $f(3) = c$.

It is clear that f is homeomorphism function and so $(X, \tau) \cong (Y, \tau')$.

<u>Definition</u>: A property "P" of a topological space (X, τ) is called "a topological Property" if every space (Y, τ') homeomorphic to (X, τ) also has the same Property.

i.e., if $(X, \tau) \cong (Y, \tau')$ and (X, τ) has a property "P", then (Y, τ') has the same property.

Subspace or Induced space

Definition: Let (X,τ) be a topological space, and $W \subseteq X$. Then $\tau_W = \{W \cap U : U \in \tau\}$ is a topology on W, and (W, τ_W) is called a subspace of (X,τ). τ_W is called the induced topology for W.

Theorem: If (X,τ) is a topological space, and $W \subseteq X$, then $\tau_W = \{W \cap U : U \in \tau\}$ is a topology on W.

Proof: ① * $W = W \cap X$; $X \in \tau \Rightarrow W \in \tau_W$.

* $\phi = W \cap \phi$; $\phi \in \tau \Rightarrow \phi \in \tau_W$.

② Let $A, B \in \tau_W$. To prove $A \cap B \in \tau_W$.

$A \in \tau_W \Rightarrow \exists U \in \tau ; A = W \cap U$.

$B \in \tau_W \Rightarrow \exists V \in \tau ; B = W \cap V$.

Then $U \cap V \in \tau$, and

$$A \cap B = (W \cap U) \cap (W \cap V)$$
$$= W \cap (U \cap V) \in \tau_W.$$

③ Let $V_\alpha \in \tau_W$ $\forall \alpha \in \Lambda$.

$\forall \alpha \in \Lambda$, $\exists U_\alpha \in \tau ; V_\alpha = W \cap U_\alpha$.

$$\bigcup_{\alpha \in \Lambda} V_\alpha = \bigcup_{\alpha \in \Lambda} (W \cap U_\alpha) = W \cap \Big(\bigcup_{\alpha \in \Lambda} U_\alpha\Big).$$

Since $\bigcup_{\alpha \in \Lambda} U_\alpha \in \tau$, then $\bigcup_{\alpha \in \Lambda} V_\alpha \in \tau_W$.

Hence, τ_W is a topology on W.

Example: Let $X=\{1,2,3,4\}$ and $\tau=\{X,\phi,\{1\},\{2\},\{1,2\}\}$.
Then (X,τ) is a topological space.
Let $W=\{1,2,3\}$, $A=\{1,4\}$, $B=\{2,3,4\}$
Find τ_W, τ_A and τ_B.

$$\tau_W=\{W\cap U : U\in\tau\}$$
$$=\{W,\phi,\{1\},\{2\},\{1,2\}\}$$

①
$W\cap X=W$
$W\cap\phi=\phi$
$W\cap\{1\}=\{1\}$
$W\cap\{2\}=\{2\}$
$W\cap\{1,2\}=\{1,2\}$

$$\tau_A=\{A,\phi,\{1\}\}$$

②
$A\cap X=A$
$A\cap\phi=\phi$
$A\cap\{1\}=\{1\}$
$A\cap\{2\}=\phi$
$A\cap\{1,2\}=\{1\}$

$$\tau_B=\{B,\phi,\{2\}\}$$

③
$B\cap X=B$.
$B\cap\phi=\phi$
$B\cap\{1\}=\phi$
$B\cap\{2\}=\{2\}$
$B\cap\{1,2\}=\{2\}$

Example: Prove or disprove the following statements:

(1) If (W, τ_W) is a subspace of (X, τ), then $\tau_W \subseteq \tau$.

Answer: Disprove, For example:

Let $X = \{1, 2, 3, 4\}$

$\tau = \{X, \phi, \{1, 2\}\}$

Then (X, τ) is a topological space.

Let $W = \{1, 3\}$.

then $\tau_W = \{W, \phi, \{1\}\}$ is the induced topology for W.

$\{1\} \in \tau_W$ but $\{1\} \notin \tau$.

Hence, $\tau_W \nsubseteq \tau$.

(2) Let (X, τ) be a topological space, and (W, τ_W) be a subspace for (X, τ).
If $\tau_W = I = \{W, \phi\}$, then $\tau = I = \{X, \phi\}$.

Answer: Disprove, For example:

Let $X = \{1, 2, 3, 4\}$, $\tau = \{X, \phi, \{1\}, \{2\}, \{1, 2\}\}$

Let $W = \{3, 4\}$.

Then $\tau_W = \{W, \phi\}$

Hence, (W, τ_W) is a subspace for (X, τ),

where $\tau_W = \{W, \phi\}$ but $\tau \neq \{X, \phi\}$.

(3) If (W, τ_W) is a subspace for (X, τ) and $\tau_W = D_W$ (the Discreat topology on W), then $\tau = D_X$ (the Discreat topology on X.

Answer; Disprove, for example:

Let $X = \{1,2,3\}$, $\tau = \{X, \phi, \{1\}, \{2\}, \{1,2\}\}$

then $\tau \neq D_X$.

Let $W = \{1,2\}$.

$\tau_W = \{W, \phi, \{1\}, \{2\}\} = D_W$.

(4) If (W, τ_W) is a subspace for (X, τ), and $\tau = D_X$ (the discreat topology on X), then $\tau_W = D_W$ = the discreat topology on W.

Proof: To Prove $\tau_W = D_W$, it's enough to prove that $(\forall y \in W \Rightarrow \{y\} \in \tau_W)$

let $y \in W \rightarrow y \in X$, since $W \subseteq X$.

$$\Rightarrow \{y\} \in \tau = D_X$$
$$\Rightarrow \{y\} \cap W \in \tau_W$$
$$\Rightarrow \{y\} \in \tau_W$$

$\therefore \forall y \in W \Rightarrow \{y\} \in \tau_W$.

Hence, τ_W is the discreat topology on W.

(5) If $\tau = I_X = \{X, \phi\}$, then $\tau_W = I_W = \{W, \phi\}$

Prove: $\tau_W = \{W \cap U : U \in \tau\}$
$$= \{W \cap X, W \cap \phi\}$$
$$= \{W, \phi\}$$
is the indescreate topology on W.

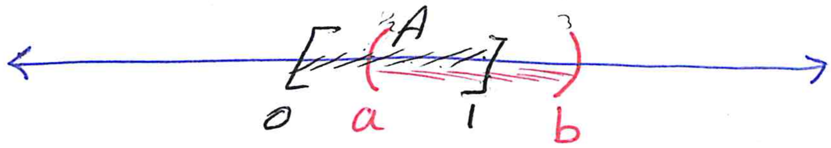
Example: In $(\mathbb{R}, \tau_u)$, Find the induced topology for the following sets:

① $A = [0, 1]$, ② $B = [0, 4)$, ③ $C = (1, 3)$

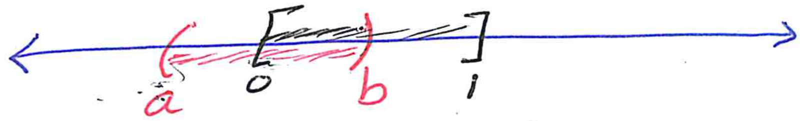
④ $D = \{1, 2, 3\}$ ⑤ $\mathbb{N}$ ⑥ E^+ ⑦ $\mathbb{Q}$

① $A = [0, 1]$

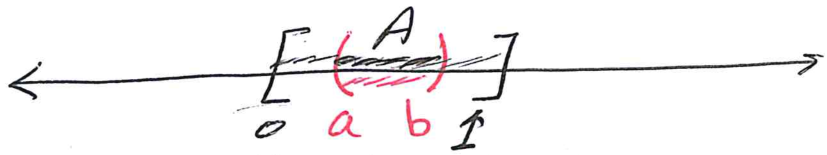
$\circledast$ $A \cap (a, b) = \phi$, $A \cap (c, d) = \phi$



$\circledast$ $A \cap (a, b) = (a, 1]$



$\circledast$ $A \cap (a, b) = [0, b)$



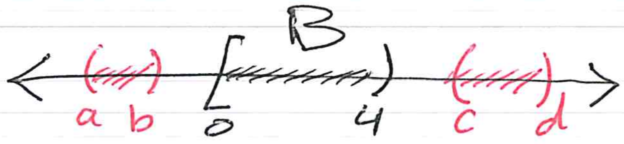
$\circledast$ $A \cap (a, b) = (a, b)$

$\circledast$ $A \cap (a, b) = A = [0, 1]$

Then $\mathcal{B}_A = \{\phi, (a, 1], [0, b), (a, b), [0, 1]\}$ is the basis for induce topology τ_{u_A}.

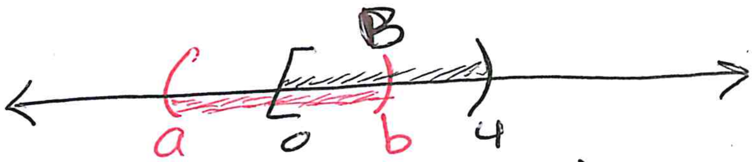
where $0 < a < b < 1$

② $B = [0,4)$

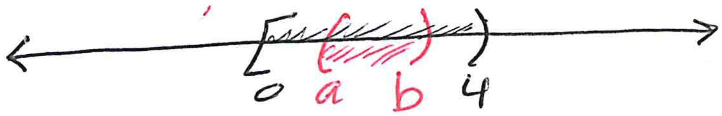


⊛ $[0,4) \cap (a,b) = \phi$,
$[0,4) \cap (c,d) = \phi$.

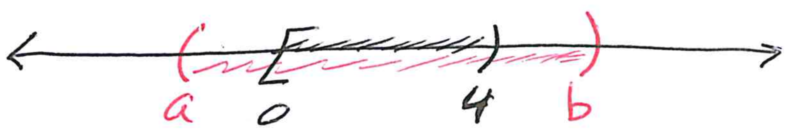
⊛ $B \cap (a,b) = (a,4)$



⊛ $B \cap (a,b) = [0,b)$



⊛ $B \cap (a,b) = (a,b)$

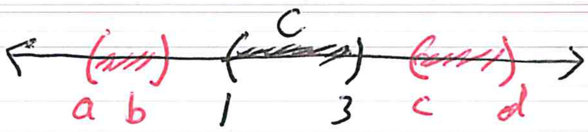


⊛ $B \cap (a,b) = [0,4)$

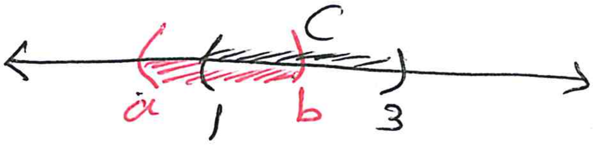
Then, $\mathcal{B}_B = \{\phi, (a,4), [0,b), (a,b), [0,4)\}$ is the basis for the induced topology τ_{u_B}

where $0 < a < b < 4$

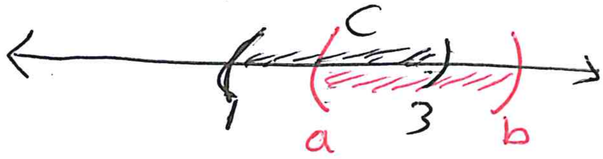
③ $C = (1,3)$



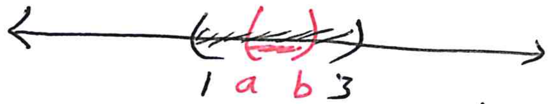
$C \cap (a,b) = \phi$, $C \cap (c,d) = \phi$



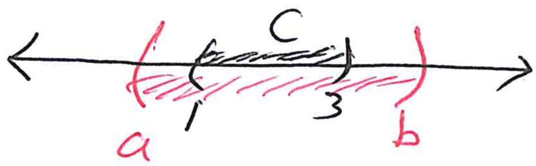
$C \cap (a,b) = (1,b)$



$C \cap (a,b) = (a,3)$



$C \cap (a,b) = (a,b)$



$C \cap (a,b) = C$

then $\mathcal{B}_C = \{\phi, (1,b), (a,3), (a,b), (1,3)\}$

is the basis for τ_C .

where $1 < a < b < 3$

(4) $D = \{1, 2, 3\}$

$\therefore \tau_D =$ Discreat topology $= \{D, \phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$

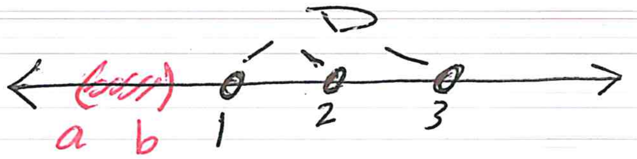
Remark: We can show that $\tau_D = D$ by Prove $\{x\} \in \tau_D$ $\forall\, x \in D$.

$D \cap (0, \frac{3}{2}) = \{1\} \in \tau_D$

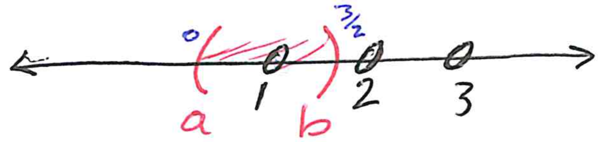
$D \cap (\frac{3}{2}, \frac{5}{2}) = \{2\} \in \tau_D$

$D \cap (\frac{5}{2}, \frac{7}{2}) = \{3\} \in \tau_D$

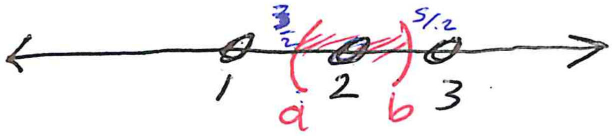
$\therefore \tau_D =$ Discreat topology on D



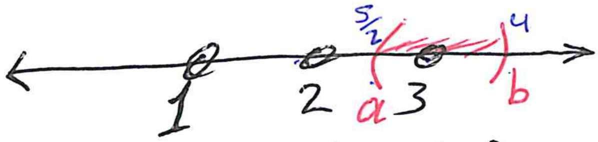
$D \cap (a, b) = \phi$



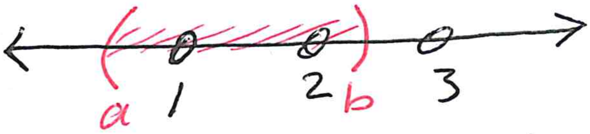
$D \cap (a, b) = \{1\}$



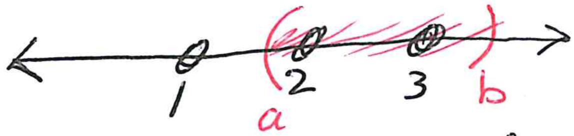
$D \cap (a, b) = \{2\}$



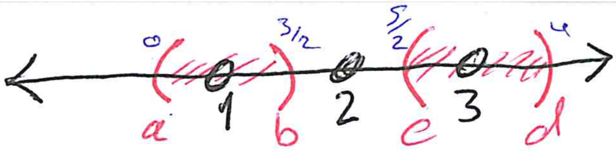
$D \cap (a, b) = \{3\}$



$D \cap (a, b) = \{1, 2\}$

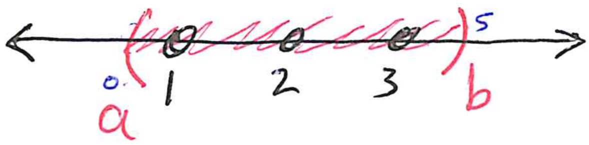


$D \cap (a, b) = \{2, 3\}$



let $U = (a, b) \cup (c, d) \in \tau_u$

$U \cap D = \{1, 3\}$



$U \cap D = D$

⑤ $\mathbb{N}$.

$\forall n \in \mathbb{N}$, Let $U_n = (n-1, n+1) \in \tau_u$.

$U_n \cap \mathbb{N} = \{n\} \in \tau_{\mathbb{N}}$.

$\therefore \tau_{u_{\mathbb{N}}} = \mathbb{P}(\mathbb{N})$ is the discreat $= D_{\mathbb{N}}$ topology on $\mathbb{N}$.

⑥ $\mathbb{Q}$ the set of all rational numbers.

$\mathbb{Q} \cap (a,b) =$ the rational numbers in (a,b).

$\therefore \mathcal{B}_{\mathbb{Q}} = \{\mathbb{Q} \cap (a,b) : a,b \in \mathbb{R}\}$ is a basis for $\tau_{u_{\mathbb{Q}}}$

⑦ E^+.

$\forall n \in E^+$, Let $U_n = (n-1, n+1) \in \tau_u$.

$U_n \cap E^+ = \{n\} \in \tau_{E^+}$.

$\therefore \tau_{u_{E^+}}$ is the discreat topology on E^+.

Theorem: Let (X, τ) be a topological space, and (W, τ_W) be a subspace of (X, τ). If $W \in \tau$, then τ_W is a subfamily of τ.

Proof: Let $V \in \tau_W \Rightarrow \exists U \in \tau ; V = W \cap U$.

Since $U \in \tau$ and $W \in \tau$ (by hypothesis), then $U \cap W \in \tau$ (by def. of topology).

$\Rightarrow V \in \tau$.

$\therefore \tau_W \subseteq \tau$.

Example Let $X = \{1, 2, 3, 4\}$
$\tau = \{X, \phi, \{1\}, \{1,2\}, \{1,2,3\}, \{1,2,4\}\}$

Let $W = \{1, 2\} \in \tau$

Then $\tau_W = \{W, \phi, \{1\}\} \subseteq \tau$

Theorem: If K is a subspace from W and W is a subspace from X, then K is a subspace from X.

Must prove that:

① $K \subseteq X$

② $\forall A \in \tau_K \; \exists U \in \tau ; A = K \cap U$.

Proof: ① Since $K \subseteq W$ and $W \subseteq X$, then $K \subseteq X$.

② Let $A \in \tau_K \Rightarrow \exists V \in \tau_W ;$
$A = V \cap K$.

$V \in \tau_W \Rightarrow \exists U \in \tau ; V = W \cap U$.

$\Rightarrow A = V \cap K = (W \cap U) \cap K$

$\Rightarrow A = (W \cap K) \cap U$

$\boxed{W \cap K = K \quad K \subseteq W}$

$\Rightarrow A = K \cap U$

$\therefore K$ is a subspace from X.

Restriction Function.

Definition: Let $f: X \to Y$ be a function and let $A \subseteq X$. We say the function $g: A \to Y$ such that $g(a) = f(a)$ $\forall a \in A$ is the restriction function on the set A and denoted by $g = f|_A$.

Example: Let $f: \mathbb{R} \to \mathbb{R}$ such that $f(x) = x$.

$\mathbb{N} \subseteq \mathbb{R}$. $\quad f|_{\mathbb{N}}: \mathbb{N} \to \mathbb{R}$; $f|_{\mathbb{N}}(x) = x$.

Theorem: If $f: (X, \tau) \to (Y, \tau')$ is a continuous function and W be a subspace of X, then $f|_W : (W, \tau_W) \to (Y, \tau')$ is continuous.

Proof: Let $V \in \tau' \Rightarrow f^{-1}(V) \in \tau$

$\Rightarrow W \cap f^{-1}(V) \in \tau_W$

$(f|_W)^{-1}(V) \in \tau_W$.

$\Rightarrow$

$\therefore$ $f|W$ is continuous.

Theorem: Let (X, τ) be a topological space and (W, τ_W) be a subspace of X. Then the inclusion function $i: (W, \tau_W) \to (X, \tau)$ such that $i(x) = x$ for all $x \in W$ is continuous.

Proof: Let $V \in \tau \Rightarrow i^{-1}(V) = V \cap W$

$\rightarrow i^{-1}(V) \in \tau_W$

$\therefore$ i^{-1} is continuous.

Product Space

Definition Let X and Y be any two sets. The Cartesian product, or simply Product of X by Y is denoted by $X \times Y$, and define as: $X \times Y = \{(x,y) : x \in X \wedge y \in Y\}$.

Definition Let (X,τ) and (Y,τ') be two topological spaces. We say the topology has a base $\mathcal{B} = \{U \times V : U \in \tau \wedge V \in \tau'\}$ is the Product topology on the set $X \times Y$ and denoted by $\tau_{X \times Y}$ and called the spaces $(X \times Y, \tau_{X \times Y})$ is the Product space of X by Y.

Example: Let $X = \{1,2,3\}$, $\tau = \{X, \phi, \{1\}\}$,
$Y = \{a,b\}$, $\tau' = \{Y, \phi, \{b\}\}$

then $X \times Y = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$

$$\mathcal{B} = \{U \times V : U \in \tau \wedge V \in \tau'\}$$

$$= \{X \times Y,\ X \times \phi,\ X \times \{b\},\ \phi \times Y,\ \phi \times \phi,\ \phi \times \{b\},\ \{1\} \times Y,\ \{1\} \times \phi,\ \{1\} \times \{b\}\}$$

$$= \{X \times Y,\ \phi,\ \{(1,b),(2,b),(3,b)\},\ \{(1,a),(1,b)\},\ \{(1,b)\}\}$$

$$\tau_{X \times Y} = \{X \times Y,\ \phi,\ \{(1,b),(2,b),(3,b)\},\ \{(1,a),(1,b)\},\ \{(1,b)\},\ \{(1,a),(1,b),(2,b),(3,b)\}\}$$

Remark If (X,τ) and (Y,τ') are any two topological spaces.

1) If $\tau = I_X$ and $\tau' = I_Y$ then $\tau_{X\times Y} = I_{X\times Y}$.

$$\mathcal{B} = \{X\times Y,\ X\times\phi,\ \phi\times Y,\ \phi\times\phi\}$$

$$\tau_{X\times Y} = \{X\times Y, \phi\} = I_{X\times Y}.$$

2) If $\tau = I_X$ and $\tau' \neq I_Y$ then $\tau_{X\times Y} \neq I_{X\times Y}$.

So if $\tau \neq I_X$ and $\tau' = I_Y$ then $\tau_{X\times Y} \neq I_{X\times Y}$.

Example: $X = \{1,2\}$, $\tau = \{X, \phi, \{1\}\}$

$Y = \{a,b\}$, $\tau' = \{Y, \phi\}$

Then, $\mathcal{B} = \{X\times Y,\ \underline{X\times\phi},\ \underline{\phi\times Y},\ \underline{\phi\times\phi},\ \{1\}\times Y,\ \underline{\{1\}\times\phi}\}$ $\to \phi$

$$\mathcal{B} = \{X\times Y, \phi, \{1\}\times Y\}$$

$$\tau_{X\times Y} = \mathcal{B} = \{X\times Y, \phi, \{(1,a),(1,b)\}\} \neq I_{X\times Y}.$$

3) If $\tau = D_X$ and $\tau' = D_Y$, then $\tau_{X\times Y} = D_{X\times Y}$.

To Prove $\tau_{X\times Y} = D_{X\times Y}$, must Prove that $\{(x,y)\} \in \tau_{X\times Y}$ $\forall x\in X, \forall y \in Y$.

Let $x \in X \wedge y \in Y \Rightarrow \{x\} \in D_X \wedge \{y\} \in D_Y$

$\Rightarrow \{x\}\times\{y\} \in \mathcal{B} \quad \forall (x,y) \in X\times Y$

$\Rightarrow \{(x,y)\} \in \mathcal{B} \quad \forall (x,y) \in X\times Y$

$\Rightarrow \{(x,y)\} \in \tau_{X\times Y} \quad \forall (x,y) \in X\times Y$.

$\therefore \tau_{X\times Y} = D_{X\times Y}$.

4) If $\tau \neq D_X$ or $\tau' \neq D_X$, then $\tau_{X\times Y} \neq D_{X\times Y}$.

For example: If $X = \{1, 2\}$, $\tau = \{X, \phi, \{1\}, \{2\}\} = D_X$.
and $Y = \{a, b\}$, $\tau' = \{Y, \phi\} \neq D_Y$, then:

$$\mathcal{B} = \{X\times Y, X\times\phi, \phi\times Y, \phi\times\phi, \{1\}\times Y, \{1\}\times\phi, \{2\}\times Y, \{2\}\times\phi\}$$

$$\mathcal{B} = \{X\times Y, \phi, \{1\}\times Y, \{2\}\times Y\} = \tau_{X\times Y} \neq D_{X\times Y}$$

5) If $A \subseteq X$ and $B \subseteq Y$, then $A\times B \subseteq X\times Y$ and

$$\overline{A\times B} = \overline{A} \times \overline{B}.$$

$$(A\times B)^{\circ} = A^{\circ} \times B^{\circ}.$$

6) There are two natural Projection functions:

$P_X : X\times Y \to X$; $P_X((x,y)) = x$ — The first Project function

$P_Y : X\times Y \to Y$; $P_Y((x,y)) = y$ — The second Project function.

// The Projection maps P_X and P_Y are continuous and open functions. //

To Prove P_X is Continuous function:

Let $U \in \tau \Rightarrow P_X^{-1}(U) = U\times Y$

Since $U \in \tau \wedge Y \in \tau' \Rightarrow U\times Y \in B_{X\times Y}$

$\Rightarrow U\times Y \in \tau_{X\times Y}$

$\Rightarrow P_X^{-1}(U) \in \tau_{X\times Y}$.

$\therefore P_X$ is Continuous.

In similar way, we can prove P_Y is continuous.

$P_Y : X \times Y \longrightarrow Y \; ; \; P_Y((x,y)) = y.$

Let $V \in \tau' \Rightarrow P_Y^{-1}(V) = X \times V$

Since $X \in \tau \wedge V \in \tau' \Rightarrow X \times V \in B_{X \times Y}$

$\Rightarrow X \times V \in \tau_{X \times Y}$

$\Rightarrow P_Y^{-1}(V) \in \tau_{X \times Y}.$

$\therefore P_Y$ is continuous.

So, both P_X and P_Y are open functions.

To prove $P_X : X \times Y \longrightarrow X \; ; \; P_X((x,y)) = x$ is open function.

Let $U \times V \in \tau_{X \times Y} \Rightarrow U \in \tau \wedge V \in \tau'$

$\Rightarrow P_X(U \times V) = U \in \tau$

$\therefore P_X$ is open function.

Similarly; P_Y is open function.

$P_Y : X \times Y \longrightarrow Y \; ; \; P_Y((x,y)) = y$

Let $U \times V \in B_{X \times Y} \Rightarrow U \times V \in \tau_{X \times Y}$

$\Rightarrow U \in \tau \wedge V \in \tau'$

$P_Y(U \times V) = V \in \tau'.$

$\therefore P_Y$ is open function.

⑦ In general $X \times Y \neq Y \times X$. But $X \times Y \cong X \times Y$.

$X \times Y$ and $Y \times X$ are homeomorphic.

Define $f: X \times Y \to Y \times X$; $f((x,y)) = (y,x)$.

f is 1-1 function

Since, Let $f((x_1,y_1)) = f((x_2,y_2))$

$\Rightarrow (y_1,x_1) = (y_2,x_2)$

$\Rightarrow x_1 = x_2$ and $y_1 = y_2$

$\Rightarrow (x_1,y_1) = (x_2,y_2)$

$\therefore$ f is 1-1.

f is onto function

$\forall (y,x) \in Y \times X \Rightarrow y \in Y \wedge x \in X$

$\Rightarrow (x,y) \in X \times Y$

$f((x,y)) = (y,x)$.

f is Continuous

Let B be a base for $X \times Y$ and B' be a base for $Y \times X$.

Let $V \times U \in B' \Rightarrow V \in \tau' \wedge U \in \tau$

$\Rightarrow U \times V \in B$

$\Rightarrow U \times V \in \tau_{X \times Y}$

$f^{-1}(V \times U) = U \times V$ open in $X \times Y$

f is open function

Let $U \times V \in B_{X \times Y} \Rightarrow U \times V \in \tau_{X \times Y}$.

$\Rightarrow U \in \tau \wedge V \in \tau'$

$\Rightarrow V \times U \in B'_{Y \times X} \Rightarrow V \times U \in \tau_{Y \times X}$

$f(U \times V) = V \times U$ $\therefore$ f is open function.

(8) If $y_0 \in Y$, then the Product Space $X \times \{y_0\}$ topological equivalent the space X

i.e. $X \times \{y_0\} \cong X$

$X \times \{y_0\} = \{(x, y_0) : x \in X\}$

Proof: Define $f : X \to X \times \{y_0\}$; $f(x) = (x, y_0)$ $\forall x \in X$.

Must Prove f is a homeomorphism function

* f is 1-1 function.

$f(x_1) = f(x_2) \Rightarrow (x_1, y_0) = (x_2, y_0)$

$\Rightarrow x_1 = x_2 \quad \forall x_1, x_2 \in X$.

* f is onto function.

$\forall (x, y_0) \in X \times \{y_0\} \Rightarrow x \in X \wedge y_0 \in \{y_0\}$

$\Rightarrow \exists x \in X$; $f(x) = (x, y_0)$

* f is Continuous function

Let $\mathcal{B}$ is the base of the space $(X \times \{y_0\}, \tau_{X \times \{y_0\}})$

Let $V \in \mathcal{B} \Rightarrow V = U \times \{y_0\}$; $U \in \tau$

or $V = \phi$

$\Rightarrow f^{-1}(U \times \{y_0\}) = U \in \tau$ or $f^{-1}(\phi) = \phi \in \tau$.

* f is open function

Let $U \in \tau \Rightarrow f(U) = U \times \{y_0\} \in \mathcal{B}_{\tau_{X \times \{y_0\}}}$

$\Rightarrow f(U)$ open in $\tau_{X \times \{y_0\}}$.

$\therefore$ f is homeomorphism function.

* $X \cong X \times \{y_0\}$

(9) If $x_0 \in X$, then $\{x_0\} \times Y \cong Y$

$\{x_0\} \times Y = \{(x_0, y) : y \in Y\}$.

Proof: Define $f : Y \longrightarrow \{x_0\} \times Y$;

$f(y) = (x_0, y) \quad \forall y \in Y$.

⊛ f is 1-1 function.

$f(y_1) = f(y_2) \Rightarrow (x_0, y_1) = (x_0, y_2)$

$\Longrightarrow y_1 = y_2 \quad \forall y_1, y_2 \in Y$.

⊛ f is onto function.

$\forall (x_0, y) \in \{x_0\} \times Y \Rightarrow x_0 \in \{x_0\} \wedge y \in Y$

$\Longrightarrow \exists y \in Y ; f(y) = (x_0, y)$

⊛ f is continuous.

Let $\mathcal{B}$ is the base for the topology $\tau_{\{x_0\} \times Y}$

Let $V \in \mathcal{B} \Rightarrow V = \{x_0\} \times U \; ; \; U \in \tau$

or $V = \phi$

$\Rightarrow f^{-1}(\{x_0\} \times U) = U \in \tau$ or $f^{-1}(\phi) = \phi \in \tau$

⊛ f is open function.

Let $U \in \tau \Rightarrow f(U) = \{x_0\} \times U \in \mathcal{B}_{\tau_{\{x_0\} \times Y}}$.

$\therefore f(U)$ open in $\{x_0\} \times Y$.

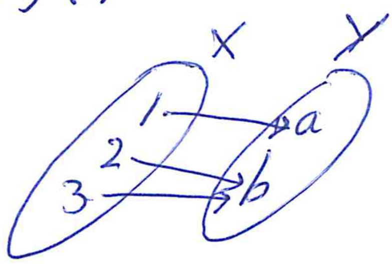
$\therefore f$ is homeomorphism function.

$\therefore \{x_0\} \times Y \cong Y$.

Quotient space

<u>Definition</u>: Let (X, τ) be a topological space, and Y be a non-empty subset of X. let $f: X \to Y$ be a Surjective (onto) function. then $\tau_f = \{G \subseteq Y : f^{-1}(G) \in \tau\}$ is a topology on Y, which is called the quotient topology on Y genereted by f and (X, τ). So, (Y, τ_f) is called the quotient space.

<u>Example</u>: let $X = \{1, 2, 3\}$
and $\tau = \{X, \phi, \{1\}, \{1,2\}, \{1,3\}\}$
Let $Y = \{a, b\}$ and $f: X \to Y \ni f(1) = a$,
$f(2) = f(3) = b$



Now;

$f^{-1}(Y) = X \in \tau \Rightarrow Y \in \tau_f$

$f^{-1}(\phi) = \phi \in \tau \Rightarrow \phi \in \tau_f$

$f^{-1}(\{a\}) = \{1\} \in \tau \Rightarrow \{a\} \in \tau_f$

$f^{-1}(\{b\}) = \{2,3\} \notin \tau \Rightarrow \{b\} \notin \tau_f$

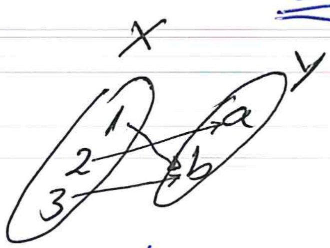
$\therefore \tau_f = \{Y, \phi, \{a\}\}$ is a topology on Y.

(Y, τ_f) is called Quotient space.

Example: Let $X=\{1,2,3\}$,

$\tau=\{X,\phi,\{1\},\{2,3\}\}$

$Y=\{a,b\}$



$f: X\longrightarrow Y \ni f(1)=f(3)=b$

and $f(2)=a$

onto function

Then:-

$f^{-1}(Y)=X\in\tau$ and $f^{-1}(\phi)=\phi\in\tau$

$\therefore Y\in\tau_f$ $\therefore \phi\in\tau_f$

$f^{-1}(\{a\})=\{2\}\notin\tau \Rightarrow \{a\}\notin\tau_f$

$f^{-1}(\{b\})=\{1,3\}\notin\tau \Rightarrow \{b\}\notin\tau_f$

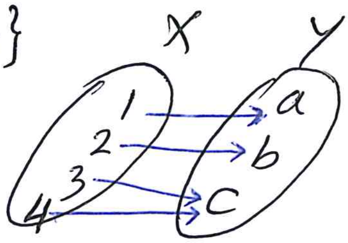
$\therefore \tau_f=\{Y,\phi\}=I_Y$

(Y,τ_f) is the quotient space.

Example: Let $X=\{1,2,3,4\}$

$\tau=\{X,\phi,\{1,2\},\{3,4\}\}$

Let $Y=\{a,b,c\}$



$f: X\longrightarrow Y \ni f(1)=a$,

$f(2)=b$ and $f(3)=f(4)=c$

* $f^{-1}(Y)=X\in\tau \Rightarrow Y\in\tau_f$
* $f^{-1}(\phi)=\phi\in\tau \Rightarrow \phi\in\tau_f$
* $f^{-1}(\{a\})=\{1\}\notin\tau$
* $f^{-1}(\{b\})=\{2\}\notin\tau$
* $f^{-1}(\{c\})=\{3,4\}\in\tau \Rightarrow \{c\}\in\tau_f$
* $f^{-1}(\{a,b\})=\{1,2\}\in\tau \Rightarrow \{a,b\}\in\tau_f$
* $f^{-1}(\{a,c\})=\{1,3,4\}$
* $f^{-1}(\{b,c\})=\{2,3,4\}$

$\tau_f=\{Y,\phi,\{c\},\{a,b\}\}$.

is the quotient topology on Y

Then (Y,τ_f) is called the quotient space.

Remark: The topology $\tau_f = \{G \subseteq Y : f^{-1}(G) \in \tau\}$ is the Largest topology on Y make the function f Continuous

Proof: Let τ' be another topology on Y making f Continuous.

$\Rightarrow f^{-1}(G)$ open in X $\forall G \in \tau'$

$\Rightarrow G \in \tau_f$ (by definition of τ_f)

$\Rightarrow \tau' \subseteq \tau_f$

$\therefore$ τ_f is the Largest topology on Y making f continuous.

هذه الملاحظة تُبين انه اذا كان (X,τ) فضاء تبولوجي
وان $f : X \longrightarrow Y$ حيث ان (Y,τ_f) هو
فضاء القسمة المتولد على Y باستخدام الدالة f.
وكان τ' فضاء توبولوجي آخر على Y حيث تكون
الدالة $f : (X,\tau) \longrightarrow (Y,\tau')$ مستمرة.
فان $\tau' \subseteq \tau_f$

$$\left.\begin{array}{l} \circledast\ (X,\tau) \text{ is a top-sp.} \\ \circledast\ f : X \longrightarrow Y,\ \tau_f \text{ is the quotient topology on } Y \text{ by } f \\ \circledast\ \tau' \text{ another topology on } Y \\ \quad ; f(X,\tau) \longrightarrow (Y,\tau') \text{ is Continuous} \end{array}\right\} \therefore \tau' \subseteq \tau_f.$$

Theorem: Let $f:(X,\tau)\to(Y,\tau')$ be a continuous, surjective function, if f either open or closed, then $\tau_f=\tau'$

Proof: Since τ_f is the Largeast topology on Y make f continuous, and $f:(X,\tau)\to(Y,\tau')$ is continuous. Then, $\tau'\subseteq\tau_f$.

Now, to prove $\tau_f\subseteq\tau'$.

Let $G\in\tau_f \Rightarrow f^{-1}(G)\in\tau$ (by def. of τ_f).

$\Rightarrow f(f^{-1}(G))\in\tau'$ is open in Y. (since f is open).

$f(f^{-1}(G))=G$ (since f is surjective).

$\Longrightarrow G\in\tau'$ $\therefore \tau_f\subseteq\tau'$

$\therefore \tau_f=\tau'$

Similarly; if f is closed function, then $\tau_f=\tau'$.

Theorem: Let Y has the quotient space generated by the surjective function $f: X \to Y$, then $g: Y \to Z$ is continuous function if and only if $g \circ f$ is continuous function.

Proof: $\Rightarrow$: If $g: Y \to Z$ is continuous and $f:(X,\tau) \to (Y,\tau_f)$ is continuous $\Big\}$ $\therefore$ $g \circ f: X \to Z$ is continuous.

Since (the composition of continuous functions is continuous)

$\Leftarrow$: Let $g \circ f$ continuous. To prove g is continuous

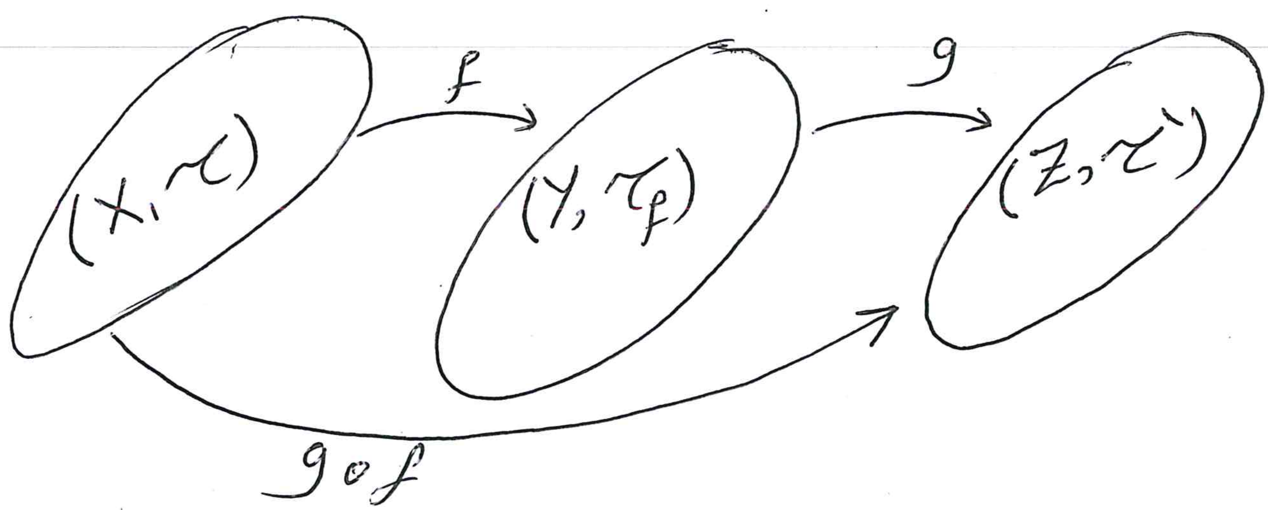
Let G be an open set in Z.

Since, $g \circ f$ is continuous $\Rightarrow (g \circ f)^{-1}(G)$ open in X.

$\Rightarrow f^{-1}(g^{-1}(G))$ open in X.

$\Rightarrow g^{-1}(G)$ open in Y (by definition of τ_f)

$\Rightarrow g$ is continuous.



Remark:

[1] Let X be a nonempty set. The Partition or decomposition on X with the relation R is the family of disjoint nonempty subsets of X and their union equal X. The elements of this Partition called equivalence classes and denoted by $[x]$.

[2] the set of equivalence classes for X is called quotient set for X with the relation R and denoted by $X|R = \{[x] : x \in X\}$

[3] The mapping $P : X \longrightarrow X|R$; $P(x) = [x]$ is called quotient mapping.

Example: Let $X = \{1, 2, 3\}$

$R = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$ is an equivalence relation on X.

$[1] = \{1, 2\} = [2]$, $[3] = \{3\}$

$X/R = \{\{1,2\}, \{3\}\}$ quotient set.

$P : X \longrightarrow X/R$; $P(1) = P(2) = \{1,2\}$.

$P(3) = \{3\}$.

Definition Quotient Space

Let (X, τ) be a topological space and R be an equivalence relation on X. Let $p: X \to X/R$; $p(x) = [x]$ be surjective quotient mapping from X to X/R, then the quotient topology on X/R is the largest topology make the function f continuous and the space $(X/R, \tau_{X/R})$ is called quotient space.