Now, $(a+b)-(a_1+b_1) = a+b-a_1-b_1$

$$= \underbrace{(a-a_1)}_{\in I} + \underbrace{(b-b_1)}_{\in I} \in I$$

Thus $\oplus$ is well-defined.

To prove $\odot$ is well-defined

let $a+I = a_1+I$ and $b+I = b_1+I$

$\Rightarrow a-a_1 \in I$ and $b-b_1 \in I$

We want to prove that $(a+I)\odot(b+I) = (a_1+I)\odot(b_1+I)$,
That is, $(a.b)-(a_1.b_1) \in I$

$a.b - a_1b_1 = ab - a_1b + a_1b - a_1b_1$

$$= (a-a_1)b + a_1(b-b_1) \in I$$

Since $a-a_1 \in I \Rightarrow (a-a_1)\cdot b \in I$ (I is an ideal)
$b-b_1 \in I \Rightarrow a_1(b-b_1) \in I$

Thus $\odot$ is well-defined

It is clear that $(R\backslash I, \oplus)$ is Comm. gp.

and $\odot$ is associative, also $\odot$ distributive over $\oplus$

Therefore $(R\backslash I, \oplus, \odot)$ is a ring. This ring is called the quotient ring of R by I.

Remarks:

Let $(R,+,\cdot)$ be a ring and I be an ideal of R. Then:

1. If R is Comm. ring, then R/I is Comm. ring
2. $R\backslash I$ is Comm. $\Leftrightarrow$ $a.b - ba \in I \quad \forall a,b \in R$.
3. If R has unity 1, then $R\backslash I$ has unity $1+I$

(b) Find all maximal ideals of $(\mathbb{Z}_{10}, +_{10}, \cdot_{10})$

Proofs: (b) " " " " " $(\mathbb{Z}_{12}, +_{12}, \cdot_{12})$

1. $\forall (a+I), (b+I) \in R\backslash I$ is Comm. ring

$$
\begin{aligned}
(a+I)\odot(b+I) &= a.b + I \quad (\text{by definition of } \odot)\\
&= b.a + I \quad (\cdot \text{ Comm. operation})\\
&= (b+I)\odot(a+I) \quad (\text{by def. of } \odot)
\end{aligned}
$$

2. $R\backslash I$ is Comm. $\overset{\text{by}}{\underset{\text{Def.}}{\Longleftrightarrow}} (a+I)\odot(b+I) = (b+I)\odot(a+I)$

$$\Leftrightarrow ab + I = ba + I$$

$$\Leftrightarrow ab - ba \in I$$

3. It is clear from definition of identity of R and $R\backslash I$

Example:

Consider $(\mathbb{Z}_4, +_4, \cdot_4)$ be a ring.

let $I = \{\bar{0}, \bar{2}\}$

$\mathbb{Z}_4\backslash I = \{a + I : a \in \mathbb{Z}_4\}$

$\bar{0} + I = I$

$\bar{1} + I = \{\bar{1}, \bar{3}\}$

$\bar{2} + I = I$ and $\bar{3} + I = \bar{1} + \underbrace{(\bar{2} + I)}_{I} = \bar{1} + I$

$\therefore \mathbb{Z}_4 \setminus I = \{\bar{0} + I, \bar{1} + I\}$

$\oplus$	$\bar{0}+I$	$\bar{1}+I$
$\bar{0}+I$	$\bar{0}+I$	$\bar{1}+I$
$\bar{1}+I$	$\bar{1}+I$	$\bar{0}+I$

محايد جمعي

$\odot$	$\bar{0}+I$	$\bar{1}+I$
$\bar{0}+I$	$\bar{0}+I$	$\bar{0}+I$
$\bar{1}+I$	$\bar{0}+I$	$\bar{1}+I$

محايد ضربي

Notice

1. $\mathbb{Z}_4$ has zero divisor $\bar{2}$, but $\mathbb{Z}_4 \setminus I$ has no zero divisor.

2. $\mathbb{Z}_4$ is not integral domain (not field), but $\mathbb{Z}_4 \setminus I$ is a field (and so it is integral domain)

3. $Ch(\mathbb{Z}_4) = 4$

4. $Ch(\mathbb{Z}_4 \setminus I) = 2$ (since $(\bar{1}+I) \oplus (\bar{1}+I) = \bar{2} + I = \bar{0} + I$)

Examples

1. $\mathbb{Z}_n \setminus \{\bar{0}\} = \{\bar{0}\}$ $\forall n > 1$

2. $\mathbb{Z}_n \setminus \mathbb{Z}_n = \mathbb{Z}_n$ $\forall n > 1$

3. Consider the ring $(\mathbb{Z}, +, \cdot)$, $I = id(4)$.

$\mathbb{Z} \setminus id(4) = \{a + id(4) : a \in \mathbb{Z}\}$

$0+I = I = \{0, \mp 4, \mp 8, \ldots\}$

$1+I = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$

$2+I = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$

$3+I = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

$\Rightarrow \ldots -8+I = -4+I = 0+I = 4+I = 8+I = \ldots$

$5+I = 1+(4+I) = 1+I$

$6+I = (2+(4+I)) = 2+I$

$7+I = 3+(4+I) = 3+I$

وهكذا البقية

$-1+I = 3+(-4+I) = 3+I$

$-2+I = 2+(-4+I) = 2+I$

وهكذا البقية

$\therefore \forall a \in Z \Rightarrow \exists\, k, r \in Z$ s.t. $a = 4k+r$

where $0 \leq r < 4$

$\therefore a+I = 4k+r+I = r+(4k+I) = r+I$

$\therefore Z \backslash I = \{0+I, 1+I, 2+I, 3+I\} = Z_4$

Therefore $(Z\backslash I, \oplus, \odot) = (Z_4, +_4, \cdot_4)$

Notice that $(Z, +, \cdot)$ is an integral domain and $ch(Z) = 0$, but $(Z_4, +_4, \cdot_4)$ is not integral domain , $ch(\mathbb{Z}_4) = 4$.

تقسيم لهذا المثال

Consider the ring $(\mathbb{Z}, +, \cdot)$ and let $I = id(n)$ where n is a fixed positive integer. Then

$$(\mathbb{Z} \setminus I, \oplus, \odot) = (\mathbb{Z}_n, +_n, \cdot_n)$$

Problem: (H.W)

let $(\mathbb{Z}_6, +_6, \cdot_6)$ be a ring and let $I = \{\bar{0}, \bar{2}, \bar{4}\}$, $J = \{\bar{0}, \bar{3}\}$.

Then find all element of

1. $\mathbb{Z}_6 \setminus I$
2. $\mathbb{Z}_6 \setminus J$
3. $\mathbb{Z}_6 \setminus I + J$
4. $\mathbb{Z}_6 \setminus I . J$

Chapter Five - Ring homomorphism - التشاكل الحلقي

Definition : let $(R,+,.)$ and $(R',\acute{+},\acute{.})$ be two ring. A mapping $f:R \longrightarrow R'$ is called ring homomorphism

$\Longleftrightarrow$ 1. $f(a+b) = f(a) \acute{+} f(b)$ $\quad \forall a,b \in R$
2. $f(a.b) = f(a) \acute{.} f(b)$

Examples :

1. let $(R,+,.)$ and $(R',\acute{+},\acute{.})$ be two rings. If $f: R \longrightarrow R'$ s.t. $f(x) = \bar{o}$ $\forall x \in R$ where $\bar{o}$ = Zero element of R'.

Solution $f(a+b) = \bar{o}$
$f(a) \acute{+} f(b) = \acute{o} \acute{+} \acute{o} = \bar{o}$
$f(a.b) = \bar{o}$, $f(a) \acute{.} f(b) = \acute{o} \acute{.} \acute{o} = \acute{o}$

2. let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be a ring homo. s.t. $f(x) = x$

3. let $f: (Z,+,.) \longrightarrow (Z,+,.)$ s.t. $f(x) = 3x$ $\forall x \in Z$
Is f homo. ?

Solution : $\forall a,b \in Z$

$f(a+b) = 3(a+b) = 3a+3b = f(a) + f(b)$

$f(a.b) = 3ab$, but $f(a).f(b) = 3a.3b = 9ab$

$\therefore f$ is not homo.

④ let $f: (\mathbb{Z}[\sqrt{3}], +, \cdot) \longrightarrow (\mathbb{Z}[\sqrt{2}], +, \cdot)$ s.t.

$$f(a+b\sqrt{3}) = a+b\sqrt{2} \quad \forall a, b \in \mathbb{Z}$$

Solution :

let $a+b\sqrt{3}$, $c+d\sqrt{3} \in \mathbb{Z}[\sqrt{3}]$

$f[(a+b\sqrt{3})+(c+d\sqrt{3})] = f[(a+c)+(b+d)\sqrt{3}]$

$= (a+c)+(b+d)\sqrt{2} = (a+b\sqrt{2}) + (c+d\sqrt{2})$

$= f(a+b\sqrt{3}) + f(c+d\sqrt{3})$

$f[(a+b\sqrt{3})(c+d\sqrt{3})] = f[(ac+3bd)+(bc+ad)\sqrt{3}]$

$= (ac+3bd)+(bc+ad)\sqrt{2}$

But $f(a+b\sqrt{3}) \cdot f(c+d\sqrt{3}) = (a+b\sqrt{2}) \cdot (c+d\sqrt{2})$

$= (ac+2bd)+(bc+ad)\sqrt{2}$

Since $f[(a+b\sqrt{3})\cdot(c+d\sqrt{3})] \neq f(a+b\sqrt{3}) \cdot f(c+d\sqrt{3})$

$\therefore$ f is not homo.

Problems : ① which of the following mappings from $(\mathbb{Z}, +, \cdot)$ into $(\mathbb{Z}, +, \cdot)$ is homo.

a. $f(x) = x^2$

b. $f(x) = 2x+1$

c. $f(x) = 2^x$

d. $f(x) = 1$

② let $f : (\mathbb{C}, +, \cdot) \longrightarrow (\mathbb{C}, +, \cdot)$ s.t.
$f(a+ib) = a - ib$ (where $i^2 = -1$). Is f homo.

Theorem :
let $f : (R, +, \cdot) \longrightarrow (R', +', \cdot')$ be any homo. Then

① $f(0) = \bar{0}$ where 0 = zero element of R
$\bar{0}$ = zero element of R'

② $f(-a) = -' f(a) \quad \forall a \in R$

③ $f(a-b) = f(a) -' f(b) \quad \forall a, b \in R$

④ If S is a subring of R, then $f(S)$ is a subring of R'

⑤ If S' is a subring of R', then $f^{-1}(S')$ is a subring of R.

⑥ If I is an ideal of R, f is onto, then $f(I)$ is an ideal of R'.

⑦ If J is an ideal of R', then $f^{-1}(J)$ is an ideal of R.

⑧ If R has unity 1, then $f(1)$ is unity of $f(R)$.

⑨ If R has unity 1, f is onto, then $f(1)$ is unity of R'

Proofs : ① $f(0) = f(0+0) = f(0) +' f(0)$
$\therefore f(0) +' 0' = f(0) +' f(0)$
$\therefore f(0) = 0'$

② $f(0)=\grave{0} \Rightarrow f(a+(-a))=\grave{0}$

$f(a)\grave{+}f(-a)=\grave{0} \xrightarrow[\text{للطرفين } -f(a)]{\text{نضيف}} f(a)\grave{-}f(a)\grave{+}f(-a)=\grave{0}\grave{-}f(a)$

$f(-a)=-f(a)$

③ $f(a-b)=f(a+(-b))=f(a)\grave{+}f(-b)$

$=f(a)\grave{+}(\grave{-}f(b))=f(a)\grave{-}f(b)$

⑤ S' is subring of R'. Then $\grave{0}\in S'$

but $f(0)=\grave{0}$ by (1)
$f(0)\in S' \Rightarrow \therefore 0\in f^{-1}(S')$

$\therefore f^{-1}(S')\neq\emptyset$ Also $f^{-1}(S')\subseteq R$ by def.

let $a,b\in f^{-1}(S') \Rightarrow f(a)\in S' \wedge f(b)\in S'$

$f(a)\grave{-}f(b)=f(a-b)$ by ③

$\therefore f(a-b)\in S' \Rightarrow a-b\in f^{-1}(S')$.

Also, $f(a)\grave{\cdot}f(b)=f(a.b)\in S'$

$\Rightarrow a.b\in f^{-1}(S')$

$\therefore f^{-1}(S')$ is a subring of R.

⑥

$\bar{0}=f(0)\in f(I) \quad \therefore f(I)\neq\emptyset$

Also, $f(I)\subseteq R'$ by def.

let $y_1, y_2 \in f(I)$ means $\exists\ x_1, x_2 \in I$ s.t.

$y_1 = f(x_1)$ and $y_2 = f(x_2)$

$\therefore y_1 - y_2 = f(x_1) - f(x_2) = f(x_1 - x_2)$

but $x_1 - x_2 \in I$, since $x_1, x_2 \in I$ and I is an ideal of R.

$\therefore y_1 - y_2 = f(\underbrace{x_1 - x_2}_{\in I}) \in f(I)$

Now, $\because f: R \to R'$ onto, $c \in R'$

$\therefore \exists\ r \in R$ s.t. $c = f(r)$

$\therefore c \cdot y_1 = f(r) \cdot f(x_1) = f(r \cdot x_1) \in f(I)$

$\therefore f(I)$ is an ideal of R'

Definition:

let $f: (R, +, \cdot) \to (R', +', \cdot')$ be a ring homo. The set $\{x : x \in R, f(x) = \bar{0}\}$ where $\bar{0}$ is the zero element of R' is called the kernal of f and denoted by $\ker(f)$. Since $\ker f = f^{-1}\{\bar{0}\}$

Remark:

$\ker f$ is an ideal of R.

Since $\ker f = f^{-1}\{0'\}$ and $\{0'\}$ is an ideal of R' Then by Theorem of properties of homo number 7 $\ker f$ is an ideal of R.

Theorem :- let $f:(R,+,\cdot) \longrightarrow (R',+',\cdot')$ be a ring homo. Then $\ker f = \{0\} \Longleftrightarrow f$ is (1-1).

Proof :- $\Rightarrow$) Suppose that $\ker f = \{0\}$

Now, let $x, y \in R$ and $f(x) = f(y) \Rightarrow f(x) - f(y) = \bar{0}$

$f(x-y) = \grave{0}$ (since f is homo.) $\Rightarrow x-y \in \ker f = \{0\}$

$\therefore x-y = 0 \Longrightarrow x = y$. Therefore f is (1-1).

$\Leftarrow$) Suppose f is (1-1).

let $x \in \ker f \Longrightarrow f(x) = 0 \Rightarrow f(x) = f(0)$

$\Longrightarrow x = 0 \Longrightarrow \ker f = \{0\}$

Example :-

let $f: R \longrightarrow R$ be homo. and $f(x,y) = (x,0)$ $\forall (x,y) \in R$, Where $R = (\mathbb{Z},+,\cdot) \times (\mathbb{Z},+,\cdot)$.
direct sum

Then find $\ker f$. $f(x,y) = (x,0)$

$\ker f = \{(x,y) : f(x,y) = (\grave{0},\grave{0})\} = \{(x,y) : (x,0) = (\grave{0},\grave{0})\}$

$= \{(x,y) : x = 0\} = \{(0,y) : y \in \mathbb{Z}\}$

Definition :

let $f: R \longrightarrow R$ be a ring homo.

① f is Called monomorphism $\Longleftrightarrow f$ is (1-1)

② f is Called epimorphism $\Longleftrightarrow f$ is onto

③ f is called isomorphism $\Leftrightarrow$ f is (1-1) and onto.

Definition: let $(R,+,\cdot)$, $(R',+',\cdot')$ be two rings

R is called isomorphic to R' $(R \cong R') \Leftrightarrow$
$\exists f: R \to R'$ s.t. f is an isomorphism.

Examples:

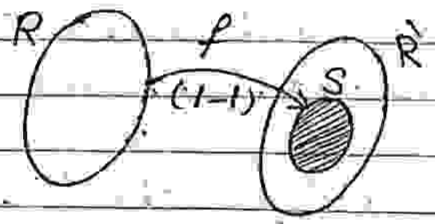
① $(Z_4, +_4, \cdot_4)$ is not isom. to $(Z_2, +_2, \cdot_2)$

since $\nexists f: Z_4 \to Z_2$ s.t. f is (1-1)

② If R has n element, $\bar{R}$ has m element s.t. $m \neq n$, then $R \ncong R'$.

Definition: (تعريف حلقة مغمورة في حلقة)

let $(R,+,\cdot)$ and $(R',+',\cdot')$ be two rings, R is said to be embeded in R' iff $\exists (S,+',\cdot')$ subring of $(R',+',\cdot')$ s.t. $R \cong S$.



Examples:

① Let $R = Z \times Z$ be a ring. Then show that $(Z,+,\cdot)$ can be embeded in R.

Solution:

let $S = \{(a,0) : a \in Z\}$. Then S is a subring of R.

Define $f: \mathbb{Z} \to S$ by $f(a) = (a, 0)$

$\therefore f$ is an isomorphism

$\therefore \mathbb{Z}$ Can be embeded in R.

② $(\mathbb{Z}_2, +_2, \cdot_2)$ Cannot be embeded in $(\mathbb{Z}_3, +_3, \cdot_3)$

لان $\mathbb{Z}_3$ لا توجد لديها subring تحوي على عنصرين وتحقق شروط ان تكون الدالة بينهما isomorphism

Theorem:

let $(R, +, \cdot)$ be a ring. Then there exists a ring R' with unity such that R is embeded in R'.

Proof:-

let $R' = \{(a, n) : a \in R \wedge n \in \mathbb{Z}\} = R \times \mathbb{Z}$

Define $+', \cdot'$ on R' by:

$(a, n) +' (b, m) = (a + b, n + m) \in R'$

$(a, n) \cdot' (b, m) = (\underbrace{a \cdot b + nb + ma}_{\in R}, \underbrace{mn}_{\in \mathbb{Z}}) \in R'$

$\therefore (R', +', \cdot')$ is a ring (check).

The unity of R' is $(0, 1)$, since

$(a, n) \cdot' (0, 1) = (a \cdot 0 + n \cdot 0 + 1 \cdot a, n) = (a, n) \quad \forall (a, n) \in R'$.

let $S = \{(a, 0) : a \in R\} \subseteq R'$. Then S is a subring of R' (check).

Define $f: (R, +, \cdot) \to (S, +', \cdot')$ s.t. $f(a) = (a, 0)$

$\therefore f$ is an isomo. (check). which Complets the proof.

Definition :-

let $(R,+,\cdot)$ be a ring, I is an ideal of R, the map, $f:(R,+,\cdot)\longrightarrow(R/I,+,\cdot)$ s.t. $f(x)=x+I$ is called the natural mapping (denoted by nat_I) $\forall x\in R$.

nat_I is ring homo, onto.

Theorem :- let $f:(R,+,\cdot)\longrightarrow(R',+',\cdot')$ be a ring homo, onto. If R is principle ideal ring, then R' is a P.I.R.

Proof :-

let $f:(R,+,\cdot)\longrightarrow(R',+',\cdot')$ be a ring homo. onto, and let I' be any ideal of R'.

To prove that R' is a P.I.R, we must prove that I' is a principle ideal of R'.

$\because I'$ is an ideal of $R'\Longrightarrow f^{-1}(I')$ is an ideal of R

let $f^{-1}(I')=I$. But R is P.I.R. then I is a P.I.

means $\exists\, a\in R$ s.t. $I=id(a)$.

we claim that $I'=id(f(a))$

let $y\in id(f(a))\Longrightarrow y=r'\cdot' f(a)$ for some $r'\in R'$

but f is onto, $r'\in R'$, so $\exists\, r\in R$ s.t. $r'=f(r)$

$\therefore y=f(r)\cdot' f(a)=f(ra)\in f(I)$

$\therefore y\in f(I)\Longrightarrow y\in I'\Longrightarrow\therefore id(f(a))\subseteq I'$

Now, To prove $I'\subseteq id(f(a))$

let $y\in I'\Longrightarrow y\in f(I)$ since $f(I)=I'$, f is onto and I' ideal of $R'\Longrightarrow y=r'\cdot' f(a)=f(ra)$

$\therefore y\in id(f(a))$.

The fundamental theorems of ring homo.

(المبرهنات الاساسية للتشاكل الحلقي)

Theorem(1): First fund. theorem (المبرهنة الاولى)

If $f: R \longrightarrow R'$ is an R-homo., then $R/\ker f \cong \operatorname{Im} f$

Proof: Define $g: R/\ker f \longrightarrow \operatorname{Im} f$ s.t

$$g(x + \ker f) = f(x) \quad \forall x \in R$$

T.P. g is well define ?

let $x_1 + \ker f = x_2 + \ker f$ where $x_1 + \ker f$ and $x_2 + \ker f \in R/\ker f$

$\Rightarrow x_1 - x_2 \in \ker f \Rightarrow f(x_1 - x_2) = 0$

$f(x_1) - f(x_2) = 0 \Rightarrow f(x_1) = f(x_2) \Rightarrow g(x_1 + \ker f) = g(x_2 + \ker f)$

$\therefore$ g is well define.

T.P. g is homo. ?

let $x_1 + \ker f,\ x_2 + \ker f \in R/\ker f$

① $g[(x_1 + \ker f) \oplus (x_2 + \ker f)] = g[(x_1 + x_2) + \ker f]$

$= f(x_1) \oplus f(x_2) = g(x_1 + \ker f) \oplus g(x_2 + \ker f)$

② $x_2 + \ker f, x_1 + \ker f \in R/\ker f$

$g[(x_1 + \ker f) \odot (x_2 + \ker f)] = g(x_1 x_2 + \ker f) = f(x_1 x_2)$

$= f(x_1) \odot f(x_2) = g(x_1 + \ker f) \odot g(x_2 + \ker f)$

$\therefore f$ is homo.

T.P. g is $(1-1)$?

عكس برهان well-define من اسفل الى اعلى
اذا بغت

let $g(x_1 + \ker f) = g(x_2 + \ker f)$

$\Rightarrow f(x_1) = f(x_2) \Rightarrow f(x_1) - f(x_2) = 0 \Rightarrow f(x_1 - x_2) = 0$

$\therefore x_1 - x_2 \in \ker f \Rightarrow x_1 + \ker f = x_2 + \ker f$.

T.P. g is onto?

let y be any element of $\operatorname{Im} f$

$\therefore \exists x \in R$ s.t. $f(x) = y$

$\Rightarrow g(x + \ker f) = y$ $\quad \therefore g$ is onto.

Corollary :- If $f : R \to R'$ is an epimorphism, then $R/\ker f \cong R'$

Theorem (2) : Second Fun. Theorem (المبرهنة الثانية)

If K and L are two ideals of a ring R, then

$K/L \cap K \cong K + L/L$

Proof : Define $f : K \to (K+L)/L$ s.t.
$f(x) = x + L \quad \forall x \in K$

$\forall x \in K \Rightarrow x + 0 \in K + L \Rightarrow x + L \in K + L/L$

T.p. f is well define

if $x_1 = x_2 \Rightarrow x_1 + L = x_2 + L \Rightarrow f(x_1) = f(x_2)$

$\therefore f$ is well define.

T.P. f is homo?

let $x_1, x_2 \in K$

$f(x_1+x_2) = (x_1+x_2)+L = (x_1+L)+(x_2+L) = f(x_1)+f(x_2)$

$f(x_1 x_2) = x_1 x_2 + L = (x_1+L)\odot(x_2+L) = f(x_1)\cdot f(x_2)$

T.P. f is onto?

let $(x+y)+L \in (K+L)/L$ where $x \in K \wedge y \in L$

$(x+y)+L = (x+L)+(y+L) = (x+L)+L = x+L = f(x)$.

By Theorem (1) $K/\ker f \cong (K+L)/L$

T.P. $\ker f = L \cap K$

$\ker f = \{x \in K : f(x) = 0\} = \{x \in K : x+L = L\}$

$= \{x \in K \wedge x \in L\} = K \cap L$.

Therefore $K/L\cap K \cong K+L/L$.

Theorem (3) Third theorem (المبرهنة الثالثة)

If K and L two ideals of a ring R and $K \subseteq L$, then $R/K / L/K \cong R/L$.

Proof: Define $f : R/K \to R/L$ s.t. $f(x+K) = x+L$ $\forall x \in R$.

T.P. f is well define?

$\forall x+K \in R/K \Rightarrow x+L \in R/L \Rightarrow f(x+K) \in R/L$.

If $x_1+K = x_2+K \Rightarrow x_1 - x_2 \in K \Rightarrow x_1 - x_2 \in L$

$\Rightarrow x_1+L = x_2+L \Rightarrow f(x_1+K) = f(x_2+K)$.

T.P. f is homo?

$\forall\ x_1+K,\ x_2+K \in R/K$

① $f[(x_1+K)\oplus(x_2+K)] = f[(x_1+x_2)+K]$

$= (x_1+x_2)+L = (x_1+L)+(x_2+L) = f(x_1+K)+f(x_2+K)$

② $f[(x_1+K)\odot(x_2+K)] = f[(x_1x_2)+K]$

$= x_1x_2+L = (x_1+L)\odot(x_2+L) = f(x_1+K)\odot f(x_2+K)$.

T.P. f is onto?

$\mathrm{Im}f = \{f(x+K) : x\in R\} = \{x+L \mid x\in R\} = R/L$

$\therefore f$ is onto.

By theorem (1) $\Longrightarrow R/K/\ker f \cong R/L$.

T.P. $\ker f = L/K$

$\ker f = \{(x+K)\in R/K : f(x+K) = L\}$

$= \{(x+K)\in R/K \mid x+L = L\}$

$= \{x+K \in R/K : x\in L\} = L/K$

$\therefore R/K / L/K \cong R/L$.