التفاوت The Variance

يعرف التفاوت (اللاتحديد) بأنه الجذر التربيعي لمتوسط مربع انحراف القياس عن القيمة المتوقعة.

$$\Delta A = \langle (A - \langle A \rangle)^2 \rangle^{1/2}$$

$$\therefore (\Delta A)^2 = \langle (A - \langle A \rangle)^2 \rangle$$
$$= \int \psi^* (A - \langle A \rangle)^2 \psi \, dv$$
$$= \int \psi^* (A^2 - 2A\langle A \rangle + \langle A \rangle^2) \psi \, dv$$
$$= \int \psi^* A^2 \psi \, dv - \int \psi^* 2A\langle A \rangle \psi \, dv + \int \psi^* \langle A \rangle^2 \psi \, dv$$
$$= \langle A^2 \rangle - 2\langle A \rangle \langle A \rangle + \langle A \rangle^2$$
$$\therefore (\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2$$

For Examples

1 Position

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 \quad \text{--- ①}$$

and $\langle x^2 \rangle = \int \psi^* x^2 \psi \, dx$

$\langle x \rangle^2 = \left(\int \psi^* x \psi \, dx \right)^2$ أي نحسب القيمة المتوقعة $\langle x \rangle$ ثم نربع النتيجة

وبعد حساب $\langle x^2 \rangle$ ، $\langle x \rangle^2$ نعوض في ① لحساب التفاوت في الموقع Δx بعد جذر النتيجة الأجمالية

2 Momentum

$$(\Delta P)^2 = \langle P^2 \rangle - \langle P \rangle^2 \quad \text{--- ②}$$

$\langle p^2 \rangle = \int \psi^* \hat{p}^2 \psi \, dv$

and $\langle p \rangle^2 = \left(\int \psi^* \hat{p} \psi \, dv \right)^2$ أي نحسب القيمة المتوقعة $\langle p \rangle$ ثم نربع النتيجة.

وبعد حساب $\langle p^2 \rangle$ ، $\langle p \rangle^2$ نعوض في معادلة ② لحساب التفاوت في الزخم Δp بعد جذر النتيجة الأجمالية.

المعادلة الذاتية وثوابت الحركة Eigen function and Constant of motion

اذا أثر المؤثر $\hat{A}$ على الدالة ψ فحولها الى عدد a مضروباً في الدالة ψ كما هو في المعادلة التالية :

$$\hat{A}\psi = a\psi$$

عندئذ تسمى ψ بالدالة الذاتية eigenfunction للمؤثر $\hat{A}$ وان a هي القيمة الذاتية eigen Value ، وان القيمة المتوقعة Expectation Value للكمية A هي :

$$\langle A \rangle = \int \psi^* \hat{A} \psi \, dv$$

but $\hat{A}\psi = a\psi$ نعوض في المعادلة اعلاه

$$\therefore \langle A \rangle = \int \psi^* a \psi \, dv = a \int \psi^* \psi \, dv$$

but $\int \psi^* \psi \, dv = 1$ Normalization Condition باعتبار أن الدالة عيارية .

$$\therefore \langle A \rangle = a$$

وهذا يعني أن كل القياسات التي اجريت لقياس المتغير الفيزيائي A تعطي نفس القيمة a ويكون المتغير في هذه الحالة أي الكمية A هي ثابت الحركة Constant of the motion والمقصود بثابت الحركة هو الكمية الفيزيائية التي لا تتغير مع الزمن أي أن :

$$\frac{\partial \langle A \rangle}{\partial t} = 0$$

Q/ Derive the Motion Equation?

Sol

$$\because \langle A \rangle = \int \psi^* \hat{A} \psi \, dV$$

$$\therefore \frac{\partial}{\partial t} \langle A \rangle = \frac{\partial}{\partial t} \int \psi^* \hat{A} \psi \, dv$$

$$\therefore \frac{\partial}{\partial t} \langle A \rangle = \int \left(\frac{\partial \psi^*}{\partial t} \hat{A} \psi + \psi^* \hat{A} \frac{\partial \psi}{\partial t} \right) dv \quad \text{---} (1)$$

$$\left.\begin{aligned} &\because i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi \\ &-i\hbar \frac{\partial \psi^*}{\partial t} = \hat{H}^* \psi^* \quad \text{Complex Conjucate} \end{aligned}\right\} \text{----} (2) \quad \begin{aligned} \frac{\partial \psi}{\partial t} &= -\frac{i}{\hbar} \hat{H} \psi \\ \frac{\partial \psi^*}{\partial t} &= +\frac{i}{\hbar} \hat{H}^* \psi^* \end{aligned}$$

(2) in (1) $\Rightarrow$

$$\frac{\partial}{\partial t} \langle A \rangle = \int \left(\frac{i}{\hbar} (\hat{H}^* \psi^*) \hat{A} \psi - \frac{i}{\hbar} \psi^* \hat{A} \hat{H} \psi \right) dv$$

$$\int (\hat{H}\psi)^* \hat{A}\psi \, dv = \int \psi^* \hat{H} \hat{A}\psi \, dv$$ من تعريف المؤثر الهيرميتي لدينا :- Hermition Operator

$$\frac{\partial}{\partial t}\langle A\rangle = \frac{i}{\hbar}\int (\psi^* \hat{H}\hat{A}\psi - \psi^* \hat{A}\hat{H}\psi)\, dv$$ ⟸ بالتعويض بالمعادلة ينتج

$$= \frac{i}{\hbar}\int \psi^* (\hat{H}\hat{A} - \hat{A}\hat{H})\psi \, dv$$

but $\hat{H}\hat{A} - \hat{A}\hat{H} = [\hat{H}, \hat{A}]$
and $\int \psi^* [\hat{H}, \hat{A}]\psi \, dv = \langle [\hat{H}, \hat{A}] \rangle$

$$\therefore \frac{\partial}{\partial t}\langle A\rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{A}] \rangle$$

Q/ Show that $\hat{P}_x$ of a free particle is constant of the motion?

<u>Sol</u>:-

$$\because \frac{\partial}{\partial t}\langle A\rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{A}] \rangle$$ Motion Equation

$$\therefore \frac{\partial}{\partial t}\langle P_x\rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{P}_x] \rangle$$

$$\because \hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V \Rightarrow \hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$ for a free particle جسيم

and $\hat{P}_x = -i\hbar \frac{\partial}{\partial x}$

$$\therefore \frac{\partial}{\partial t}\langle P_x\rangle = \frac{i}{\hbar}\left\langle \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}, -i\hbar\frac{\partial}{\partial x} \right] \right\rangle$$

$$= \frac{i}{\hbar}\left\langle \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi - \left(-i\hbar\frac{\partial}{\partial x}\right)\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\psi \right\rangle$$

$$= \frac{i}{\hbar}\left\langle \frac{i\hbar^3}{2m}\frac{\partial^3}{\partial x^3}\psi - \frac{i\hbar^3}{2m}\frac{\partial^3}{\partial x^3}\psi \right\rangle = 0$$

$$\because \frac{\partial}{\partial t}\langle P_x\rangle = 0$$ $\therefore P_x$ is constant of the motion

نظرية اهرنفست Ehrenfist's theorem

أن النتيجة التي يتنبأ بها الميكانيك الكلاسيكي هي في الواقع القيمة المتوقعة Expectation value لتلك الكمية تبعاً لقوانين الميكانيك الكمي

① $\frac{d}{dt}\langle x\rangle = \frac{1}{m}\langle p\rangle$

To Prove ①

$\because \frac{d}{dt}\langle x\rangle = \frac{i}{\hbar}\langle [H, x]\rangle$ --- ① motion Equation

Now we use $H = \frac{p^2}{2m} + V(x)$

$$[H, x] = \left[\frac{p^2}{2m} + V(x), x\right]$$

$$= \left[\frac{p^2}{2m}, x\right] + [V(x), x] \quad --- ②$$

but $[V(x), x] = 0$ --- ⓐ

and $\left[\frac{\hat{p}^2}{2m}, x\right] = \left[\frac{\hat{p}}{2m}, x\right]\hat{p} + \hat{p}\left[\frac{\hat{p}}{2m}, x\right]$ --- ⓑ

$$\left[\frac{\hat{p}}{2m}, x\right]\psi = -\frac{i\hbar}{2m}\left(\frac{\partial}{\partial x}x\psi - x\frac{\partial}{\partial x}\psi\right)$$

$$= -\frac{i\hbar}{2m}\left(\psi + \cancel{x\frac{\partial\psi}{\partial x}} - \cancel{x\frac{\partial\psi}{\partial x}}\right)$$

$$= -\frac{i\hbar}{2m}\psi$$

$\therefore \left[\frac{\hat{p}}{2m}, x\right] = -\frac{i\hbar}{2m}$ in ⓑ

$$\therefore \left[\frac{\hat{p}^2}{2m}, x\right] = -\frac{i\hbar}{2m}\hat{p} + p\left(-\frac{i\hbar}{2m}\right)$$

$$= -\frac{i\hbar}{m}\hat{p} \quad --- ©$$

© and ⓐ in ② $\Rightarrow$

$[\hat{H}, \hat{x}] = -\frac{i\hbar}{m}\hat{p}$

$\therefore \frac{d}{dt}\langle x\rangle = \frac{i}{\hbar}\left\langle -\frac{i\hbar}{m}\hat{p}\right\rangle$

$\frac{d}{dt}\langle x\rangle = \frac{1}{m}\langle p_x\rangle$ او يمكن كتابتها $\langle p_x\rangle = m\frac{d\langle x\rangle}{dt}$ وهنا

يقابله لمعادلة نيوتن الكلاسيكي في الحركة $p = m\frac{dx}{dt}$ نلاحظ انه في ميكانيك الكم

② $\frac{d}{dt}\langle p\rangle = -\left\langle\frac{dV}{dx}\right\rangle$

$\because \frac{d}{dt}\langle p\rangle = \frac{i}{\hbar}\left\langle[\hat{H},\hat{P}]\right\rangle$ — ① motion Equation

Now we use $H = \frac{P^2}{2m} + V$

$\therefore [\hat{H},\hat{P}] = \left[\frac{\hat{P}^2}{2m} + \hat{V}, \hat{P}\right]$

$[\hat{H},\hat{P}] = \left[\frac{\hat{P}^2}{2m}, P\right] + [\hat{V},\hat{P}]$ --- ②

but $\left[\frac{\hat{P}^2}{2m},\hat{P}\right]\psi = \frac{1}{2m}\left(\hat{P}^2\hat{P}\psi - \hat{P}\hat{P}^2\psi\right) = 0$ --- ⓐ

and $[\hat{V},\hat{P}]\psi = \hat{V}\hat{P}\psi - PV\psi$ $\quad (\hat{p} = -i\hbar\frac{\partial}{\partial x})$

$\therefore [\hat{V},\hat{P}]\psi = -i\hbar\left(V\frac{\partial}{\partial x}\psi - \frac{\partial}{\partial x}V\psi\right)$

$= -i\hbar\left(V\cancel{\frac{\partial\psi}{\partial x}} - V\cancel{\frac{\partial\psi}{\partial x}} - \psi\frac{\partial V}{\partial x}\right) = +i\hbar\frac{\partial V}{\partial x}\psi$

$\therefore [\hat{V},\hat{P}] = i\hbar\frac{\partial V}{\partial x}$ --- ⓑ

ⓐ and ⓑ in ② $\Longrightarrow$

$[\hat{H},\hat{P}] = 0 + i\hbar\frac{\partial V}{\partial x} = i\hbar\frac{\partial V}{\partial x}$ $\quad$ in eq ① $\Longrightarrow$

$\frac{d}{dt}\langle p\rangle = \frac{i}{\hbar}\left\langle i\hbar\frac{dV}{dx}\right\rangle$

$\therefore \frac{d}{dt}\langle p\rangle = -\left\langle\frac{dV}{dx}\right\rangle$

These results together (① and ②) give us Ehrenfest theorem, Which states that the laws of classical mechanics embodied in Newton's Law hold for the Expectation Values of quantum Operators x and p,
This establishes a Correspondence between classical and quantum dynamics.

Q/ Prove that $\langle E \rangle = \langle T \rangle + \langle V \rangle$ and What is the Physical meaning of the result?

Sol

$$\because \quad i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi \qquad \text{TDSE}$$

but $\hat{E} = i\hbar \frac{\partial}{\partial t}$ and $\hat{T} = \frac{-\hbar^2}{2m} \nabla^2$ (interms of operators)

$$\therefore \quad \hat{E}\psi = \hat{T}\psi + V\psi$$

$\times\, \psi^*$ بالضرب، ونأخذ التكامل

$$\int \psi^* E \psi \, dV = \int \psi^* \hat{T} \psi \, dv + \int \psi^* V \psi \, dv$$

$$\therefore \quad \langle E \rangle = \langle T \rangle + \langle V \rangle$$

نلاحظ أن النتيجة تماثل نتيجة أساسية في الميكانيك الكلاسيكي حيث تماثل في ميكانيك الكم بالقيمة المتوقعة Expectation Value.

حفظ الاحتمالية The Conservation of probability

$\because P(x,t) = \psi^*(x,t)\,\psi(x,t)$ The probability density

$\therefore \frac{\partial}{\partial t} P = \frac{\partial}{\partial t}(\psi^* \psi)$

$$\frac{\partial}{\partial t} P = \psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t} \quad \text{---- ①}$$

$$\because i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}\psi + V\psi \qquad \div i\hbar \Rightarrow$$

$$\frac{\partial \psi}{\partial t} = \frac{-\hbar}{2mi} \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{i\hbar} V \psi \quad \text{---- ⓐ}$$

and $\frac{\partial \psi^*}{\partial t} = \frac{\hbar}{2mi} \frac{\partial^2 \psi^*}{\partial x^2} - \frac{1}{i\hbar} V \psi^*$ By taking Complex Conjucate -- ⓑ

$*\,\psi^*$ from left to eq. ① and $*\,\psi$ from left to eq ⓑ $\Rightarrow$

$$\left.\begin{aligned} \psi^* \frac{\partial \psi}{\partial t} &= -\frac{\hbar}{2mi} \psi^* \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{i\hbar} V \psi^* \psi \\ \text{and } \psi \frac{\partial \psi^*}{\partial t} &= \frac{\hbar}{2mi} \psi \frac{\partial^2 \psi^*}{\partial x^2} - \frac{1}{i\hbar} V \psi \psi^* \end{aligned}\right\} \text{--- ②}$$

eq ② in eq ① $\Rightarrow$

$$\frac{\partial P}{\partial t} = -\frac{\hbar}{2mi} \psi^* \frac{\partial^2 \psi}{\partial x^2} + \cancel{\frac{1}{i\hbar} V \psi^* \psi} + \frac{\hbar}{2mi} \psi \frac{\partial^2 \psi^*}{\partial x^2} - \cancel{\frac{1}{i\hbar} V \psi \psi^*}$$

$$\therefore \frac{\partial P}{\partial t} = \frac{-\hbar}{2mi} \left(\psi^*(x,t) \frac{\partial^2}{\partial x^2} \psi(x,t) - \psi(x,t) \frac{\partial^2}{\partial x^2} \psi^*(x,t) \right)$$

We see that the result on the right side is $-\partial j/\partial x$

$\therefore \frac{\partial P}{\partial t} + \frac{\partial j}{\partial x} = 0$ The continuity equation of probability

and In three dimension this generalizes to :-

$$J(r,t) = \frac{\hbar}{2mi} \left[\psi^* \nabla \psi - \psi \nabla \psi^* \right] \quad \text{Probability Current}$$

and $\frac{\partial P}{\partial t} + \nabla \cdot J = 0$ The continuity equation of probability

والتي تعني أن احتمالية ايجاد الجسيم داخل حجم قدره V يساوي سالب احتمالية كثافة التيار خلال

Ex: Find the probability current to the wave function $\psi(x,t) = 3k\,e^{ikx}$ where k is a Constant ?

Sol

$$\because J = \frac{\hbar}{2mi}\left[\psi^{*}\nabla\psi - \psi\nabla\psi^{*}\right] \quad \text{Probability Current eq.}$$

$$\because \psi = 3k\,e^{ikx} \qquad \therefore \psi^{*} = 3k\,e^{-ikx}$$

$$\therefore \nabla\psi = 3k(ik)\,e^{ikx} \qquad \text{and} \quad \nabla\psi^{*} = 3k(-ik)\,e^{-ikx}$$

$$= 3ik^{2}e^{ikx} \qquad \text{and} \qquad = -3ik^{2}e^{-ikx}$$

$$\Rightarrow J = \frac{\hbar}{2mi}\left[3k\,e^{-ikx}(3ik^{2}e^{ikx}) - 3k\,e^{ikx}(-3ik^{2}e^{-ikx})\right]$$

$$= \frac{\hbar}{2mi}\left[9ik^{3}e^{0} + 9ik^{3}e^{0}\right] = \frac{9\hbar k^{3}}{m}$$

Ex: Find the probability Current to the wave function $\psi = 5e^{2x}$?

Sol

$$\because J = \frac{\hbar}{2mi}\left[\psi^{*}\nabla\psi - \psi\nabla\psi^{*}\right]$$

$$\because \psi = 5e^{2x} = \psi^{*}$$

$$\text{and } \nabla\psi = 10e^{2x} = \nabla\psi^{*}$$

$$\therefore J = \frac{\hbar}{2mi}\left[5e^{2x}(10e^{2x}) - 5e^{2x}(10e^{2x})\right]$$

$$= \frac{\hbar}{2mi}\left[50e^{4x} - 50e^{4x}\right]$$

$$= 0$$

Q/ The wave function for a particle Confined to $0 \leq x \leq 0.3$ was $\psi(x) = e^{3i}$ find the normalization Constant ?

Sol

$\because \int \psi^*(x)\,\psi(x)\,dx = 1$ Normalization Condition

and $\psi(x) = e^{3i} \Rightarrow \psi^*(x) = e^{-3i}$

$\therefore \int_0^{0.3} A e^{-3i} A e^{3i}\,dx = 1$

$\Rightarrow A^2 \int_0^{0.3} e^0\,dx = 1 \Rightarrow A^2 (x\big|_0^{0.3}) = 1 \Rightarrow A^2 (0.3) = 1$

$\Rightarrow A = \frac{1}{\sqrt{0.3}} \Rightarrow \psi(x) = \frac{1}{\sqrt{0.3}} e^{3i}$

Q/ A particle move with a wave function $\psi(x) = A e^{ix^2}$ at interval $(0, 0.5)$ find ⓐ The normalization Constant A ⓑ The expectation value x, p ?

Sol ⓐ $\because \int |\psi(x)|^2\,dx = 1$ Normalization Condition

and $\psi(x) = A e^{ix^2} \Rightarrow \psi^*(x) = A e^{-ix^2}$

$\therefore \int_0^{0.5} A e^{-ix^2} A e^{ix^2}\,dx = 1 \Rightarrow |A^2| \int_0^{0.5} e^{-ix^2+ix^2}\,dx = 1$

$\therefore |A^2|\, x \big|_0^{0.5} = 1 \Rightarrow A = \sqrt{2} \Rightarrow \psi(x) = \sqrt{2}\, e^{ix^2}$

Sol ⓑ $\because \langle A \rangle = \int \psi^* \hat{A} \psi\,dv$

$\therefore \langle x \rangle = \int_0^{0.5} \sqrt{2} e^{-ix^2}\, x\, \sqrt{2} e^{ix^2}\,dx = 2 \frac{x^2}{2} \Big|_0^{0.5} =$

and $\langle p \rangle = \int_0^{0.5} \sqrt{2} e^{-ix^2} \left(-i\hbar \frac{\partial}{\partial x}\right) \sqrt{2}\, e^{ix^2}\,dx$

$= -2i\hbar \int_0^{0.5} e^{-ix^2} (2ix)\, e^{ix^2}\,dx = 4\hbar \int_0^{0.5} x\,dx$

$= \hbar 4 \frac{x^2}{2} \Big|_0^{0.5}$

H.W: find the expectation value for T ?

Q/ A particle move in One - dimension with a Wave function $\psi = Ae^{-x^2}$ at interval $0 \to \infty$ find the normalization Constant?

Sol

$$\because \int_{-\infty}^{+\infty} \psi^*(x)\, \psi(x)\, dx = 1 \qquad \text{and} \qquad \psi(x) = Ae^{-x^2} = \psi^*(x)$$

$$\therefore A^2 \int_{-\infty}^{+\infty} e^{-2x^2} dx = 1 \qquad = \underline{\underline{2}}|A|^2 \int_{\underline{\underline{0}}}^{\infty} e^{-2x^2} dx = 1 \quad \text{--- } ①$$

$$\text{let } u = 2x^2 \quad \therefore x^2 = \frac{u}{2} \Rightarrow x = \frac{\sqrt{u}}{\sqrt{2}} \quad \therefore dx = \frac{1}{\sqrt{2}} \frac{1}{2} u^{-1/2} du$$

وبالتعويض في معادلة ①

$$2|A|^2 \int_0^{\infty} e^{-u} \frac{1}{\sqrt{2}} \frac{1}{2} u^{-1/2} du = 1$$

$$\therefore \frac{1}{\sqrt{2}} |A|^2 \int_0^{\infty} u^{-1/2} e^{-u} du = 1$$

$$\text{but} \quad \int_0^{\infty} u^{-1/2} e^{-u} du = \sqrt{\pi}$$

بالتعويض بالدالة الرياضية

$$\therefore \frac{1}{\sqrt{2}} |A|^2 (\sqrt{\pi}) = 1 \qquad \therefore |A|^2 = \frac{\sqrt{2}}{\sqrt{\pi}} \qquad \therefore A = \left(\frac{2}{\pi}\right)^{1/4}$$

Q/ suppose that a Certain probability distribution is given by $P(x) = \frac{9}{4} \frac{1}{x^3}$ for $1 \le x \le 3$ Find the probability in $\frac{5}{2} \le x \le 3$?

Sol

$$\because P(x) = \int |\psi(x)|^2 dx$$

$$\therefore P(x) = \frac{9}{4} \int_{5/2}^{3} \frac{1}{x^3} dx \quad = \frac{9}{4} \frac{x^{-2}}{-2} \Big|_{5/2}^{3}$$

$$= +\frac{9}{8} \left(\frac{1}{9} - \frac{4}{25} \right) = 0.055$$

$\therefore$ 5.5 percent chance to find the particle in $5/2 \le x \le 3$.

30

Q/ A free particle with a wave function $\psi(x) = e^{ikx}$ is forced to move along $0 \leq x \leq 1$ and $\psi(x) = 0$ elsewhere, determine the probability for finding the particle in interval $(0.25, 0.75)$ and find Δx (the variance for position)?

Sol

$$\therefore P(x) = \int \psi^*_{(x)} \psi(x) \, dx$$

and $\psi(x) = e^{ikx} \Rightarrow \psi^*(x) = e^{-ikx}$

$$\therefore P(x) = \int_{0.25}^{0.75} e^{-ikx} e^{ikx} dx = \int_{0.25}^{0.75} dx$$

$$= x \Big|_{0.25}^{0.75} = (0.75 - 0.25) = 0.5$$

$$\therefore (\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 \qquad \text{The Variance}$$

$$\langle x \rangle = \int \psi^* x \psi \, dx = \int_0^1 e^{-ikx} x \, e^{ikx} dx$$

$$= \frac{x^2}{2} \Big|_0^1 = \frac{1}{2} \Rightarrow \langle x \rangle^2 = \frac{1}{4}$$

and $\langle x^2 \rangle = \int \psi^* x^2 \psi \, dx$

$$= \int_0^1 e^{-ikx} x^2 e^{ikx} dx = \int_0^1 x^2 dx$$

$$= \frac{x^3}{3} \Big|_0^1 = \frac{1}{3}$$

$$\therefore (\Delta x)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\therefore \Delta x = \sqrt{\frac{1}{12}}$$