Formation of Partial Differential Equations

Saad Al-Momen

1

MATHEMATICAL PHYSICS I

Master Degree Class
Department of Astronomy and Space
College of Science - University of Baghdad

1. Formation of PDE by elimination of arbitrary constants

“

- If the number of arbitrary constants to be eliminated is **equal** to the number of independent variables, we obtained a **1st** order PDE.

$$z = (x^2+a^2)(y^2+b^2)$$

$$z = (x^2+a^2)(y^2+b^2)$$

$$z_x = (y^2+b^2)(2x) \quad \cdots \quad (1)$$

$$z_y = (x^2+a^2)(2y) \quad \cdots \quad (2)$$

$$z = (x^2+a^2)(y^2+b^2)$$

$$z_x = (y^2+b^2)(2x) \quad \cdots \text{①}$$

$$z_y = (x^2+a^2)(2y) \quad \cdots \text{②}$$

$$\text{①} \Rightarrow \quad y^2+b^2 = \frac{z_x}{2x}$$

$$z = (x^2+a^2)(y^2+b^2)$$

$$z_x = (y^2+b^2)(2x) \quad \cdots \quad (1)$$

$$z_y = (x^2+a^2)(2y) \quad \cdots \quad (2)$$

$$(1) \Rightarrow \quad y^2+b^2 = \frac{z_x}{2x}$$

$$(2) \Rightarrow \quad x^2+a^2 = \frac{z_y}{2y}$$

$$z = (x^2+a^2)(y^2+b^2)$$

$$z_x = (y^2+b^2)(2x) \quad \cdots \quad (1)$$

$$z_y = (x^2+a^2)(2y) \quad \cdots \quad (2)$$

$$(1) \Rightarrow \quad y^2+b^2 = \frac{z_x}{2x}$$

$$(2) \Rightarrow \quad x^2+a^2 = \frac{z_y}{2y}$$

$$\therefore \quad z = \frac{z_y}{2y} \cdot \frac{z_x}{2x} \Rightarrow \boxed{z_x z_y - 4xyz = 0}$$

“

- If the number of arbitrary constants to be eliminated is **more** than the number of independent variables, we get PDE of **2nd** or **higher** order.

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

Diff w.r.t. x $\Rightarrow$ $2(x-a)+2(z-c)z_x=0$ ... (1)

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

Diff w.r.t. x $\Rightarrow$ $2(x-a)+2(z-c)z_x=0$ ---- (1)

Diff w.r.t. y $\Rightarrow$ $2(y-b)+2(z-c)z_y=0$ ---- (2)

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

Diff w.r.t. $x \Rightarrow 2(x-a)+2(z-c)z_x=0$ ... (1)

Diff w.r.t. $y \Rightarrow 2(y-b)+2(z-c)z_y=0$ ... (2)

Diff (1) w.r.t. $x \Rightarrow 1+(z-c)z_{xx}+z_x^2=0$ ... (3)

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

Diff w.r.t. x $\Rightarrow$ $2(x-a)+2(z-c)z_x=0$ --- (1)

Diff w.r.t. y $\Rightarrow$ $2(y-b)+2(z-c)z_y=0$ --- (2)

Diff (1) w.r.t. x $\Rightarrow$ $1+(z-c)z_{xx}+z_x^2=0$ --- (3)

Diff (2) w.r.t. y $\Rightarrow$ $1+(z-c)z_{yy}+z_y^2=0$ --- (4)

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

Diff w.r.t. $x \Rightarrow 2(x-a)+2(z-c)z_x=0$ --- (1)

Diff w.r.t. $y \Rightarrow 2(y-b)+2(z-c)z_y=0$ --- (2)

Diff (1) w.r.t. $x \Rightarrow 1+(z-c)z_{xx}+z_x^2=0$ --- (3)

Diff (2) w.r.t. $y \Rightarrow 1+(z-c)z_{yy}+z_y^2=0$ --- (4)

from 3 we get $(z-c)=-\frac{(1+z_x^2)}{z_{xx}}$

from 4 we get $(z-c)=-\frac{(1+z_y^2)}{z_{yy}}$

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

Diff w.r.t. x $\Rightarrow$ $2(x-a)+2(z-c)z_x=0$ --- (1)

Diff w.r.t. y $\Rightarrow$ $2(y-b)+2(z-c)z_y=0$ --- (2)

Diff (1) w.r.t. x $\Rightarrow$ $1+(z-c)z_{xx}+z_x^2=0$ --- (3)

Diff (2) w.r.t. y $\Rightarrow$ $1+(z-c)z_{yy}+z_y^2=0$ --- (4)

from 3 we get $(z-c)=-\dfrac{(1+z_x^2)}{z_{xx}}$

from 4 we get $(z-c)=-\dfrac{(1+z_y^2)}{z_{yy}}$

$$\therefore \frac{(1+z_x^2)}{z_{xx}}=\frac{(1+z_y^2)}{z_{yy}}$$

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

Diff w.r.t. x $\Rightarrow$ $2(x-a)+2(z-c)z_x=0$ ... (1)

Diff w.r.t. y $\Rightarrow$ $2(y-b)+2(z-c)z_y=0$ ... (2)

Diff (1) w.r.t. x $\Rightarrow$ $1+(z-c)z_{xx}+z_x^2=0$ ... (3)

Diff (2) w.r.t. y $\Rightarrow$ $1+(z-c)z_{yy}+z_y^2=0$ ... (4)

from 3 we get $(z-c)=-\dfrac{(1+z_x^2)}{z_{xx}}$

from 4 we get $(z-c)=-\dfrac{(1+z_y^2)}{z_{yy}}$

$$\therefore \frac{(1+z_x^2)}{z_{xx}}=\frac{(1+z_y^2)}{z_{yy}}$$

$$\Rightarrow \boxed{(1+z_x^2)z_{yy}-(1+z_y^2)z_{xx}=0}$$

2. Formation of PDE by elimination of arbitrary functions

“

- The elimination of **one** arbitrary function from a given relation gives a PDE of **1st** order.

$$z = x f\left(\frac{x}{y}\right) \quad \text{---} \quad (*)$$

$$z = x f\left(\frac{x}{y}\right) \quad \text{---} \quad (*)$$

$$z_x = x f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right) + f\left(\frac{x}{y}\right) = \frac{x}{y} f' + f \quad \text{---} \quad (1)$$

$$z_y = x f'\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right) + 0 = -\frac{x^2}{y^2} f' \quad \text{---} \quad (2)$$

$$z = x f\left(\frac{x}{y}\right) \quad \text{--- } (*)$$

$$z_x = x f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right) + f\left(\frac{x}{y}\right) = \frac{x}{y} f' + f \quad \text{--- } (1)$$

$$z_y = x f'\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right) + 0 = -\frac{x^2}{y^2} f' \quad \text{--- } (2)$$

from $(*)$ $f = \frac{z}{x}$

from (2) $f' = -\frac{y^2}{x^2} z_y$

$$z = x f\left(\frac{x}{y}\right) \quad \cdots \circledast$$

$$z_x = x f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right) + f\left(\frac{x}{y}\right) = \frac{x}{y} f' + f \quad \cdots ①$$

$$z_y = x f'\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right) + 0 = -\frac{x^2}{y^2} f' \quad \cdots ②$$

from $\circledast$ $f = \frac{z}{x}$

from ② $f' = -\frac{y^2}{x^2} z_y$

$$① \Rightarrow z_x = \frac{x}{y}\left(-\frac{y^2}{x^2}\right) z_y + \frac{z}{x}$$

$$z = x f\left(\frac{x}{y}\right) \quad \cdots \circledast$$

$$z_x = x f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right) + f\left(\frac{x}{y}\right) = \frac{x}{y} f' + f \quad \cdots ①$$

$$z_y = x f'\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right) + 0 = -\frac{x^2}{y^2} f' \quad \cdots ②$$

from $\circledast$ $f = \frac{z}{x}$

from ② $f' = -\frac{y^2}{x^2} z_y$

① $\Rightarrow z_x = \frac{x}{y}\left(-\frac{y^2}{x^2}\right) z_y + \frac{z}{x}$

$$\Rightarrow z_x = -\frac{y}{x} z_y + \frac{z}{x}$$

$$z = x f\left(\frac{x}{y}\right) \quad \cdots \circledast$$

$$z_x = x f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right) + f\left(\frac{x}{y}\right) = \frac{x}{y} f' + f \quad \cdots (1)$$

$$z_y = x f'\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right) + 0 = -\frac{x^2}{y^2} f' \quad \cdots (2)$$

from $\circledast$ $f = \frac{z}{x}$

from (2) $f' = -\frac{y^2}{x^2} z_y$

(1) $\Rightarrow$ $z_x = \frac{x}{y}\left(-\frac{y^2}{x^2}\right) z_y + \frac{z}{x}$

$\Rightarrow$ $z_x = -\frac{y}{x} z_y + \frac{z}{x}$

$\Rightarrow$ $\boxed{x z_x + y z_y = z}$

“

- Elimination of **two** arbitrary functions from a given relation give a **2nd** or **higher** order PDE.

$$Z = x^2 f(y) + y^2 g(x) \quad \cdots \quad (1)$$

$$Z = x^2 f(y) + y^2 g(x) \quad \cdots \quad (1)$$

Diff (1) w.r.t. x $\Rightarrow$ $Z_x = 2x f(y) + y^2 g'(x) \quad \cdots \quad (2)$

$$Z = x^2 f(y) + y^2 g(x) \quad \text{---} \quad (1)$$

Diff (1) w.r.t. x $\Rightarrow$ $Z_x = 2x f(y) + y^2 g'(x) \quad \text{---} \quad (2)$

Diff (1) w.r.t. y $\Rightarrow$ $Z_y = x^2 f'(y) + 2y g(x) \quad \text{---} \quad (3)$

$$Z = x^2 f(y) + y^2 g(x) \quad \text{---} \ (1)$$

Diff (1) w.r.t. x $\Rightarrow$ $Z_x = 2x f(y) + y^2 g'(x)$ --- (2)

Diff (1) w.r.t. y $\Rightarrow$ $Z_y = x^2 f'(y) + 2y g(x)$ --- (3)

Diff (2) w.r.t. y $\Rightarrow$ $Z_{xy} = 2x f'(y) + 2y g'(x)$ ---- (4)

$$z = x^2 f(y) + y^2 g(x) \quad \text{---} \quad (1)$$

Diff (1) w.r.t. $x \Rightarrow z_x = 2x f(y) + y^2 g'(x) \quad \text{---} \quad (2)$

Diff (1) w.r.t. $y \Rightarrow z_y = x^2 f'(y) + 2y g(x) \quad \text{---} \quad (3)$

Diff (2) w.r.t. $y \Rightarrow z_{xy} = 2x f'(y) + 2y g'(x) \quad \text{----} \quad (4)$

multiply (2) by x and (3) by y and add them togather

$$x z_x + y z_y = 2x^2 f(y) + x y^2 g'(x) + x^2 y f'(y) + 2y^2 g(x) \quad \text{---} \quad (4)$$

$$Z = x^2 f(y) + y^2 g(x) \quad \cdots \text{①}$$

Diff ① w.r.t. x $\Rightarrow$ $Z_x = 2x f(y) + y^2 g'(x) \quad \cdots \text{②}$

Diff ① w.r.t. y $\Rightarrow$ $Z_y = x^2 f'(y) + 2y g(x) \quad \cdots \text{③}$

Diff ② w.r.t. y $\Rightarrow$ $Z_{xy} = 2x f'(y) + 2y g'(x) \quad \cdots \text{④}$

multiply ② by x and ③ by y and add them togather

$$xZ_x + yZ_y = 2x^2 f(y) + xy^2 g'(x) + x^2 y f'(y) + 2y^2 g(x) \quad \cdots \text{④}$$

$$\Rightarrow xZ_x + yZ_y = 2(\underbrace{x^2 f(y) + y^2 g(x)}_{Z}) + xy(\underbrace{y g'(x) + x f'(y)}_{\frac{Z_{xy}}{2}})$$

$$Z = x^2 f(y) + y^2 g(x) \quad \cdots \; ①$$

Diff ① w.r.t. x $\Rightarrow$ $Z_x = 2x f(y) + y^2 g'(x) \quad \cdots \; ②$

Diff ① w.r.t. y $\Rightarrow$ $Z_y = x^2 f'(y) + 2y g(x) \quad \cdots \; ③$

Diff ② w.r.t. y $\Rightarrow$ $Z_{xy} = 2x f'(y) + 2y g'(x) \quad \cdots \; ④$

multiply ② by x and ③ by y and add them togather

$$xZ_x + yZ_y = 2x^2 f(y) + xy^2 g'(x) + x^2 y f'(y) + 2y^2 g(x) \quad \cdots \; ④$$

$$\Rightarrow xZ_x + yZ_y = 2(\underbrace{x^2 f(y) + y^2 g(x)}_{Z}) + xy(\underbrace{y g'(x) + x f'(y)}_{\frac{Z_{xy}}{2}})$$

$$\Rightarrow \boxed{2xZ_x + 2yZ_y - xy Z_{xy} = 4Z}$$

3. Formation of PDE by elimination of arbitrary function $\phi(u,v)=0$

- Formation of PDE by eliminating of arbitrary function ϕ from $\phi(u,v)$, where u and v are functions of x and y.

$$g\left(\frac{y}{x}, x^2+y^2+z^2\right)=0$$

$$g\left(\frac{y}{x}, x^2+y^2+z^2\right)=0$$

let $u=\frac{y}{x}$, $v=x^2+y^2+z^2$

$$g(u,v)=0$$

$$g\left(\frac{y}{x}, x^2+y^2+z^2\right)=0$$

let $u=\frac{y}{x}$, $v=x^2+y^2+z^2$

$$g(u,v)=0$$

$u_x=-\frac{y}{x^2}$ $u_y=\frac{1}{x}$

$$g\left(\frac{y}{x}, x^2+y^2+z^2\right)=0$$

let $u=\frac{y}{x}$, $v=x^2+y^2+z^2$

$$g(u,v)=0$$

$u_x=-\frac{y}{x^2}$ $u_y=\frac{1}{x}$

$v_x=2x+2zz_x$ $v_y=2y+2zz_y$

$$g\left(\frac{y}{x}, x^2+y^2+z^2\right)=0$$

let $u=\frac{y}{x}$, $v=x^2+y^2+z^2$

$$g(u,v)=0$$

$u_x=-\frac{y}{x^2}$ $\qquad$ $u_y=\frac{1}{x}$

$v_x=2x+2zz_x$ $\qquad$ $v_y=2y+2zz_y$

$$\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}=0$$

$$g\left(\frac{y}{x}, x^2+y^2+z^2\right)=0$$

let $u=\frac{y}{x}$, $v=x^2+y^2+z^2$

$$g(u,v)=0$$

$u_x=-\frac{y}{x^2}$ $\quad$ $u_y=\frac{1}{x}$

$v_x=2x+2zz_x$ $\quad$ $v_y=2y+2zz_y$

$$\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}=0$$

$$\begin{vmatrix} -\frac{y}{x^2} & 2x+2zz_x \\ \frac{1}{x} & 2y+2zz_y \end{vmatrix}=0$$

$$g\left(\frac{y}{x}, x^2+y^2+z^2\right)=0$$

let $u=\frac{y}{x}$, $v=x^2+y^2+z^2$

$$g(u,v)=0$$

$u_x=-\frac{y}{x^2}$ $u_y=\frac{1}{x}$

$v_x=2x+2zz_x$ $v_y=2y+2zz_y$

$$\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}=0$$

$$\begin{vmatrix} -\frac{y}{x^2} & 2x+2zz_x \\ \frac{1}{x} & 2y+2zz_y \end{vmatrix}=0$$

$$-\frac{y}{x^2}(2y+2zz_y)-\frac{1}{x}(2x+2zz_x)=0$$

$$g\left(\frac{y}{x}, x^2+y^2+z^2\right)=0$$

let $u=\frac{y}{x}$, $v=x^2+y^2+z^2$

$$g(u,v)=0$$

$$u_x = -\frac{y}{x^2} \qquad u_y=\frac{1}{x}$$

$$v_x = 2x+2zz_x \qquad v_y=2y+2zz_y$$

$$\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}=0$$

$$\begin{vmatrix} -\frac{y}{x^2} & 2x+2zz_x \\ \frac{1}{x} & 2y+2zz_y \end{vmatrix}=0$$

$$-\frac{y}{x^2}(2y+2zz_y)-\frac{1}{x}(2x+2zz_x)=0$$

$$y(2y+2zz_y)+x(2x+2zz_x)=0$$

$$g\left(\frac{y}{x}, x^2+y^2+z^2\right)=0$$

let $u=\frac{y}{x}$, $v=x^2+y^2+z^2$

$$g(u,v)=0$$

$u_x=-\frac{y}{x^2}$ $\quad u_y=\frac{1}{x}$

$v_x=2x+2zz_x$ $\quad v_y=2y+2zz_y$

$$\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}=0$$

$$\begin{vmatrix} -\frac{y}{x^2} & 2x+2zz_x \\ \frac{1}{x} & 2y+2zz_y \end{vmatrix}=0$$

$$-\frac{y}{x^2}(2y+2zz_y)-\frac{1}{x}(2x+2zz_x)=0$$

$$y(2y+2zz_y)+x(2x+2zz_x)=0$$

$$2y^2+2yzz_y+2x^2+2xzz_x=0 \quad /2$$

$$g\left(\frac{y}{x}, x^2+y^2+z^2\right)=0$$

let $u=\frac{y}{x}$, $v=x^2+y^2+z^2$

$$g(u,v)=0$$

$$u_x=-\frac{y}{x^2} \qquad u_y=\frac{1}{x}$$

$$v_x=2x+2zz_x \qquad v_y=2y+2zz_y$$

$$\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}=0$$

$$\begin{vmatrix} -\frac{y}{x^2} & 2x+2zz_x \\ \frac{1}{x} & 2y+2zz_y \end{vmatrix}=0$$

$$-\frac{y}{x^2}(2y+2zz_y)-\frac{1}{x}(2x+2zz_x)=0$$

$$y(2y+2zz_y)+x(2x+2zz_x)=0$$

$$2y^2+2yzz_y+2x^2+2xzz_x=0 \quad /2$$

$$\boxed{(xz_x+yz_y)z+x^2+y^2=0}$$

Thank You

Any questions?

You can find me at:
saad.m@sc.uobaghdad.edu.iq

Lagrange's Linear Equations

Saad Al-Momen

2

MATHEMATICAL PHYSICS I

Master Degree Class
Department of Astronomy and Space
College of Science - University of Baghdad

$$P\,z_x + Q\,z_y = R$$

where P, Q and R are functions of x, y and z

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Lagrange's auxiliary equations

If $u = a$ and $v = b$ are two solutions of Lagrange's auxiliary equations then the solution given by $\varphi(u, v) = 0$

1.

Method of **Grouping**

Solve $xz_x + yz_y = z$

$$P = x, \quad Q = y, \quad R = z$$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Taking $\frac{dx}{x} = \frac{dy}{y} \Rightarrow \ln x = \ln y + c$

$$\Rightarrow \boxed{\frac{x}{y} = a}$$

taking $\frac{dy}{dy} = \frac{dz}{z}$ $\Rightarrow \ln y = \ln z + c$

$\Rightarrow \boxed{\frac{y}{z} = b}$

Hence the general Solution is $\phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0$

Solve $\frac{y^2 z}{x} z_x + xz z_y = y^2$

$$\frac{x\,dx}{y^2 z} = \frac{dy}{xz} = \frac{dz}{y^2} \Rightarrow \frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{xy^2}$$

Taking $\frac{dx}{y^2 z} = \frac{dy}{x^2 z} \Rightarrow \frac{dx}{y^2} = \frac{dy}{x^2}$

$$\Rightarrow x^2 dx = y^2 dy$$

$$\Rightarrow \frac{x^3}{3} = \frac{y^3}{3} + c_1 \Rightarrow \boxed{x^3 - y^3 = a}$$

Taking $\frac{dx}{y^2 z} = \frac{dz}{xy^2} \Rightarrow \frac{dx}{z} = \frac{dz}{x} \Rightarrow x\,dx = z\,dz$

$$\Rightarrow \frac{x^2}{2} = \frac{z^2}{2} + c_2 \Rightarrow \boxed{x^2 - z^2 = b}$$

$\therefore$ The general solution is $\phi(x^3 - y^3, x^2 - z^2) = 0$

2. Method of Multipliers

Single Multipliers

Double Multipliers

Single Multipliers

- Choose any three multipliers (l, m, n) may be constants or functions of x, y and z we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx+mdy+ndy}{lP+mQ+nR}$$

If it is possible to choose (l, m, n) such that $lP + mQ + nR = 0$ then

$$ldx + mdy + ndy = 0$$

If $ldx + mdy + ndy$ is perfect differential of some function, say $u(x, y, z)$ then $du = 0$
$\Rightarrow \ u = a$ is the solution.

If $lP + mQ + nR \neq 0$ for any (l, m, n) then go to **double multipliers**.

Solve $(y-z)z_x + (z-x)z_y = x-y$

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{dx+dy+dz}{y-z+z-x+x-y} = \frac{dx+dy+dz}{0}$$

$$\Rightarrow dx+dy+dz=0 \Rightarrow \boxed{x+y+z=a}$$

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{x\,dx+y\,dy+z\,dz}{x(y-z)+y(z-x)+z(x-y)} = \frac{x\,dx+y\,dy+z\,dz}{0}$$

$$\Rightarrow x\,dx+y\,dy+z\,dz=0 \Rightarrow \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c \Rightarrow \boxed{x^2+y^2+z^2=b}$$

$\therefore$ The general is $f(x+y+z,\ x^2+y^2+z^2)=0$

Double Multipliers

"

- Choose double multipliers (l_1, m_1, n_1) and (l_2, m_2, n_2)may be constants or functions of x, y and z we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l_1 dx+m_1 dy+n_1 dy}{l_1 P+m_1 Q+n_1 R} = \frac{l_2 dx+m_2 dy+n_2 dy}{l_2 P+m_2 Q+n_2 R}$$

Integrating both sides of

$$\frac{l_1 dx + m_1 dy + n_1 dy}{l_1 P + m_1 Q + n_1 R} = \frac{l_2 dx + m_2 dy + n_2 dy}{l_2 P + m_2 Q + n_2 R}$$

Solve $(y+z)z_x + (x+z)z_y = x+y$

$$\frac{dx}{y+z} = \frac{dy}{x+z} = \frac{dz}{x+y} \quad \text{--- (1)}$$

Each fraction of (1) $= \dfrac{dx+dy+dz}{2(x+y+z)} = \dfrac{dx-dy}{-(x-y)} = \dfrac{dy-dz}{-(y-z)}$

$$\frac{d(x-y)}{(x-y)} = \frac{d(y-z)}{(y-z)}$$

Integrating, we get $\ln(x-y) = \ln(y-z) + c$

i.e. $\boxed{\dfrac{x-y}{y-z} = a}$

Taking $\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{-(x-y)}$, we have

$$\frac{1}{2}\frac{d(x+y+z)}{(x+y+z)} = -\frac{d(x-y)}{(x-y)}$$

Integrating, we get $\frac{1}{2}\ln(x+y+z) = -\ln(x-y) + c_2$

$$\boxed{\sqrt{x+y+z}\,(x-y) = b}$$

The general Solution $\phi\left(\frac{x-y}{y-z}, \sqrt{x+y+z}\,(x-y)\right) = 0$

Thank You

Any questions?

You can find me at:

saad.m@sc.uobaghdad.edu.iq

Solution of PDE by Direct Integration

Saad Al-Momen

3

MATHEMATICAL PHYSICS I

Master Degree Class
Department of Astronomy and Space
College of Science - University of Baghdad

- A partial differential equation can be solved by successive integration in all cases where the dependent variable occurs **only** in the partial derivatives.

Example 1

Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x$

Integrating w.r.t. "x" : $\frac{\partial z}{\partial y} = -\cos x + f(y)$

Integrating w.r.t. "y" : $z = -y\cos x + F(y) + g(x)$

where $F(y) = \int f(y)\,dy$

Example 2

Solve $\frac{\partial^2 z}{\partial x \partial y} = 0$

Integrating w.r.t. 'x': $\frac{\partial z}{\partial y} = f(y)$

Integrating w.r.t. 'y': $z = F(y) + g(x)$

Example 3

Solve $\frac{\partial z}{\partial x} = 2x + 3y$; $\frac{\partial z}{\partial y} = 3x + \cos y$

$\frac{\partial z}{\partial x} = 2x + 3y$, Integrating w.r.t. 'x': $z = x^2 + 3xy + f(y)$

diffrentiating with respect to y we get:

$$\frac{\partial z}{\partial y} = 3x + f'(y)$$

but we know that

$$\frac{\partial z}{\partial y} = 3x + \cos y$$

$$\therefore f'(y) = \cos y \Rightarrow f(y) = \sin y + c$$

$$\therefore \boxed{z = x^2 + 3xy + \sin y + c}$$

Example 4

$$\frac{\partial^2 z}{\partial y^2} - z = 0; \text{ when } y = 0, z = e^x \text{ and } \frac{\partial z}{\partial y} = e^{-x}$$

$$\frac{\partial^2 z}{\partial y^2} - z = 0$$

$$m^2 - 1 = 0 \Rightarrow m = \mp 1$$

$$\therefore z = c_1 e^{y} + c_2 e^{-y}$$

When $y = 0, z = e^x \Rightarrow e^x = c_1 + c_2$

$$\frac{\partial z}{\partial y} = c_1 e^{y} - c_2 e^{-y}$$

When $y = 0, \frac{\partial z}{\partial y} = e^{-x} \Rightarrow e^{-x} = c_1 - c_2$

Now,

$$c_1 + c_2 = e^x$$

$$c_1 - c_2 = e^{-x}$$

$$(+) \quad 2c_1 = e^x + e^{-x}$$

$$\Rightarrow c_1 = \frac{e^x + e^{-x}}{2} = \sinh x$$

Example 4

and

$$c_1 + c_2 = e^{x}$$
$$c_1 - c_2 = e^{-x}$$

$\ominus$

$$-2c_2 = e^{-x} - e^{x}$$

$$c_2 = \frac{e^{x} - e^{-x}}{2} = \cosh x$$

$$\therefore \; z = e^{y} \sinh x + e^{-y} \cosh x$$

Thank You

Any questions?

You can find me at:

saad.m@sc.uobaghdad.edu.iq

Solution of Standard Types of 1st Order PDEs

Saad Al-Momen

4

MATHEMATICAL PHYSICS I

Master Degree Class
Department of Astronomy and Space
College of Science - University of Baghdad

“

- Any solution of PDE in which the number of arbitrary constants is **equal** to the number of independent variables is called the **Complete Integral**.

“

- Any solution obtained from the complete integral by given **particular values** to the arbitrary constants is called a **Particular Integral**.

“

- Any solution obtained from the complete integral by **eliminating** the arbitrary constants is called a **Singular Integral**.

“

- A PDE which involves only 1st order partial derivatives $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$ is called a first order PDE, and the general form is $\boldsymbol{f(x, y, z, p, q) = 0}$.

1.

TYPE I

$$f(p, q) = 0$$

TYPE I

Given $f(p,q)=0$ ---(1)

Let $z=ax+by+c$ be a trial solution of equation (1)

Then $p=\frac{\partial z}{\partial x}=a$ and $q=\frac{\partial z}{\partial y}=b$

From (1), we get

$$f(a,b)=0$$

Hence the Complete integral of (1) is

$$z=ax+by+c$$

Solving for b from $f(a,b)=0$, we get $b=\beta(a)$

∴ The Complete integral of ① is

$$z = ax + \beta(a)y + c \quad \text{--- } ②$$

Since [number of arbitrary Constant (a, c) = number of independent variables (x, y) = 2].

To obtain the singular integral we have to eliminate a and c from equation $z = ax + \beta(a)y + c$, $\frac{\partial z}{\partial a} = 0$, $\frac{\partial z}{\partial c} = 0$

i.e. $z = ax + \beta(a)y + c$

$$x + \beta'(a)y = 0$$

$$1 = 0$$

The last equation being absurd and hence there is no singular integral

TYPE I

Example 1

Solve $\sqrt{p}+\sqrt{q}=1$

let $z=ax+by+c$, $p=\frac{\partial z}{\partial x}=a$, $q=\frac{\partial z}{\partial y}=b$

$\Rightarrow \sqrt{a}+\sqrt{b}=1 \Rightarrow b=(1-\sqrt{a})^2$

$\therefore$ The Complete integral is

$$z=ax+(1-\sqrt{a})^2y+c$$

2.

TYPE II

$$z = px + qy + f(p, q)$$

TYPE II

Suppose a PDE of the form

$$z = px + qy + f(p,q)$$

which said to be Clairaut's form

Put $p = a$ and $q = b$, we get

$$z = ax + by + f(a,b)$$

where a and b are arbitrary constants.

Complete integral

TYPE II

differentiate $z = ax + by + f(a,b)$ w.r.t. a and b and then equating to zero, we get

$$x + \frac{\partial f}{\partial a} = 0$$

$$y + \frac{\partial f}{\partial b} = 0$$

By eliminating a and b from the last three equations we get the singular integral

TYPE II

Example 2

Solve $z = px + qy + p^2 + pq + q^2$

$\Rightarrow$ The general integral is $z = ax + by + a^2 + ab + b^2$

To find the singular integral, diff w.r.t. a and b and equating to zero

$$x + 2a + b = 0 \longrightarrow 2a + b = -x$$
$$y + 2b + a = 0 \longrightarrow a + 2b = -y$$

$\Rightarrow a = \frac{1}{3}(y - 2x)$ and $b = \frac{1}{3}(x - 2y)$

$$\Rightarrow 3z = xy - x^2 - y^2$$

which is the singular integral

3. TYPE III

$$f(z, p, q) = 0$$

TYPE III

Given $f(z,p,q)=0$ --- ①

Let $z=f(x+ay)$ be the solution of ①

Put $x+ay=u$ then

$$z=f(u) \text{ and } \frac{\partial u}{\partial x}=1, \quad \frac{\partial u}{\partial y}=a$$

$$p=\frac{\partial z}{\partial x}=\frac{dz}{du}\cdot\frac{\partial u}{\partial x}=\frac{dz}{du}$$

$$q=\frac{\partial z}{\partial y}=\frac{dz}{du}\cdot\frac{\partial u}{\partial y}=a\frac{dz}{du}$$

Subsituting in ① we get

$$f\left(z,\frac{dz}{du},a\frac{dz}{du}\right)=0$$

TYPE III

This is an ordinary differential equation of 1st order

Solving for $\frac{dz}{du}$, we obtained

$$\frac{dz}{du} = g(z,a)$$

i.e. $$\frac{dz}{g(z,a)} = du$$

Integrating $$\int \frac{dz}{g(z,a)} = u + c$$

$$\phi(z,a) = u + c$$

i.e. $$\phi(z,a) = x + ay + c$$

which is the complete integral

TYPE III

Example 3

Solve $q^2 = z^2 p^2 (1 - p^2)$

let $z = f(x + ay)$, $u = x + ay \Rightarrow z = f(u)$, $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$

Substituting in the equation

$$a^2\left(\frac{dz}{du}\right)^2 = z^2\left(\frac{dz}{du}\right)^2\left(1 - \left(\frac{dz}{du}\right)^2\right) \Rightarrow \frac{a^2}{z^2} = 1 - \left(\frac{dz}{du}\right)^2$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 = \frac{z^2 - a^2}{z^2} \Rightarrow \frac{dz}{du} = \frac{\sqrt{z^2 - a^2}}{z} \Rightarrow \frac{z}{\sqrt{z^2 - a^2}}\, dz = du$$

Integrating $\int \frac{z}{\sqrt{z^2 - a^2}}\, dz = \int du$, put $z^2 - a^2 = t \Rightarrow 2z\,dz = dt$

$\Rightarrow z\,dz = \frac{1}{2} dt$

$$\therefore \frac{1}{2}\int \frac{dt}{\sqrt{t}} = \int du \Rightarrow \frac{1}{2}\, 2\sqrt{t} = u + b \Rightarrow \sqrt{t} = u + b$$

$\Rightarrow \sqrt{z^2 - a^2} = x + ay + b$ which is the Complete integral

4.

TYPE IV

$$f(x,p) = g(y,q)$$

TYPE IV

A first order partial differential equation is called separable if it can be written as $f(x,p) = g(y,q)$.

Let $f(x,p) = g(y,q) = a$, where a is an arbitrary constant.

$\therefore$ $p(x,p) = a$, $g(y,q) = a$, Solving for p and q from these two equations we get

$$p = f_1(x,a) \quad \text{and} \quad q = g_1(y,a)$$

Since $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

$$= p\,dx + q\,dy$$

TYPE IV

$$\therefore dz = f_1(x,a)\,dx + g_1(x,a)\,dy$$

Integrating, we get

$$z = \int f_1(x,a)\,dx + \int g_1(y,a)\,dy + b$$

which gives the Complete integral.

TYPE IV

Example 4

Solve $\sqrt{p} + \sqrt{q} = 2x$

let $\sqrt{p} - 2x = -\sqrt{q} = a$

$\sqrt{p} - 2x = a \Rightarrow p = (a+2x)^2 = f_1(x,a)$

$-\sqrt{q} = a \Rightarrow q = a^2 = f_2(y,a)$

∴ The complete integral is

$$z = \int f_1(x,a)\,dx + \int f_2(y,a)\,dy + b$$

$$z = \int (a+2x)^2 dx + \int a^2 dy + b$$

$$z = \frac{(a+2x)^3}{3*2} + a^2 y + b$$

$$z = \frac{(a+2x)^3}{6} + a^2 y + b$$

Thank You

Any questions?

You can find me at:

saad.m@sc.uobaghdad.edu.iq

Charpit's Method

Saad Al-Momen

5

MATHEMATICAL PHYSICS I

Master Degree Class
Department of Astronomy and Space
College of Science - University of Baghdad

- Charpit’s method is a general method to solve non-linear PDEs of the 1st order. It is used when it is difficult to put the PDE in one of the standard forms.

$Let\ the\ given\ equation\ br\ f(x, y, z, p, q) = 0 \quad \ldots(1)$

$If\ we\ succeed\ to\ find\ another\ relation$

$$F(x, y, z, p, q) = 0 \qquad \ldots(2)$$

$then\ we\ can\ solve\ (1)\ and\ (2) for\ p\ and\ q$

$Since\ z\ consists\ of\ two\ independent\ variables$
$x\ and\ y\ then$

$$dz = pdx + qdy \qquad \ldots(3)$$

For determining F, diffrentiate (1) and (2) w.r.t. x and y respectively

$$\left.\begin{aligned}\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p+\frac{\partial f}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial x}=0\\ \frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}p+\frac{\partial F}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial x}=0\end{aligned}\right\}\quad \dots(4)$$

$$\left.\begin{aligned}\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q+\frac{\partial f}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial y}=0\\ \frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}q+\frac{\partial F}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial y}=0\end{aligned}\right\}\quad \dots(5)$$

$Eliminating\ \frac{\partial p}{\partial x} from\ the\ first\ pair (4), we\ get$

$$\left(\frac{\partial f}{\partial x}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial x}\frac{\partial f}{\partial p}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial p}\right) p + \boxed{\left(\frac{\partial f}{\partial q}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial q}\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x}} = 0 \ldots(6)$$

$Eliminating\ \frac{\partial q}{\partial y} from\ the\ second\ pair (5), we\ get$

$$\left(\frac{\partial f}{\partial y}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial y}\frac{\partial f}{\partial q}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial q}\right) q + \boxed{\left(\frac{\partial f}{\partial p}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y}} = 0 \ldots(7)$$

$since\ \frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y}, hence\ the\ last\ terms\ in$ $(6) and\ (7)\ are\ opppsite\ signs. Adding\ (6) and\ (7)$

$$\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial p}+\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial q}+\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial z}+\left(-\frac{\partial f}{\partial p}\right)\frac{\partial F}{\partial x}+\left(-\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial y}=0 \quad \ldots(8)$$

Clearly this Lagrange's equation with x, y, z, p, q as independent vriables and F as dependent variable, so

The **Auxiliary** Equations

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{dF}{0}$$

Example 1

Using charpit's method find Complete integral of $pxy + pq + qy = zy$

$$f(x, y, z, p, q) = pxy + pq + qy - yz = 0 \quad \text{--- (1)}$$

then by charpit's method, the auxiliary equations is

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{d\phi}{0}$$

or

$$\frac{dx}{-(xy+q)} = \frac{dy}{-(p+y)} = \frac{dz}{-p(xy+q) - q(p+y)} = \frac{dp}{py + p(-y)} = dz$$

$$= \frac{dq}{(px + q - z) + q(-y)}$$

Example 1

implying $dp=0$ or $p=a$ ---- (2)

Putting $p=a$ in (1), $axy+aq+qy=yz$

$$q(a+y)=y(z-ax)$$

$$q=\frac{y(z-ax)}{(a+y)} \quad \text{----} \quad (3)$$

Also we know that for $z(x,y)$,

$$dz=p\,dx+q\,dy \quad \text{----} \quad (4)$$

On substituting the values of p and q from (2) and (3)

Example 1

$$dz = a\,dx + \frac{y(z-ax)}{a+y}\,dy$$

or

$$\frac{dz - a\,dx}{z-ax} = \left(1 - \frac{a}{a+y}\right)dy$$

Integrating,

$$\ln(z-ax) = y - a\ln(a+y) + C_1$$

$$\boxed{z - ax = b\,e^{y}(y+a)^{-a}}$$

Example 2

Solve the equation $z^2 = pqxy$

$$z^2 - pqxy = 0 = f(x, y, z, p, q)$$

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{d\phi}{0}$$

$$\frac{dx}{qxy} = \frac{dy}{pxy} = \frac{dz}{pqxy + qpxy} = \frac{dp}{-pqy + 2pz} = \frac{dq}{-pqx + 2qz} = \frac{d\phi}{0}$$

Example 2

$$\frac{x\,dp + p\,dx}{-pqxy + 2pxz + pqxy} = \frac{y\,dq + q\,dy}{-pqxy + 2qzy + pqxy}$$

$$\Rightarrow \frac{x\,dp + p\,dx}{2pxz} = \frac{y\,dq + q\,dy}{2qzy}$$

$$\Rightarrow \frac{d(xp)}{xp} = \frac{d(yq)}{yq} \Rightarrow \ln px = \ln qy + b$$

$$\Rightarrow px = cqy \qquad \therefore p = cq\frac{y}{x}$$

Example 2

Substituting this in the original equation, we get

$$z^2 - \left(cq\frac{y}{x}\right)qyx = 0$$

$$z^2 - cq^2y^2 = 0 \Rightarrow q^2 = \frac{1}{c}\left(\frac{z}{y}\right)^2 \Rightarrow q = b\left(\frac{z}{y}\right)$$

$$\Rightarrow \quad z^2 - p\left(b\frac{z}{y}\right)xy = 0$$

$$\Rightarrow \quad z^2 - bxzp = 0 \qquad \Rightarrow p = \frac{1}{b}\left(\frac{z}{x}\right)$$

Example 2

Now $dz = p\,dx + q\,dy = \frac{1}{b}\left(\frac{z}{x}\right)dx + b\left(\frac{z}{y}\right)dy$

$$\Rightarrow \frac{1}{z}dz = \frac{1}{b}\frac{dx}{x} + b\frac{dy}{y}$$

$$\Rightarrow \ln z = \ln x^{1/b} + \ln y^{b} + c_1$$

$$\Rightarrow z = c\,x^{1/b}y^{b}$$

Thank You

Any questions?

You can find me at:

saad.m@sc.uobaghdad.edu.iq

Homogeneous Linear Partial Differential Equation

Saad Al-Momen

6

MATHEMATICAL PHYSICS I

Master Degree Class
Department of Astronomy and Space
College of Science - University of Baghdad

“

- A linear partial differential equation in which all partial derivatives are of the same order is called **homogeneous** linear PDE.

A homogeneous linear PDE of n^th^ order with constant coefficients is of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \cdots + a_n \frac{\partial^n z}{\partial y^n} = \mathrm{F(x, y)}$$

where $a_0, a_1, \ldots, a_n$ are constants.

This equation also can be written in the form

$$a_0 D^n + a_1 D^{n-1} D' + \cdots + a_n {D'}^n = F(x, y)$$

where $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$ $\rightarrow$ $f(D, D') = F(x, y)$

The particular solution is called the particular integral function and given by:

$$P.I. = \frac{F(x, y)}{f(D, D')}$$

Hence the complete solution

Z=Complementary Function + Particular Integral

Z= C.F. + P.I.

1.

Complementary Function

Put $D = m$ and $D' = 1$ in $f(D, D') = 0$

The auxiliary equation $f(m, 1) = 0$ will be

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_n = 0$$

Let the roots of this equation be $m_1, m_2, ..., m_n$

CASE I

If the roots are real (or imaginary) and **distinct**. Then,

$$\text{C.F.} = f_1(y + m_1 x) + f_2(y + m_2 x) + \cdots + f_n(y + m_n x)$$

Put $D = m$ and $D' = 1$ in $f(D, D') = 0$

The auxiliary equation $f(m, 1) = 0$ will be

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_n = 0$$

Let the roots of this equation be $m_1, m_2, \ldots, m_n$

CASE II (a)

If any **two** roots are **equal** (i.e., $m_1 = m_2 = m$) and others are distinct. Then,

$$\text{C.F.} = f_1(y + mx) + x f_2(y + mx) + f_3(y + m_3 x) + \cdots + f_n(y + m_n x)$$

Put $D = m$ and $D' = 1$ in $f(D, D') = 0$

The auxiliary equation $f(m, 1) = 0$ will be

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_n = 0$$

Let the roots of this equation be $m_1, m_2, ..., m_n$

CASE II (b)

If any **three** roots are **equal** (i.e., $m_1 = m_2 = m_3 = m$) and others are distinct. Then,

C.F.$= f_1(y + mx) + x f_2(y + mx) + x^2 f_3(y + mx) + f_4(y + m_4 x) + \cdots + f_n(y + m_n x)$

Example 1

Solve $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$

$$(D^2 - 3DD' + 2D'^2)z = 0$$

$$m^2 - 3m + 2 = 0$$

$$(m-1)(m-2) = 0$$

$$\Rightarrow m = 1, 2$$

$$C.F. = f_1(y+x) + f_2(y+2x)$$

$\therefore$ The Solution is $z = f_1(y+x) + f_2(y+2x)$

Example 2

Solve $\dfrac{\partial^3 z}{\partial x^3} - 4\dfrac{\partial^3 z}{\partial x^2 \partial y} + 4\dfrac{\partial^3 z}{\partial x \partial y^2} = 0$

$$(D^3 - 4D^2D' + 4DD'^2)z = 0$$

$\Rightarrow \quad m^3 - 4m^2 - 4m = 0$

$\Rightarrow \quad m(m^2 - 4m - 4) = 0$

$\Rightarrow \quad m(m-2)^2 = 0$

$\Rightarrow \quad m = 0, 2, 2$

C.F. $= f_1(y + 0x) + f_2(y + 2x) + x f_3(y + 2x)$

$\therefore$ The Solution is $z = f_1(y) + f_2(y + 2x) + x f_3(y + 2x)$

2.

Particular Integral

TYPE I

If $F(x,y) = e^{ax+by}$, then

$$\text{P.I.} = \frac{1}{f(D,D')} e^{ax+by}$$

$$= \frac{1}{f(a,b)} e^{ax+by}, \text{ provided } f(a,b) \neq 0$$

If $f(a,b) = 0$, then

$$\text{P.I.} = x \frac{1}{f'(D,D')} e^{ax+by}$$

Example 3

Solve $(D^2 - D'^2) z = e^{x+2y}$

$$m^2 - 1 = 0$$

$$(m-1)(m+1) = 0$$

$$\Rightarrow m = 1, -1$$

$\therefore$ C.F. is $f_1(y+x) + f_2(y-x)$

$$\text{P.I.} = \frac{1}{D^2 - D'^2} e^{x+2y}$$

$a = 1$
$b = 2$

$$= \frac{1}{1-4} e^{x+2y} = -\frac{1}{3} e^{x+2y}$$

$\therefore$ The Complete Solution $z = f_1(y+x) + f_2(y-x) - \frac{1}{3} e^{x+2y}$

Example 4

Solve $(D^2+3DD'+2D'^2)z = e^x \cosh y$

$$m^2+3m+2=0$$

$$m=-1,-2$$

$\therefore$ C.F. is $f_1(y-x)+f_2(y-2x)$

$$\text{P.I.} = \frac{1}{D^2+3DD'+2D'^2} e^x \cosh y$$

$$= \frac{1}{D^2+3DD'+2D'^2} e^x \left(\frac{e^y+e^{-y}}{2}\right)$$

$$= \frac{1}{2}\frac{1}{D^2+3DD'+2D'^2}\left(e^{x+y}+e^{x-y}\right)$$

$$= \frac{1}{2}\left[\frac{1}{6}e^{x+y} + x\frac{1}{-1}e^{x-y}\right] = \frac{1}{2}\left[\frac{1}{6}e^{x+y} - xe^{x-y}\right]$$

The Complete Solution = C.F. + P.I.

TYPE II

If $F(x, y) = \sin(ax + by)$, then

$$\text{P.I.} = \frac{1}{f(D,D')}\sin(ax + by)$$

Replace D^2 by $-a^2$, D'^2 by $-b^2$ and DD' by $-ab$

If $f(D, D') = 0$, then

$$\text{P.I.} = x\frac{1}{f'(D,D')}\sin(ax + by)$$

Similar formal for $F(x, y) = \cos(ax + by)$

Example 5

Solve $(D^2 + DD' - 6D'^2)z = \cos(2x+y)$

$m^2 + m - 6 = 0$

$(m-2)(m+3) = 0 \Rightarrow m_1 = 2,\ m_2 = -3$

$\therefore$ C.F. is $f_1(y+2x) + f_2(y-3x)$

$$P.I. = \frac{1}{D^2 + DD' - 6D'^2} \cos(2x+y)$$

$D^2 = -4$
$D'^2 = -1$
$DD' = -2$

$$= \frac{1}{-4-2+6} \cos(2x+y)$$

$$= \frac{1}{0} \cos(2x+y)$$

Example 5

$$\therefore P.I = x \frac{1}{2D + D'} \cos(2x+y)$$

$$= x \frac{2D - D'}{4D^2 - D'^2} \cos(2x+y)$$

$$= x \frac{2D - D'}{-16+1} \cos(2x+y) = -\frac{x}{15}\left[2D - D'\right] \cos(2x+y)$$

$$= -\frac{x}{15}\left(-4\sin(2x+y) + \sin(2x+y)\right)$$

$$= \frac{x}{5} \sin(2x+y)$$

The complete Solution, $z = C.F. + P.I.$

Example 6

Solve $(D^3+D^2D'-DD'^2-D'^3)z=3\sin(x+y)$

$$m^3+m^2-m-1=0$$

$$m-1 \,\big)\, m^3+m^2-m-1 \,\big(\, m^2+2m+1$$

$$m^3-m^2$$

$$2m^2-m-1$$

$$2m^2-2m$$

$$m-1$$

$$m-1$$

$$0$$

$\Rightarrow m=1,-1,-1$

C.F. is $f_1(y+x)+f_2(y-x)+xf_3(y-x)$

Example 6

$$P.I = \frac{1}{D^3+D^2D'-DD'^2-D'^3} 3\sin(x+y)$$

$$= \frac{1}{-D-D'+D-D'} 3\sin(x+y)$$

$D^2 = -1$

$D'^2 = -1$

$$= \frac{1}{0} 3\sin(x+y)$$

$$\therefore P.I = x \frac{1}{3D^2+2DD'-D'^2} 3\sin(x+y)$$

$D^2 = -1$

$D'^2 = -1$

$DD' = -1$

$$= x \frac{1}{-3-2+1} 3\sin(x+y)$$

$$= -\frac{3}{4} x \sin(x+y)$$

The complete Solution $z = C.F. + P.I.$

TYPE III

If $F(x, y) = x^m y^n$, then

$$\text{P.I.} = \frac{1}{f(D, D')} x^m y^n$$

$$= [f(D, D')]^{-1} x^m y^n$$

Expand $[f(D, D')]^{-1}$ is ascending powers of D, D' and the operate on $x^m y^n$.

Note: if $m > n$, then try to write $f(D, D')$ as a function of $\frac{D'}{D}$ and if $m < n$ try to write it as a function of $\frac{D}{D'}$

Note: $\frac{1}{D}$ denotes integration w.r.t. x and $\frac{1}{D'}$ denotes integration w.r.t. y.

Example 7

Solve $(D^2+3DD'+2D'^2)z = x^2y^2$

$$m^2+3m+2=0$$

$$(m+1)(m+2)=0 \Rightarrow m_1=-1,\ m_2=-2$$

$$C.F. = f_1(y-x)+f_2(y-x)$$

$$P.I. = \frac{1}{D^2+3DD'+2D'^2}x^2y^2$$

$$= \frac{1}{D^2\left[1+3\frac{D'}{D}+\frac{2D'^2}{D^2}\right]}x^2y^2$$

$$= \frac{1}{D^2}\left[1+\left(3\frac{D'}{D}+\frac{2D'^2}{D^2}\right)\right]^{-1}x^2y^2$$

$$\frac{1}{1+x} = 1-x+x^2-x^3+x^4\ldots$$

$$\frac{1}{1-x} = 1+x+x^2+x^3+\cdots$$

Example 7

$$= \frac{1}{D^2}\left[1 - \left(3\frac{D'}{D} + 2\frac{D'^2}{D^2}\right) + \left(\frac{3D'}{D} + \left(2\frac{D'^2}{D^2}\right)\right)^2 - \cdots\right] x^2y^2$$

$$= \frac{1}{D^2}\left[1 - 3\frac{D'}{D} - 2\frac{D'^2}{D^2} + \frac{9D'^2}{D^2}\right] x^2y^2$$

$$= \frac{1}{D^2}\left[1 - 3\frac{D'}{D} + 7\frac{D'^2}{D^2}\right] x^2y^2$$

$$= \frac{1}{D^2}\left[x^2y^2 - \frac{3}{D}(2x^2y) + \frac{7}{D^2}(2x^2)\right]$$

$$= \frac{1}{D^2}(x^2y^2) - \frac{1}{D^3}(6x^2y) + \frac{1}{D^4}(14x^2)$$

$$= \frac{1}{12}x^4y^2 - \frac{1}{10}x^5y + \frac{7}{180}x^6$$

Note

$$\frac{1}{D^n}x^m = \frac{x^{m+n}}{(m+1)(m+2)\cdots(m+n)}$$

Example 8

Solve $(D^2+4DD'-5D'^2)z = x+y^2$

$$m^2+4m-5=0$$

$$(m+5)(m-1)=0$$

$$m_1=-5, \quad m_2=1$$

$$\therefore \text{C.F.} = f_1(y-5x)+f_2(y-x)$$

$$\text{P.I} = \frac{1}{D^2+4DD'-5D'^2}(x+y^2)$$

$$= \frac{1}{-5D'^2\left(\frac{-D^2}{5D'}-\frac{4D}{5D'}+1\right)}(x+y^2)$$

$$= \frac{1}{-5D'^2}\left[1-\left(\frac{4D}{5D'}+\frac{D^2}{5D'}\right)\right]^{-1}(x+y^2)$$

Example 8

$$= -\frac{1}{5D'^2}\left[1 + \left(\frac{4D}{5D'} + \frac{D^2}{5D'}\right) + \left(\frac{4D}{5D'} + \frac{D^2}{5D'}\right)^2 \cdots\right](x+y^2)$$

$$= -\frac{1}{5D'^2}\left[1 + \frac{4D}{5D'}\right](x+y^2) = -\frac{1}{5D'^2}\left[x+y^2 + \frac{4}{5D'}(x+y^2)\right]$$

$$= -\frac{1}{5}\frac{1}{D'^2}(x) - \frac{1}{5D'^2}(y^2) - \frac{4}{5}\frac{1}{D'^3}(1)$$

$$= -\frac{1}{5}\left(\frac{xy^2}{2} + \frac{y^4}{12} + \frac{2y^3}{15}\right)$$

The complete solution $z = C.F. + P.I$

TYPE IV

If $F(x, y) = e^{ax+by}\varphi(x, y)$, then

$$\text{P.I.} = \frac{1}{f(D, D')} e^{ax+by}\varphi(x, y)$$

$$= e^{ax+by} \frac{1}{f(D+a, D'+b)} \varphi(x, y)$$

Example 9

Solve $(D^2-4DD'+4D'^2)z = e^{x-2y}\cos(2x-y)$

$$m^2-4m+4=0 \Rightarrow m_1=m_2=2$$

$$C.F. = f_1(y+2x)+x f_2(y+2x)$$

$$P.I. = \frac{1}{D^2-4DD'+4D'^2} e^{x-2y}\cos(2x-y)$$

$$= e^{x-2y}\frac{1}{(D+1)^2-4(D+1)(D'-2)+4(D'-2)^2}\cos(2x-y)$$

$$= e^{x-2y}\frac{1}{D^2+10D-4DD'-20D'+4D'^2+25}\cos(2x-y)$$

$$= e^{x-2y}\frac{1}{-4+10D-8-20D'-4+25}\cos(2x-y)$$

Example 9

$$= e^{x-2y} \frac{1}{10D - 20D' + 9} \cos(2x-y)$$

$$= e^{x-2y} \frac{((10D-20D')-9)}{(10D-20D')^2 - 81} \cos(2x-y)$$

$$= e^{x-2y} \frac{(10D-20D'-9)\cos(2x-y)}{100D^2 - 400DD' + 400D'^2 - 81}$$

$$= e^{x-2y} \frac{(10D-20D'-9)\cos(2x-y)}{-400-800-400-81}$$

$$= \frac{-e^{x-2y}}{1681}\left(-20\sin(2x-y) - 20\sin(2x-y) - 9\cos(2x-y)\right)$$

$$= \frac{e^{x-2y}}{1681}\left(40\sin(2x-y) + 9\cos(2x-y)\right)$$

The Complete Solution $z = C.F. + P.I.$

TYPE V

If $F(x,y)$ is any function, resolve $f(D,D')$ into linear factors say $(D-m_1D'), (D-m_2D'), \ldots, (D-m_nD')$

$$\frac{1}{D-mD'}F(x,y) = \int F(x, c-mx)dx, \text{ Where } y = c-mx$$

$$\frac{1}{D+mD'}F(x,y) = \int F(x, c+mx)dx, \text{ Where } y = c+mx$$

Example 10

Solve $(D^2+DD'-6D'^2)z = y\cos x$

$$m^2+m-6=0$$
$$(m+3)(m-2)=0$$
$$m_1=2,\quad m_2=-3$$

$$C.F. = f_1(y+2x)+f_2(y-3x)$$

$$P.I. = \frac{1}{D^2+DD'-6D'^2}\, y\cos x$$

$$= \frac{1}{(D-2D')(D+3D')}\, y\cos x$$

$$= \frac{1}{D-2D'}\int (c+3x)\cos x\,dx \quad \text{where } y = c+3x$$

$$= \frac{1}{D-2D'}\left[(c+3x)\sin x + 3\cos x\right]$$

Example 10

$$= \frac{1}{D-2D'}\left[(y-3x+3x)\sin x + 3\cos x\right] \quad \because c = y-3x$$

$$= \frac{1}{D-2D'}(y\sin x + 3\cos x)$$

$$= \int \left[(c_1 - 2x)\sin x + 3\cos x\right] dx \quad \text{where } y = c_1 - 2x$$

$$= \left[(c_1 - 2x)(-\cos x) - (-2)(-\sin x)\right] + 3\sin x$$

$$= -(c_1 - 2x)\cos x - 2\sin x + 3\sin x$$

$$= -y\cos x + \sin x$$

and $z = C.F + P.I.$

Thank You

Any questions?

You can find me at:
saad.m@sc.uobaghdad.edu.iq

8

Separation of Variables
for Partial Deferential Equations

Saad Al-Momen

MATHEMATICAL PHYSICS I

Master Degree Class
Department of Astronomy and Space
College of Science - University of Baghdad

“

- The basic idea of this method is that the solution is assumed to consist of product of two or more functions. Each function being the function of one independent variable only.

1.

Finite Vibrating String Problem

Finite Vibrating String Problem

For a point x on the string we let

$u(x,t)$ = displacement of the point x at time t

Assuming small displacements this is well modeled by PDE called the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad x \in [0, L],\ t \geq 0 \quad \text{---}(1)$$

with the boundary conditions (BC)

$$u(0,t) = 0\ ,\ u(L,t) = 0 \quad \forall t \geq 0 \quad \text{---}(2)$$

and initial conditions (IC) for $0 \leq x \leq L$

$$u(x,0) = \phi(x)\ ,\ u_t(x,0) = \psi(x) \quad \text{---}(3)$$

Now, Assume that

$$u(x,t) = X(x)T(t) \quad \text{---(4)}$$

here $X(x)$ is function of x alone and $T(t)$ is a function of t alone. Substituting (4) in equation (1)

$$XT'' = c^2 X'' T$$

Separating the variables $$\frac{X''}{X} = \frac{T''}{c^2 T}$$

The left side is a function of x and the right side is a function of t.

The equality will hold only if both are equal to a constant, say k.

We get two differential equations as follows:

$$X'' - kX = 0 \qquad \text{---(5)}$$

$$T'' - c^2kT = 0 \qquad \text{....(6)}$$

Since k is any constant.

- it can be zero, or
- it can be positive, or
- it can be negative.

CASE I

Case I $k=0$

In this case equation (5) and (6) reduce to

$$X''=0 \quad \text{and} \quad T''=0$$

$$\Rightarrow \quad X(x) = Ax+B \qquad \text{and} \qquad T(t)=Ct+D$$

But the solution $u(x,t)=X(x)T(t)$ is a trivial solution if it has to satisfy the boundary conditions

$$u(0,t)=u(L,t)=0$$

So, this case is rejected since it gives rise to trivial solutions only.

CASE II

Case II $\quad k>0$, let $k=\lambda^2$ for some $\lambda>0$

$$\Rightarrow \quad X''-\lambda^2 X=0 \quad \text{and} \quad T''-c^2\lambda^2 T=0$$

Giving rise to solutions

$$X(x)=Ae^{\lambda x}+Be^{-\lambda x}$$
$$T(t)=Ce^{c\lambda t}+De^{-c\lambda t}$$

Therefore

$$u(x,t)=\left(Ae^{\lambda x}+Be^{-\lambda x}\right)\left(Ce^{c\lambda t}+De^{-c\lambda t}\right)$$

Using boundary condition $u(0,t)=0$

$$A+B=0 \Rightarrow A=-B$$

Using boundary Condition $u(L,t)=0$

$$(Ae^{\lambda L}-Ae^{-\lambda L})(Ce^{c\lambda t}+De^{-c\lambda t})=0$$

$$A(Ae^{\lambda L}-e^{-\lambda L})(Ce^{c\lambda t}+De^{-c\lambda t})=0$$

The t part of the Solution Cannot be Zero as it will lead to a trivial Solution. Then we must have

$$A(e^{\lambda L}-e^{-\lambda L})=0$$

which leads to $A=0$ as $\lambda\neq 0$

$k>0$ also gives rise to trivial Solution and also rejected.

Finite Vibrating String Problem

CASE III

Case III $K < 0$, let $K = -\lambda^2$ for some $\lambda > 0$

$\Rightarrow$ $X'' + \lambda^2 X = 0$ and $T'' + c^2\lambda^2 T = 0$

giving rise to solutions

$$X(x) = A\cos\lambda x + B\sin\lambda x$$

$$T(t) = C\cos(c\lambda t) + D\sin(c\lambda t)$$

Hence

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos(c\lambda t) + D\sin(c\lambda t))$$

using boundary condition $u(0,t) = 0$, $\boxed{A = 0}$

using boundary condition $u(L,t) = 0$,

$$B\sin\lambda X = 0$$

$B \neq 0$ as that will lead to a trivial solution

Hence we must have

$$\sin \lambda L = 0$$

which give us

$$\boxed{\lambda = \lambda_n = \frac{n\pi}{L}}, \quad n = 1, 2, 3, \cdots$$

These λ_n's are called eigenvalues and note that corresponding to each n there will be an eigenfunctio[n]

According, the Solution is

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos(c\lambda t) + D\sin(c\lambda t))$$

$$= \sin\lambda_n x \left(BC\cos(c\lambda_n t) + BD\sin(c\lambda_n t)\right)$$

$$= \sin\frac{n\pi}{L}x\left(A_n\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi ct}{L}\right)$$

$$= u_n(x,t)$$

The Solution corresponding to each eigenvalue is called an eigenfunction.

Since the wave equation is linear and homogeneos, any linear Combination will also be a Solution

Hence, we can expect the Solution in the following form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left[A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right] \quad \text{---(7)}$$

Using the initial Condition $u(x,0) = \phi(x)$

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$$

This series can be recognized as the half-range sine expansion of a function $\phi(x)$ defined in the range $(0,L)$.

A_n can be obtained by multipling the last equation by $\sin\frac{n\pi x}{L}$ and integrating with respect to x from 0 to L.

Therefore

$$A_n = \frac{2}{L}\int_0^L \phi(x) \sin\frac{n\pi x}{L}\, dx \quad , \quad n=1,2,3,\ldots$$

Here, we have used the fact that

$$\int_{-L}^{L} \sin\frac{n\pi x}{L} \sin\frac{n\pi x}{L}\, dx = L$$

To use the other initial condition $u_t(x,0) = \psi(x)$, we need to differentiate (7) w.r.t. t to get

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(\frac{n\pi c}{L}\right)\left[-A_n \sin\frac{n\pi ct}{L} + B_n \cos\frac{n\pi ct}{L}\right]$$

Then

$$\psi(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin\frac{n\pi x}{L}$$

Similarly

$$B_n = \frac{2}{n\pi c}\int_0^L \psi(x) \sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots$$

Now, Consider the infinite Series

$$\sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left[A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right]$$

with

$$A_n = \frac{2}{L}\int_0^L \phi(x)\sin\frac{n\pi x}{L}\,dx\ , \quad n=1,2,3\ldots$$

$$B_n = \frac{2}{n\pi c}\int_0^L \psi(x)\sin\frac{n\pi x}{L}\,dx\ , \quad n=1,2,3,\ldots$$

2.

One Dimensional
Heat Flow

Thank You

Any questions?

You can find me at:

saad.m@sc.uobaghdad.edu.iq

Separation of Variables
for Partial Deferential Equations

Saad Al-Momen

8

MATHEMATICAL PHYSICS I

Master Degree Class
Department of Astronomy and Space
College of Science - University of Baghdad

"

- The basic idea of this method is that the solution is assumed to consist of product of two or more functions. Each function being the function of one independent variable only.

1.

Finite Vibrating String Problem

Finite Vibrating String Problem

For a point x on the string we let

$u(x,t)$ = displacement of the point x at time t

Assuming small displacements this is well modeled by PDE called the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad x \in [0, L], \ t \geq 0 \quad \text{---} (1)$$

with the boundary conditions (BC)

$$u(0,t) = 0 \ , \ u(L,t) = 0 \qquad \forall t \geq 0 \quad \text{---} (2)$$

and initial conditions (IC) for $0 \leq x \leq L$

$$u(x,0) = \phi(x) \ , \ u_t(x,0) = \psi(x) \quad \text{---} (3)$$

Now, Assume that

$$u(x,t) = X(x)T(t) \quad \text{---(4)}$$

here $X(x)$ is function of x alone and $T(t)$ is a function of t alone. Substituting (4) in equation (1)

$$XT'' = c^2 X'' T$$

Separating the variables

$$\frac{X''}{X} = \frac{T''}{c^2 T}$$

The left side is a function of x and the right side is a function of t.

The equality will hold only if both are equal to a Constant, Say k.

We get two differential equations as follows:

$$X'' - kX = 0 \quad \text{---} (5)$$

$$T'' - c^2 kT = 0 \quad \text{....} (6)$$

Since k is any constant.

- it can be zero, or
- it can be positive, or
- it can be negative.

Case I $\quad k=0$

In this case equation (5) and (6) reduce to

$$X''=0 \quad \text{and} \quad T''=0$$

$$\Rightarrow \quad X(x)=Ax+B \quad \text{and} \quad T(t)=Ct+D$$

But the solution $u(x,t)=X(x)T(t)$ is a trivial solution if it has to satisfy the boundary conditions

$$u(0,t)=u(L,t)=0$$

So, this case is rejected since it gives rise to trivial solutions only.

Finite Vibrating String Problem

CASE II

Case II $k>0$, let $k=\lambda^2$ for some $\lambda>0$

$\Rightarrow$ $X''-\lambda^2 X=0$ and $T''-c^2\lambda^2 T=0$

Giving rise to solutions

$$X(x)=Ae^{\lambda x}+Be^{-\lambda x}$$

$$T(t)=Ce^{c\lambda t}+De^{-c\lambda t}$$

Therefore

$$u(x,t)=\left(Ae^{\lambda x}+Be^{-\lambda x}\right)\left(Ce^{c\lambda t}+De^{-c\lambda t}\right)$$

Using boundary condition $u(0,t)=0$

$$A+B=0 \Rightarrow A=-B$$

Using boundary Condition $u(L,t)=0$

$$\left(Ae^{\lambda L}-Ae^{-\lambda L}\right)\left(Ce^{c\lambda t}+De^{-c\lambda t}\right)=0$$

$$A\left(Ae^{\lambda L}-e^{-\lambda L}\right)\left(Ce^{c\lambda t}+De^{-c\lambda t}\right)=0$$

The t part of the Solution Cannot be Zero as it will lead to a trivial Solution. Then we must have

$$A\left(e^{\lambda L}-e^{-\lambda L}\right)=0$$

which leads to $A=0$ as $\lambda\neq 0$

$K>0$ also gives rise to trivial Solution and also rejected.

Case III $K < 0$, let $K = -\lambda^2$ for some $\lambda > 0$

$\Rightarrow$ $X'' + \lambda^2 X = 0$ and $T'' + c^2\lambda^2 T = 0$

giving rise to solutions

$$X(x) = A\cos\lambda x + B\sin\lambda x$$

$$T(t) = C\cos(c\lambda t) + D\sin(c\lambda t)$$

Hence

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos(c\lambda t) + D\sin(c\lambda t))$$

using boundary condition $u(0,t) = 0$, $\boxed{A = 0}$

using boundary condition $u(L,t) = 0$,

$$B\sin\lambda x = 0$$

$B \neq 0$ as that will lead to a trivial solution

Hence we must have

$$\sin \lambda L = 0$$

which give us

$$\boxed{\lambda = \lambda_n = \frac{n\pi}{L}}, \quad n = 1, 2, 3, \cdots$$

These λ_n's are called eigenvalues and note that corresponding to each n there will be an eigenfunctio[n]

According, the Solution is

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos(c\lambda t) + D\sin(c\lambda t))$$

$$= \sin\lambda_n x \left(BC\cos(c\lambda_n t) + BD\sin(c\lambda_n t)\right)$$

$$= \sin\frac{n\pi}{L}x\left(A_n\cos\frac{n\pi c t}{L} + B_n\sin\frac{n\pi c t}{L}\right)$$

$$= u_n(x,t)$$

Finite Vibrating String Problem

CASE III

The Solution Corresponding to each eigenvalue is called an eigenfunction.

Since the wave equation is linear and homogeneos, any linear Combination will also be a Solution

Hence, we can expect the Solution in the following form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left[A_n \cos\frac{n\pi c t}{L} + B_n \sin\frac{n\pi c t}{L}\right] \quad \text{---(7)}$$

Using the initial Condition $u(x,0) = \phi(x)$

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$$

This series can be recognized as the half-range sine expansion of a function $\phi(x)$ defined in the range $(0, L)$.

A_n can be obtained by multipling the last equation by $\sin \frac{n\pi x}{L}$ and integrating with respect to x from 0 to L.

Therefore

$$A_n = \frac{2}{L} \int_0^L \phi(x) \sin \frac{n\pi x}{L} \, dx \, , \quad n = 1, 2, 3, \ldots$$

Here, we have used the fact that

$$\int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{n\pi x}{L} \, dx = L$$

To use the other initial condition $u_t(x,0) = \psi(x)$, we need to differentiate (7) w.r.t. t to get

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(\frac{n\pi c}{L}\right)\left[-A_n \sin\frac{n\pi ct}{L} + B_n \cos\frac{n\pi ct}{L}\right]$$

Then

$$\psi(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin\frac{n\pi x}{L}$$

Similarly

$$B_n = \frac{2}{n\pi c}\int_0^L \psi(x) \sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots$$

Now, Consider the infinite Series

$$\sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left[A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right]$$

with

$$A_n = \frac{2}{L}\int_0^L \phi(x)\sin\frac{n\pi x}{L}\,dx \,, \quad n=1,2,3\ldots$$

$$B_n = \frac{2}{n\pi c}\int_0^L \psi(x)\sin\frac{n\pi x}{L}\,dx \,, \quad n=1,2,3,\ldots$$

2.

One Dimensional Heat Flow

One Dimensional Heat Flow

Let $u(x,t)$ denote the temperature at position x and time t in a long, thin rod of length L that runs from $x=0$ to $x=L$. Assume that the sides of the rod are isulated so heat energy niether enters nor leaves the rod through its sides. Also assume that heat energy is neither created nor destroyed in the interior of the rod. Then $u(x,t)$ obeys the heat eq.

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{for all } 0<x<L, \text{ and } t>0$$

This equation is called (The Heat Equation (one Space Dim))

B.C. $\begin{cases} u(0,t)=0 & \text{for all } t>0 \\ u(L,t)=0 & \text{for all } t>0 \end{cases}$

I.C. $u(x,0)=f(x)$ for all $0<x<L$

Find the Solution to the heat Conduction problem

$$4u_t = u_{xx}, \quad 0 \leq x \leq 2 \quad t > 0$$

$$u(0,t) = 0$$

$$u(2,t) = 0$$

$$u(x,0) = 2\sin\left(\frac{\pi x}{2}\right) - \sin(\pi x) + 4\sin(2\pi x)$$
$$= f(x)$$

Let $u(x,t) = X(x)T(t)$

$$\Rightarrow \quad 4X(x)T'(t) = X''(x)T(t)$$

$$\Rightarrow \quad 4\frac{T'}{T} = \frac{X''}{X} = \lambda$$

where λ is Constant

$0 = u(0,t) = X(0)T(t) \Rightarrow X(0) = 0$

$0 = u(2,t) = X(2)T(t) \Rightarrow X(2) = 0$

(Since $T(t)$ won't be 0 for all t)

$\therefore X(0) = X(2) = 0$ are the BCs

Now, $T' = \frac{\lambda}{4} T \Rightarrow \frac{dT}{dt} = \frac{\lambda}{4} T \Rightarrow \frac{dT}{T} = \frac{\lambda}{4} dt$

Integrating both sides

$$\int \frac{dT}{T} = \frac{\lambda}{4} \int dt \Rightarrow \ln T = \frac{\lambda}{4} t + C_1$$

$$\Rightarrow T = C_1 e^{\lambda t/4}$$

and $X'' = \lambda X$ with $X(0) = X(2) = 0$

There are 3 cases $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$

One Dimensional Heat Flow

CASE I

$\lambda > 0 \Rightarrow T(t) = c\,e^{\lambda t/4} \longrightarrow \infty$ at $t \longrightarrow \infty$

$\Rightarrow u(x,t) = X(x)\, e^{\lambda t/4} \longrightarrow \infty$

and $\lambda > 0$ would suggest the the temperature $u \longrightarrow \infty$ which doesn't make sense.

Set $\lambda = k^2 > 0 \Rightarrow X'' - k^2 X = 0$

$\Rightarrow r^2 - k^2 = 0 \Rightarrow r_{1,2} = \mp k$

$\therefore X(x) = C_1 e^{kx} + C_2 e^{-kx}$

$X(0) = 0 \Rightarrow C_1 + C_2 = 0 \Rightarrow C_1 = -C_2$

$X(2) = 0 \Rightarrow C_1 e^{2k} + C_2 e^{-2k} = 0$

$c_1 (e^{2k} - e^{-2k}) = 0 \Rightarrow C_1 = 0 \Rightarrow C_2 = 0$

$\Rightarrow$ Trivial Solution

CASE II

$\lambda = 0$ $\quad X'' = 0 \Rightarrow X = Ax + B$

$X(0) = 0 \Rightarrow 0 = B \Rightarrow X(x) = Ax$

$X(2) = 0 \Rightarrow 0 = 2A \Rightarrow A = 0$

$\Rightarrow$ Trivial Solution

CASE III

$\lambda < 0$, set $\lambda = -k^2 < 0$

$$X'' + k^2 X = 0 \Rightarrow m^2 + k^2 = 0 \Rightarrow m = \mp ik$$

$$\Rightarrow X(x) = c_1 e^{ikx} + c_2 e^{-ikx}$$

$$\text{or} \quad X(x) = c_1 \sin(kx) + c_2 \cos(kx)$$

$X(0) = 0 \Rightarrow c_1 \sin(0) + c_2 \cos(0) = 0 \Rightarrow c_2 = 0$

$X(2) = 0 \Rightarrow c_1 \sin(2k) = 0$

Since $\sin(\theta)$ has roots at $\theta = n\pi$, $n = 1, 2, 3, \ldots$

the second condition tells us that $2k = n\pi$

$$\Rightarrow k = \frac{n\pi}{2}, \quad n = 1, 2, \ldots$$

Thus we have our eigenfunctions with eigenvalues $\lambda<0$:

$$\lambda_n = -\left(\frac{n\pi}{2}\right)^2$$

$$X_n = \sin\left(\frac{n\pi x}{2}\right)$$

$$u_n(x,t) = X_n(x)\,T_n(t) = \sin\left(\frac{n\pi x}{2}\right)\exp\left(-\frac{n^2\pi^2}{16}t\right)$$

$$n = 1, 2, 3, \ldots$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)\exp\left(-\frac{n^2\pi^2 t}{16}\right)$$

we solve for the b_n using the initial condition. That is, $u(x,0) = f(x)$ so

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

which is a Fourier Sine Series. we exploit orthogonality of the sines, that is

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & m \neq n \\ L/2 & m = n \end{cases}$$

where $L = 2$,

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx$$

Since $L=2$. Now here $f(x) = 2\sin\left(\frac{\pi x}{2}\right) - \sin(\pi x) + 4\sin(2\pi x)$

$$b_n = \int_0^2 \left(2\sin\left(\frac{\pi x}{2}\right) - \sin(\pi x) + 4\sin(2\pi x)\right)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$b_n = 2\underbrace{\int_0^2 \sin\left(\frac{\pi x}{2}\right)\sin\left(\frac{n\pi x}{2}\right)dx}_{①} - \underbrace{\int_0^2 \sin(\pi x)\sin\left(\frac{n\pi x}{2}\right)dx}_{②} + 4\underbrace{\int_0^2 \sin(2\pi x)\sin\left(\frac{n\pi x}{2}\right)dx}_{③}$$

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & m \neq n \\ L/2 & m = n \end{cases}$$

$$b_n = 2\underbrace{\int_0^2 \sin\left(\frac{\pi x}{2}\right)\sin\left(\frac{n\pi x}{2}\right)dx}_{①} - \underbrace{\int_0^2 \sin(\pi x)\sin\left(\frac{n\pi x}{2}\right)dx}_{②} + 4\underbrace{\int_0^2 \sin(2\pi x)\sin\left(\frac{n\pi x}{2}\right)dx}_{③}$$

$$① = \begin{cases} 1 & n=1 \\ 0 & \text{otherwise} \end{cases} \qquad ② = \begin{cases} 1 & n=2 \\ 0 & \text{otherwise} \end{cases} \qquad ③ = \begin{cases} 1 & n=4 \\ 0 & \text{otherwise} \end{cases}$$

So then

$$b_1 = 2\int_0^2 \sin\left(\frac{\pi x}{2}\right)\sin\left(\frac{n\pi x}{2}\right)dx - 0 + 0 = 2$$

$$b_2 = 0 - \int_0^2 \sin(\pi x)\sin\left(\frac{n\pi x}{2}\right)dx + 0 = -1$$

$$b_4 = 0 - 0 + 4\int_0^2 \sin(2\pi x)\sin\left(\frac{n\pi x}{2}\right)dx = 4$$

$$b_n = 0 \text{ if } n \neq 1, 2, 4$$

we can now re-write our solution

$$u(x,t) = 2\sin\left(\frac{\pi x}{2}\right)\exp\left(-\frac{\pi^2}{16}t\right) - \sin(\pi x)\exp\left(-\frac{\pi^2}{4}t\right)$$

$$- \sin(2\pi x)\exp(-\pi^2 t)$$

Thank You

Any questions?

You can find me at:
saad.m@sc.uobaghdad.edu.iq

Introduction to Laplace Transform

Saad Al-Momen

9

MATHEMATICAL PHYSICS I

Master Degree Class
Department of Astronomy and Space
College of Science - University of Baghdad

Definition

Definition: Given a function $f(t)$, $t \geq 0$, its Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} \doteq \int_0^\infty e^{-st} f(t)\, dt \doteq \lim_{A\to\infty} \int_0^A e^{-st} f(t)\, dt$$

We say the transform converges if the limit exists, and diverges if not.

Next we will give examples on computing the Laplace transform of given functions by definition.

Example 1

Example 1. $f(t) = 1$ for $t \geq 0$.

$$F(s) = \mathcal{L}\{f(t)\} = \lim_{A\to\infty} \int_0^A e^{-st} \cdot 1\, dt = \lim_{A\to\infty} -\frac{1}{s} e^{-st} \Big|_0^A$$

$$= \lim_{A\to\infty} -\frac{1}{s} \left[e^{-sA} - 1\right] = \frac{1}{s}, \quad (s > 0)$$

Example 2

Example 2. $f(t) = e^{at}$

$$F(s) = \mathcal{L}\{f(t)\} = \lim_{A\to\infty} \int_0^A e^{-st} e^{at} dt = \lim_{A\to\infty} \int_0^A e^{-(s-a)t} dt$$

$$= \lim_{A\to\infty} -\frac{1}{s-a} e^{-(s-a)t}\Big|_0^A = \lim_{A\to\infty} -\frac{1}{s-a}\left(e^{-(s-a)A} - 1\right)$$

$$= \frac{1}{s-a}, \quad (s > a)$$

Example 3

Example 3. $f(t) = t^n$, for $n \geq 1$ integer.

$$F(s) = \lim_{A\to\infty} \int_0^A e^{-st} t^n dt$$

$$= \lim_{A\to\infty} \left\{ t^n \frac{e^{-st}}{-s}\Big|_0^A - \int_0^A \frac{nt^{n-1}e^{-st}}{-s} dt \right\}$$

$$= 0 + \frac{n}{s} \lim_{A\to\infty} \int_0^A e^{-st} t^{n-1} dt = \frac{n}{s} \mathcal{L}\{t^{n-1}\}.$$

$$\mathcal{L}\{t^n\} = \frac{n}{s} \mathcal{L}\{t^{n-1}\}, \quad \forall n,$$

Example 3

$$\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\}, \quad \forall n,$$

$$\mathcal{L}\{t^{n-1}\} = \frac{n-1}{s}\mathcal{L}\{t^{n-2}\}, \quad \mathcal{L}\{t^{n-2}\} = \frac{n-2}{s}\mathcal{L}\{t^{n-3}\}, \cdots$$

$$\begin{aligned}
\mathcal{L}\{t^n\} &= \frac{n}{s}\mathcal{L}\{t^{n-1}\} = \frac{n}{s}\frac{(n-1)}{s}\mathcal{L}\{t^{n-2}\} \\
&= \frac{n}{s}\frac{(n-1)}{s}\frac{(n-2)}{s}\mathcal{L}\{t^{n-3}\} \\
&= \cdots = \frac{n}{s}\frac{(n-1)}{s}\frac{(n-2)}{s}\cdots\frac{1}{s}\mathcal{L}\{1\} \\
&= \frac{n!}{s^n}\frac{1}{s} = \frac{n!}{s^{n+1}}, \quad (s > 0)
\end{aligned}$$

Example 4

Example 4. Find the Laplace transform of $\sin at$ and $\cos at$.

Method 1. Compute by definition, with integration-by-parts, twice. (lots of work...)

Method 2. Use the Euler's formula

$$e^{iat} = \cos at + i \sin at, \qquad \Rightarrow \qquad \mathcal{L}\{e^{iat}\} = \mathcal{L}\{\cos at\} + i\mathcal{L}\{\sin at\}.$$

$$\mathcal{L}\{e^{iat}\} = \frac{1}{s - ia} = \frac{1(s+ia)}{(s-ia)(s+ia)} = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}.$$

Example 5

Example 5. Find the Laplace transform of

$$f(t) = \begin{cases} 1, & 0 \le t < 2, \\ t-2, & 2 \le t. \end{cases}$$

We do this by definition:

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)\,dt = \int_0^2 e^{-st}dt + \int_2^\infty (t-2)e^{-st}dt \\
&= \frac{1}{-s}e^{-st}\Big|_{t=0}^{2} + (t-2)\frac{1}{-s}e^{-st}\Big|_{t=2}^{\infty} - \int_2^A \frac{1}{-s}e^{-st}dt \\
&= \frac{1}{-s}(e^{-2s}-1) + (0-0) + \frac{1}{s}\frac{1}{-s}e^{-st}\Big|_{t=2}^{\infty} \\
&= \frac{1}{-s}(e^{-2s}-1) + \frac{1}{s^2}e^{-2s}
\end{aligned}$$

Properties of Laplace Transform

Properties

1. Linearity: $\mathcal{L}\{c_1 f(t) + c_2 g(t)\} = c_1\mathcal{L}\{f(t)\} + c_2\mathcal{L}\{g(t)\}$.
2. First derivative: $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$.
3. Second derivative: $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$.
4. Higher order derivative:

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

5. $\mathcal{L}\{-tf(t)\} = F'(s)$ where $F(s) = \mathcal{L}\{f(t)\}$. This also implies $\mathcal{L}\{tf(t)\} = -F'(s)$.
6. $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ where $F(s) = \mathcal{L}\{f(t)\}$. This implies $e^{at}f(t) = \mathcal{L}^{-1}\{F(s-a)\}$.

Example 1

$$\mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}.$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$

Example 2

$$\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2 + b^2}.$$

$$\mathcal{L}\{\sin bt\} = \frac{b}{s^2 + b^2}$$

Example 3

$$\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}.$$

$$\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}$$

Example 4

$$\mathcal{L}\{t^3 + 5t - 2\} = \mathcal{L}\{t^3\} + 5\mathcal{L}\{t\} - 2\mathcal{L}\{1\} = \frac{3!}{s^4} + 5\frac{1}{s^2} - 2\frac{1}{s}.$$

Example 5

$$\mathcal{L}\{t^3 + 5t - 2\} = \mathcal{L}\{t^3\} + 5\mathcal{L}\{t\} - 2\mathcal{L}\{1\} = \frac{3!}{s^4} + 5\frac{1}{s^2} - 2\frac{1}{s}.$$

$$\mathcal{L}\{e^{2t}(t^3 + 5t - 2)\} = \frac{3!}{(s-2)^4} + 5\frac{1}{(s-2)^2} - 2\frac{1}{s-2}.$$

Example 6

$$\mathcal{L}\{(t^2+4)e^{2t} - e^{-t}\cos t\} = \frac{2}{(s-2)^3} + \frac{4}{s-2} - \frac{s+1}{(s+1)^2+1},$$

$$\mathcal{L}\{t^2+4\} = \frac{2}{s^3} + \frac{4}{s}, \qquad \Rightarrow \mathcal{L}\{(t^2+4)e^{2t}\} = \frac{2}{(s-2)^3} + \frac{4}{s-2}.$$

Example 7

$$\mathcal{L}\{te^{at}\} = -\left(\frac{1}{s-a}\right)' = \frac{1}{(s-a)^2}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

Example 8

$$\mathcal{L}\{t\sin bt\} = -\left(\frac{b}{s^2+b^2}\right)' = \frac{-2bs}{(s^2+b^2)^2}$$

Example 9

$$\mathcal{L}\{t\cos bt\} = -\left(\frac{s}{s^2+b^2}\right)' = \cdots = \frac{s^2-b^2}{(s^2+b^2)^2}.$$

Inverse of
Laplace Transform

$$\mathcal{L}^{-1}\{F(s)\} = f(t), \qquad \text{if} \qquad F(s) = \mathcal{L}\{f(t)\}$$

Example 1

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2+4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3}{2}\cdot\frac{2}{s^2+2^2}\right\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= \frac{3}{2}\sin 2t.$$

Example 2

$$\mathcal{L}^{-1}\left\{\frac{2}{(s+5)^4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{3}\cdot\frac{6}{(s+5)^4}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3!}{(s+5)^4}\right\}$$

$$= \frac{1}{3}e^{-5t}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \frac{1}{3}e^{-5t}t^3.$$

Example 3

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$=\cos 2+\frac{1}{2}\sin 2t.$$

Example 4

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s-2)(s+2)}\right\}$$

$$\frac{s+1}{(s-2)(s+2)} = \frac{A}{s-2} + \frac{B}{s+2}, \quad A = 3/4, \quad B = 1/4.$$

$$= \mathcal{L}^{-1}\left\{\frac{3/4}{s-2} + \frac{1/4}{s+2}\right\} = \frac{3}{4}e^{2t} + \frac{1}{4}e^{-2t}.$$

Table of LT

$f(t)$ for $t \geq 0$	$\widehat{f} = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$

Table of LT

$f(t)$ for $t \geq 0$	$\widehat{f} = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$
1	$\frac{1}{s}$

Table of LT

$f(t)$ for $t \geq 0$	$\widehat{f} = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$

Table of LT

$f(t)$ for $t \geq 0$	$\hat{f} = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$ $(n = 0, 1, \ldots)$

Table of LT

$f(t)$ for $t \geq 0$	$\widehat{f} = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$ $(n = 0, 1, \ldots)$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$ $(a > 0)$

Table of LT

$f(t)$ for $t \ge 0$	$\hat{f} = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}\ (n = 0, 1, \ldots)$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}\ (a > 0)$
$\sin bt$	$\frac{b}{s^2+b^2}$

Table of LT

$f(t)$ for $t \geq 0$	$\widehat{f} = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$ $(n = 0, 1, \ldots)$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$ $(a > 0)$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$

Table of LT

$f(t)$ for $t \geq 0$	$\hat{f} = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}\ (n = 0, 1, \ldots)$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}\ (a > 0)$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$\sinh bt$	$\frac{b}{s^2-b^2}$

Table of LT

$f(t)$ for $t \geq 0$	$\widehat{f} = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$ $(n = 0, 1, \ldots)$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$ $(a > 0)$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$\sinh bt$	$\frac{b}{s^2-b^2}$
$\cosh bt$	$\frac{s}{s^2-b^2}$

Table of LT

$f(t)$ for $t \geq 0$	$\hat{f} = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$ $(n = 0, 1, \ldots)$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$ $(a > 0)$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$\sinh bt$	$\frac{b}{s^2-b^2}$
$\cosh bt$	$\frac{s}{s^2-b^2}$
$f'(t)$	$s\mathcal{L}(f) - f(0)$

Table of LT

$f(t)$ for $t \geq 0$	$\widehat{f} = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$ $(n = 0, 1, \ldots)$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$ $(a > 0)$
$\sin bt$	$\frac{b}{s^2 + b^2}$
$\cos bt$	$\frac{s}{s^2 + b^2}$
$\sinh bt$	$\frac{b}{s^2 - b^2}$
$\cosh bt$	$\frac{s}{s^2 - b^2}$
$f'(t)$	$s\mathcal{L}(f) - f(0)$
$f''(t)$	$s^2\mathcal{L}(f) - sf(0) - f'(0)$

Table of LT

$f(t)$ for $t \geq 0$	$\hat{f} = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$ $(n = 0, 1, \ldots)$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$ $(a > 0)$
$\sin bt$	$\frac{b}{s^2 + b^2}$
$\cos bt$	$\frac{s}{s^2 + b^2}$
$\sinh bt$	$\frac{b}{s^2 - b^2}$
$\cosh bt$	$\frac{s}{s^2 - b^2}$
$f'(t)$	$s\mathcal{L}(f) - f(0)$
$f''(t)$	$s^2\mathcal{L}(f) - sf(0) - f'(0)$
$t^n f(t)$	$(-1)^n \frac{d^n F}{ds^n}(s)$

Table of LT

$f(t)$ for $t \geq 0$	$\hat{f} = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$ $(n = 0, 1, \ldots)$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$ $(a > 0)$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$\sinh bt$	$\frac{b}{s^2-b^2}$
$\cosh bt$	$\frac{s}{s^2-b^2}$
$f'(t)$	$s\mathcal{L}(f) - f(0)$
$f''(t)$	$s^2\mathcal{L}(f) - sf(0) - f'(0)$
$t^n f(t)$	$(-1)^n \frac{d^n F}{ds^n}(s)$
$e^{at} f(t)$	$\mathcal{L}(f)(s-a)$

Thank You

Any questions?

You can find me at:

saad.m@sc.uobaghdad.edu.iq

Solutions of IVP's Using Laplace Transform

Saad Al-Momen

10

MATHEMATICAL PHYSICS I

Master Degree Class
Department of Astronomy and Space
College of Science - University of Baghdad

First derivative: $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$.

Second derivative: $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$.

Higher order derivative:

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Example 1

Example 1

Solve the initial value problem by Laplace transform,

$$y'' - 3y' - 10y = 2, \qquad y(0) = 1, y'(0) = 2.$$

Step 1. Take Laplace transform on both sides: Let $\mathcal{L}\{y(t)\} = Y(s)$, and then

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY - 1,$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y - s - 2.$$

Note the initial conditions are the first thing to go in!

$$\mathcal{L}\{y''(t)\} - 3\mathcal{L}\{y'(t)\} - 10\mathcal{L}\{y(t)\} = \mathcal{L}\{2\},$$

$$\Rightarrow \qquad s^2Y - s - 2 - 3(sY - 1) - 10Y = \frac{2}{s}.$$

Now we get an algebraic equation for $Y(s)$.

Example 1

Step 2: Solve it for $Y(s)$:

$$(s^2 - 3s - 10)Y(s) = \frac{2}{s} + s + 2 - 3 = \frac{s^2 - s + 2}{s},$$

$$\Rightarrow \qquad Y(s) = \frac{s^2 - s + 2}{s(s-5)(s+2)}.$$

Step 3: Take inverse Laplace transform to get $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

The main technique here is partial fraction.

$$Y(s) = \frac{s^2 - s + 2}{s(s-5)(s+2)} = \frac{A}{s} + \frac{B}{s-5} + \frac{C}{s+2}$$

$$= \frac{A(s-5)(s+2) + Bs(s+2) + Cs(s-5)}{s(s-5)(s+2)}.$$

Example 1

Compare the numerators:

$$s^2 - s + 2 = A(s-5)(s+2) + Bs(s+2) + Cs(s-5).$$

The previous equation holds for all values of s.
Set $s = 0$: we get $-10A = 2$, so $A = -\frac{1}{5}$.
Set $s = 5$: we get $35B = 22$, so $B = \frac{22}{35}$.
Set $s = -2$: we get $14C = 8$, so $C = \frac{4}{7}$.

$$\frac{s^2 - s + 2}{s(s-5)(s+2)} = \frac{A}{s} + \frac{B}{s-5} + \frac{C}{s+2}$$

Example 1

Now, $Y(s)$ is written into sum of terms which we can find the inverse transform:

$$y(t) = A\mathcal{L}^{-1}\{\frac{1}{s}\} + B\mathcal{L}^{-1}\{\frac{1}{s-5}\} + C\mathcal{L}^{-1}\{\frac{1}{s+2}\}$$

$$= -\frac{1}{5} + \frac{22}{35}e^{5t} + \frac{4}{7}e^{-2t}.$$

Example 2

Example 2

$$y'' + y = \cos 2t, \qquad y(0) = 2, \quad y'(0) = 1.$$

$$s^2Y - 2s - 1 + Y = \mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$$

$$(s^2+1)Y(s) = \frac{s}{s^2+4} + 2s + 1 = \frac{2s^3+s^2+9s+4}{s^2+4}$$

$$Y(s) = \frac{2s^3+s^2+9s+4}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}.$$

$$2s^3+s^2+9s+4 = (As+B)(s^2+4) + (Cs+D)(s^2+1).$$

$$Y(s) = \frac{7}{3}\frac{s}{s^2+1} + \frac{1}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4}$$

$$y(t) = \frac{7}{3}\cos t + \sin t - \frac{1}{3}\cos 2t.$$

$B = 1, A = \frac{7}{3}.$

$D = 0, C = -\frac{1}{3}.$

Thank You

Any questions?

You can find me at:

saad.m@sc.uobaghdad.edu.iq

Solving PDEs Using Laplace Transform

Saad Al-Momen

11

MATHEMATICAL PHYSICS I

Master Degree Class
Department of Astronomy and Space
College of Science - University of Baghdad

Introduction

Given a function $u(x,t)$ defined for all $t > 0$ and assumed to be bounded we can apply the Laplace transform in t considering x as a parameter.

$$L(u(x,t)) = \int_0^\infty e^{-st} u(x,t)\, dt \equiv U(x,s)$$

In applications to PDEs we need the following:

$$L(u_t(x,t) = \int_0^\infty e^{-st} u_t(x,t)\, dt = e^{-st} u(x,t)\Big|_0^\infty + s \int_0^\infty e^{-st} u(x,t)\, dt = sU(x,s) - u(x,0)$$

$u = e^{-st}$

$du = -se^{-st} dt$

$dv = u_t \, dt$

$v = u$

so we have

$$L(u_t(x,t) = sU(x,s) - u(x,0)$$

In exactly the same way we obtain

$$L(u_{tt}(x,t) = s^2 U(x,s) - su(x,0) - u_t(x,0).$$

We also need the corresponding transforms of the x derivatives:

$$L(u_x(x,t)) = \int_0^\infty e^{-st} u_x(x,t)\, dt = U_x(x,s)$$

$$L(u_{xx}(x,t)) = \int_0^\infty e^{-st} u_{xx}(x,t)\, dt = U_{xx}(x,s)$$

1D Wave Equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + \sin(\pi x), \quad 0 < x < 1, \quad t > 0,$$

$$u(x,0) = 0, \quad u_t(x,0) = 0$$

$$u(0,t) = 0 \quad u(1,t) = 0.$$

Taking the Laplace transform and applying the initial conditions we obtain

$$\frac{d^2 U}{dx^2}(x,s) = s^2 U(x,s) - su(x,0) - u_t(x,0) - \frac{\sin(\pi x)}{s} = s^2 U(x,s) - \frac{\sin(\pi x)}{s}.$$

$$\frac{d^2 U}{dx^2}(x,s) - s^2 U(x,s) = -\frac{\sin(\pi x)}{s}$$

$$m^2 - s^2 = 0$$

$$m_{1,2} = \mp s$$

$$U(x,s) = U_h(x,s) + U_p(x,s)$$

$$U_h(x,s) = c_1 e^{sx} + c_2 e^{-sx}$$

$$U_p(x,s) = A\cos(\pi x) + B\sin(\pi x).$$

$$U_p(x,s) = A\cos(\pi x) + B\sin(\pi x).$$

$$\frac{d}{dx}U_p(x,s) = -\pi A\sin(\pi x) + \pi B\cos(\pi x),$$

$$\frac{d^2}{dx^2}U_p(x,s) = -\pi^2 A\cos(\pi x) - \pi^2 B\sin(\pi x).$$

$$\begin{aligned}\frac{d^2}{dx^2}U_p(x,s) - s^2U_p(x,s) \\ &= (-\pi^2 - s^2)[A\cos(\pi x) + B\sin(\pi x)] \\ &= -\frac{\sin(\pi x)}{s}.\end{aligned}$$

$$-(s^2+\pi^2)A = 0, \qquad \text{and} \qquad -(s^2+\pi^2)B = -\frac{1}{s},$$

$$A = 0, \quad B = \frac{1}{s(s^2+\pi^2)}.$$

$$U_p(x,s) = \frac{\sin(\pi x)}{s(s^2+\pi^2)}$$

$$U(x,s) = c_1 e^{sx} + c_2 e^{-sx} + \frac{\sin(\pi x)}{s(s^2+\pi^2)}.$$

$u(0,t) = 0 \quad u(1,t) = 0.$

Next we apply the BCs to find c_1 and c_2.

$$0 = U(0,s) = c_1 + c_2, \quad \text{and} \quad 0 = U(1,s) = c_1 e^s + c_2 e^{-s}$$

which implies $c_1 = 0$ and $c_2 = 0$. So we arrive at

$$U(x,s) = \frac{\sin(\pi x)}{s(s^2+\pi^2)}.$$

Finally we apply the inverse Laplace transform to obtain

$$u(x,t) = L^{-1}(U(x,s)) = L^{-1}\left(\frac{1}{s(s^2+\pi^2)}\right) \sin(\pi x)$$

$$\frac{1}{s(s^2+\pi^2)} = \frac{a}{s} + \frac{bs+c}{(s^2+\pi^2)} = \frac{1}{\pi^2}\left(\frac{1}{s} - \frac{s}{(s^2+\pi^2)}\right)$$

$$= \frac{1}{\pi^2} L^{-1} \left(\frac{1}{s} - \frac{s}{(s^2 + \pi^2)} \right) \sin(\pi x)$$

$$= \frac{1}{\pi^2}(1 - \cos(\pi t)) \sin(\pi x).$$

Thank You

Any questions?

You can find me at:

saad.m@sc.uobaghdad.edu.iq