Mathematics II

INTEGRATION

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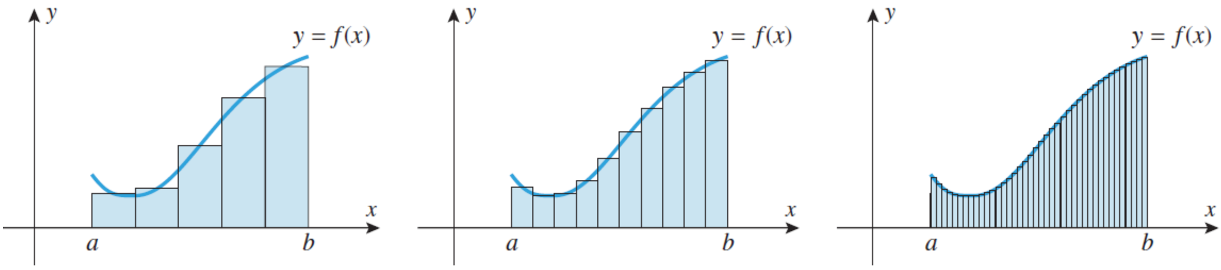
Integration

Integration is a mathematical technique used to find the area under a curve or an accumulation of a quantity over an interval by dividing it into smaller parts and adding them up. It is the inverse operation of differentiation and involves finding the antiderivative of a primitive function of a given function. In simpler terms, integration is the process of finding the whole from its parts.



☐ Indefinite Integral

Definition: A function F is called an ***antiderivative*** of a function f on a given open interval if $F'(x) = f(x)$ for all x in the interval.

For example the function $F(x) = \frac{1}{3}x^3$ is the antiderivative of $f(x) = x^2$ on the interval $(-\infty, +\infty)$ because for each x in this interval

$$F'(x) = \frac{d}{dx}\left[\frac{1}{3}x^3\right] = x^2 = f(x)$$

However, $F(x) = \frac{1}{3}x^3$ is not the only antiderivative of $f(x) = x^2$ on this interval. If we add any constant C to $\frac{1}{3}x^3$, then the function $G(x) = \frac{1}{3}x^3 + C$ is also an antiderivative of f on $(-\infty, +\infty)$, since

$$G'(x) = \frac{d}{dx}\left[\frac{1}{3}x^3 + C\right] = x^2 + 0 = f(x)$$

Definition: The ***indefinite integral*** of $f(x)$ is given by

$$\int f(x)dx = F(x) + C$$

where C any constant and $F(x)$ integration of f.

Example: Find $\int 3x^2 dx$

Solution: $\int 3x^2 dx = \frac{3x^3}{3} + C = x^3 + C$

Theorem: (Rules of Indefinite Integral)

1. $\int 1\, dx = x + C$
2. $\int a\, dx = ax + C$
3. $\int af(x)dx = a\int f(x)dx$
4. $\int (f \mp g)(x)\, dx = \int f(x)dx \mp \int g(x)dx$
5. $\int [a_1 f_1(x) \mp a_2 f_2(x) \mp \cdots \mp a_n f_n(x)]dx = a_1 \int f_1(x)dx \mp a_2 \int f_2(x)dx \mp \cdots \mp a_n \int f_n(x)dx$
6. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, where $n \neq -1$
7. $\int \cos x\, dx = \sin x + C$
8. $\int \sin x\, dx = -\cos x + C$
9. $\int \sec^2 x\, dx = \tan x + C$
10. $\int \sec x \tan x\, dx = \sec x + C$
11. $\int \cot x \csc x\, dx = -\csc x + C$

Example: Evaluate the following

1) $\int \frac{x^4+1}{x^2} dx = \int (x^2 + x^{-2})dx = \frac{x^3}{3} - \frac{1}{x} + C$
2) $\int (\sin x + x)dx = \int \sin x\, dx + \int x dx = -\cos x + \frac{x^2}{2} + C$
3) $\int \frac{\cos x}{\sin^2 x} dx = \int \frac{1}{\sin x}\frac{\cos x}{\sin x} dx = \int \csc x \cot x\, dx = -\csc x + C$
4) $\int \frac{t^2 - 2t^4}{t^4} dt = \int \left(\frac{1}{t^2} - 2\right) dt = \int (t^{-2} - 2)dt = \frac{t^{-1}}{-1} - 2t + C = -\frac{1}{t} - 2t + C$

☐ Definite Integral

Definition: ***Definite Integral*** has start and end points. In other words, there is an interval $[a, b]$

The points are put at the bottom and top of the $\int$ as $\int_a^b$

$\int_a^b f(x)\, dx$ definite integral from a to b

☐ The First Fundamental Theorem of Calculus

Theorem: Let f be a continuous function on the closed interval $[a, b]$, then

$$\int_{a}^{b} f(x)dx = F(b) - F(a)$$

Where F is any integration of f on $[a, b]$.

Example: Find $\int_{1}^{3} 2x\,dx = x^2|_1^3 = 3^2 - 1^2 = 8$

Properties of Definite Integral:

1. $\int_{a}^{a} f(x)dx = 0$
2. $\int_{a}^{b} f(x)dx = -\int_{b}^{a} f(x)dx$
3. $\int_{a}^{b} kf(x)dx = k\int_{a}^{b} f(x)dx$, where k constant.
4. $\int_{a}^{b}(f \mp g)(x)dx = \int_{a}^{b} f(x)dx \mp \int_{a}^{b} g(x)dx$
5. $\int_{a}^{b} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{b} f(x)dx$, where $c \in [a, b]$
6. If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.
7. If f is integrable on $[a, b]$ and $f(x) \geq 0\ \forall x \in [a, b]$, then $\int_{a}^{b} f(x)dx \geq 0$.
8. If f and g are integrable on $[a, b]$ and $f(x) \geq g(x) \forall x \in [a, b]$ then $\int_{a}^{b} f(x)dx \geq \int_{a}^{b} g(x)dx$

☐ The Second Fundamental Theorem of Calculus

Theorem: Let f a continouse function on the closed interval $[a, b]$, and define

$$G(x) = \int_{a}^{x} f(t)dt,\ a \leq x \leq b$$

Then,

$$G'(x) = \frac{d}{dx}\left[\int_{a}^{x} f(t)dt\right] = f(x)$$

For example,

1) $\frac{d}{dx}\int_{a}^{x} \cos t\,dt = \cos x$
2) $\frac{d}{dx}\int_{a}^{x} \frac{dt}{1+t^2} = \frac{1}{1+x^2}$

Remarks:

1) $\frac{d}{dx}\int_a^{g(x)} f(t)dt = f\big(g(x)\big)\cdot g'(x)$
2) $\frac{d}{dx}\int_{g_1(x)}^{g_2(x)} f(t)dt = f\big(g_2(x)\big)\cdot g_2'(x) - f\big(g_1(x)\big)\cdot g_1'(x)$

For Example,

1) $\frac{d}{dx}\int_1^{x^2} \sin t\, dt = (\sin x^2)(2x)$
2) $\frac{d}{dx}\int_{2x}^{x^2} (1+t)\, dt = (1+x^2)(2x) - (1+2x)(2)$

☐ Integration by Substitution

Substitution applies to integrals of the form

$$\int f\big(g(x)\big)g'(x)dx$$

Note that, we have $g(x)$ and its derivative $g'(x)$.

Let $u = g(x)$ be a differentiable function. Then $du = g'(x)dx$ and

$$\int f\big(g(x)\big)g'(x)dx = \int f(u)du = F(u) + C$$

Also

$$\int_a^b f\big(g(x)\big)g'(x)dx = \int_{g(a)}^{g(b)} f(u)du = F\big(g(b)\big) - F\big(g(a)\big)$$

Example: Find $\int \cos x^2 \cdot 2xdx$

Solution: Let $u = x^2$, then $du = 2xdx$

$$\Rightarrow \int \cos x^2 \cdot 2xdx = \int \cos u\, du = \sin u + C = \sin x^2 + C$$

Example: Find $\int \sin(x+9)\, dx$

Solution: Let $u = x + 9$, then $du = dx$

$$\Rightarrow \int \sin(x+9)\, dx = \int \sin u\, du = -\cos u + C = -\cos(x+9) + C$$

Example: Find $\int (x-8)^{23} dx$

Solution: Let $u = x - 8$, then $du = dx$

$$\Rightarrow \int (x-8)^{23}dx = \int u^{23}du = \frac{u^{24}}{24} + C = \frac{(x-8)^{24}}{24} + C$$

Example: Find $\int \cos x^2 \cdot 6x\, dx$

Solution: Let $u = x^2$, then $du = 2xdx$

$$\Rightarrow \int \cos x^2 \cdot 6x\, dx = \int \cos x^2 \cdot 2x \cdot 3dx = 3\int \cos u\, du$$

$$= 3\sin u + C = 3\sin x^2 + C$$

Example: Find $\int (\sin x)^2 \cos x \; dx$

Solution: $\int (\sin x)^2 \cos x \; dx = \int \frac{\sin^3 x}{3} + C$

Example: Find $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx$

Solution: $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx = \int \cos\sqrt{x} \frac{1}{\sqrt{x}} dx = \int \frac{2}{2} \cos\sqrt{x} \frac{1}{\sqrt{x}} dx = 2\int \cos\sqrt{x} \frac{1}{2\sqrt{x}} dx$

$$= 2\int \sin\sqrt{x}\, dx$$

Example: Find $\int t^4 \sqrt[3]{3-5t^5} dt$

Solution:

Let $u = 3 - 5t^5$, then $du = -25t^4dt$ or $-\frac{1}{25} du = t^4dt$

$$\int t^4 \sqrt[3]{3-5t^5} dt = -\frac{1}{25}\int \sqrt[3]{u}\, du = -\frac{1}{25}\int u^{\frac{1}{3}} du = -\frac{1}{25} \frac{u^{\frac{4}{3}}}{\frac{4}{3}} + C$$

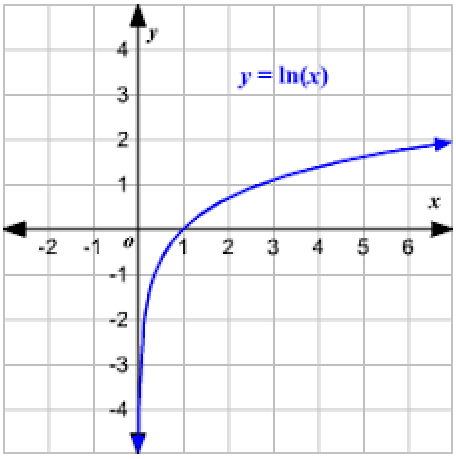
$$= -\frac{3}{100}(3-5t^5)^{\frac{4}{3}} + C$$

The Natural Logarithm Function

Definition: The natural logarithm function is defined as:

$$\ln x = \int_1^x \frac{1}{t} dt \,, x > 0$$

Remarks:



1) Domain of $\ln x = \mathbb{R}^+$
2) Range of $\ln x = \mathbb{R}$
3) $\ln 1 = \int_1^1 \frac{1}{t} dt = 0$
4) $\ln x$ is continues function
5) $\ln ab = \ln a + \ln b$
6) $\ln \frac{a}{b} = \ln a - \ln b$
7) $\ln \frac{1}{b} = -\ln b$
8) $\ln a^n = n \ln a$

Examples:

1) $\ln 6 = \ln 2 + \ln 3$
2) $\ln \frac{4}{5} = \ln 4 - \ln 5$
3) $\ln 8 = \ln 2^3 = 3 \ln 2$

☐ The Derivative of $y = \ln x$

The derivative of $y = \ln x$ is given by

$$y' = \frac{dy}{dx} = \frac{d}{dx}[\ln x] = \frac{1}{x}, (x > 0)$$

In general

$$y' = \frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx}$$

if $u = u(x) > 0$, and u is differentiable at x.

Example: Find y' for the following functions:

1) $y = \ln 3x \Rightarrow \; y' = \frac{3}{3x} = \frac{1}{x}$
2) $y = \ln(x^2 + 3) \Rightarrow \; y' = \frac{2x}{x^2+3}$

Remark:

$$\frac{d}{dx}[\ln|x|] = \frac{1}{x}, \text{ if } x \neq 0$$

In general

$$\frac{d}{dx}[\ln|u|] = \frac{1}{u}\frac{du}{dx}$$

if $u = u(x) \neq 0$, and u is differentiable at x.

Example: Find y' for the function $y = \ln|\sin x|$

Solution: $\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$

Example: Find y' for the function $y = \ln\left(\frac{x^2 \sin x}{\sqrt{1+x}}\right)$

Solution: $\frac{dy}{dx} = \frac{d}{dx}\left[2\ln x + \ln\sin x - \frac{1}{2}\ln(1+x)\right]$

$$= \frac{2}{x} + \frac{\cos x}{\sin x} - \frac{1}{2(1+x)} = \frac{2}{x} + \cot x - \frac{1}{2+2x}$$

□ Integration Related to the Natural Logarithm

1) $\int \frac{1}{x}\,dx = \ln|x| + C,\ x \neq 0$
2) $\int \frac{1}{u}\,du = \ln|u| + C$, if $u = u(x) \neq 0$, and u is differentiable at x.

Example: Find $\int \frac{3x^2}{x^3+5}\,dx$

Solution: $\int \frac{3x^2}{x^3+5}\,dx = \ln|x^3+5| + C$

Example: Find $\int \tan x\ dx$

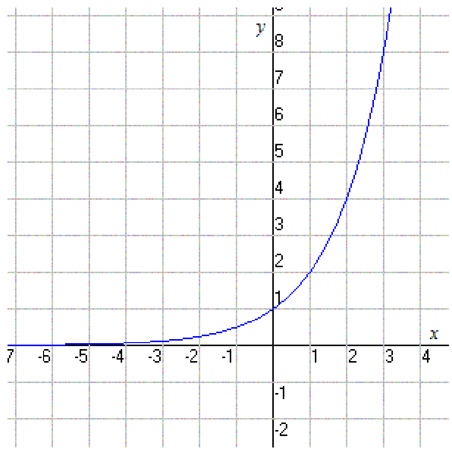
Solution: $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| + C$

The Exponential Function

Definition: The exponential function $y = \exp(x) = e^x$ is the invers of the natural logarithm function

$$e^x = \ln^{-1} x$$

Remarks:



1) Domain of $e^x = \mathbb{R}$
2) Range of $e^x = \mathbb{R}^+$
3) $e^0 = 1$
4) $\ln e^x = x$
5) $e^{\ln x} = x$
6) $e^{-x} = \frac{1}{e^x}$
7) $e^{x+y} = e^x e^y$
8) $e^{x-y} = \frac{e^x}{e^y}$

Examples:

1) $\ln e^2 = 2$
2) $\ln e^{\sin x} = \sin x$
3) $\ln \frac{e^{2x}}{5} = \ln e^{2x} - \ln 5 = 2x - \ln 5$

☐ Derivative of $y = e^x$

The derivative of $y = e^x$ is given by

$$y' = \frac{d}{dx} e^x = e^x$$

In general

$$y' = \frac{d}{dx} e^u = e^u \frac{du}{dx}$$

if $u = u(x)$ is differentiable at x.

Example: Find y' for the following functions:

1) $y = e^{\sin x} \Rightarrow y' = e^{\sin x} \cos x$
2) $y = e^{-x} \Rightarrow y' = e^{-x}(-1) = -e^{-x}$

3) $y = e^{x^3} \Rightarrow y' = e^{x^3}\frac{d}{dx}(x^3) = 3x^2e^{x^3}$

4) $y = e^{-2x} \Rightarrow y' = e^{-2x}\frac{d}{dx}(-2x) = -2e^{-2x}$

□ Integration Related to the Exponential Function

$$\int e^u du = e^u + C$$

if $u = u(x)$ is differentiable at x.

Example: Find $\int 3e^{3x+1}dx$

Solution: Let $u = 3x + 1 \Rightarrow \frac{du}{dx} = 3 \Rightarrow du = 3dx$

$$\int e^{3x+1}(3dx) = \int e^u du = e^u + C = e^{3x+1} + C$$

Example: Find $\int e^{5x}dx$

Solution: Let $u = 5x \Rightarrow \frac{du}{dx} = 5 \Rightarrow \frac{1}{5}du = dx$

$$\int e^{5x}dx = \frac{1}{5}\int e^u du = \frac{1}{5}e^u + C = \frac{1}{5}e^{5x} + C$$

Example: Find $\int \frac{e^{\tan^{-1}x}}{1+x^2}dx$

Solution: Let $u = \tan^{-1}x \Rightarrow \frac{du}{dx} = \frac{1}{1+x^2} \Rightarrow du = \frac{dx}{1+x^2}$

$$\int \frac{e^{\tan^{-1}x}}{1+x^2}dx = \int e^{\tan^{-1}x}\frac{dx}{1+x^2} = \int e^u du = e^u + C = e^{\tan^{-1}x} + C$$

Example: Find $\int_0^{\ln 3} e^x(1+e^x)^{1/2}\,dx$

Solution: Let $u = 1 + e^x \Rightarrow \frac{du}{dx} = e^x \Rightarrow du = e^x dx$

$$x = 0 \Rightarrow u = 1 + e^0 = 1 + 1 = 2$$

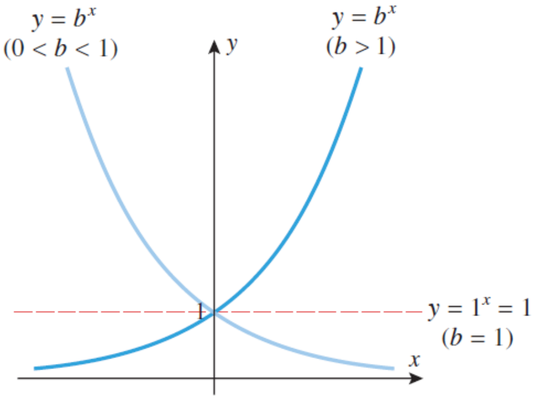
$$x = \ln 3 \Rightarrow u = 1 + e^{\ln 3} = 1 + 3 = 4$$

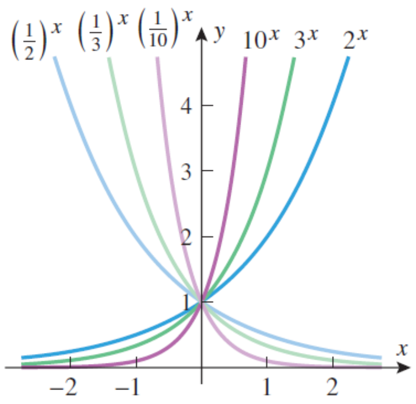
$$\int_0^{\ln 3} e^x(1+e^x)^{1/2}\,dx = \int_2^4 u^{1/2}\,du = \frac{2}{3}u^{3/2}\Big|_2^4 = \frac{2}{3}\left[4^{3/2} - 2^{3/2}\right]$$

$$= \frac{16 - 4\sqrt{2}}{3}$$

The Function $y = a^x$

Definition: For any number $a > 0$, then $a^x = e^{x \ln a}$





Examples:

1) $e^{\sqrt{3}\ln 2} = 2^{\sqrt{3}}$
2) $3^{\pi} = e^{\pi \ln 3}$

☐ Properties of a^x

1) Domain of $a^x = \mathbb{R}$
2) Range of $a^x = \mathbb{R}^+$
3) a^x is a continuous function
4) $a^0 = 1$
5) $a^1 = a$
6) $a^{x+y} = a^x \cdot a^y$
7) $a^{x-y} = \frac{a^x}{a^y}$
8) $a^{-x} = \frac{1}{a^x}$
9) $(a^x)^y = (a^y)^x = a^{xy}$

☐ Derivative and Integral of a^x

Derivative	Integral
$\frac{d}{dx}(a^x) = a^x \ln a$	$\int a^x dx = \frac{a^x}{\ln a} + C$
$\frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx}$	$\int a^u dx = \frac{a^u}{\ln a} + C$

Examples:

1) $y = 3^{-x} \qquad \Rightarrow y' = \frac{d}{dx} 3^{-x} = 3^{-x} \ln 3 \ (-1) = -3^{-x} \ln 3$

2) $y = 3^{\sin x} \qquad \Rightarrow y' = \frac{d}{dx} 3^{\sin x} = 3^{\sin x} \ln 3 \ (\cos x)$

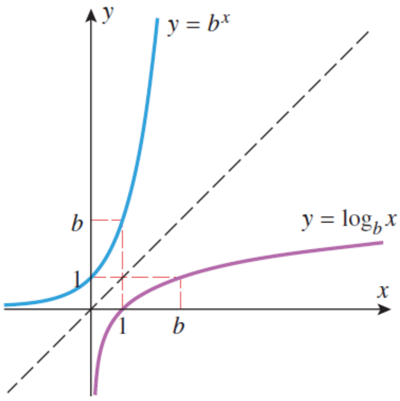
3) $\int 2^x \, dx = \frac{2^x}{\ln a} + C$

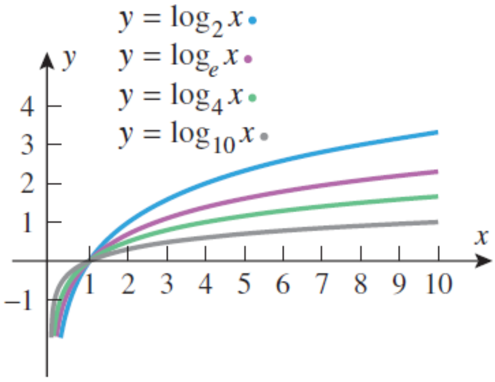
4) $\int 5^{\sin x} \cos x \, dx = \frac{5^{\sin x}}{\ln 5} + C$

The Function $y = \log_a x$

Definition: For any number $a > 0$, $a \neq 1$

$$y = \log_a x = \frac{\ln x}{\ln a}$$





Remarks:

1) $\log_a x$ is the inverse of a^x
2) If $a = 1$, then $\log_a x = \frac{\ln x}{\ln a} = \frac{\ln x}{\ln 1} = \frac{\ln x}{0} = \infty$

☐ Properties of $y = \log_a x$

1) Domain of $\log_a x = (0, \infty)$
2) Range of $\log_a x = \mathbb{R}$
3) $\log_a x$ is a continuous function
4) $\log_a xy = \log_a x + \log_a y$
5) $\log_a \frac{x}{y} = \log_a x - \log_a y$
6) $\log_a x^y = y \log_a x$

☐ Derivative and Integral of $\log_a x$

Derivative	Integral
$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$	$\int \frac{\log_a x}{x}\, dx = \int \frac{\ln x}{x \ln a}\, dx$ $= \frac{1}{\ln a} \int \frac{\ln x}{x}\, dx$ $= \frac{1}{\ln a} \frac{(\ln x)^2}{2} + C$
$\frac{d}{dx}(\log_a u) = \frac{1}{u \ln a} \frac{du}{dx}$	

Examples:

1) $\int \frac{\log_2 x}{x} dx = \int \frac{\ln x}{x \ln 2} dx = \frac{1}{\ln 2} \int \frac{\ln x}{x} dx = \frac{1}{2 \ln 2} (\ln x)^2 + C$

2) $\int_2^3 \frac{2 \log_2 (x-1)}{x-1} dx = \int_2^3 \frac{2 \ln(x-1)}{(x-1) \ln 2} dx = \frac{2}{\ln 2} \int_2^3 \frac{\ln(x-1)}{(x-1)} dx = \frac{2}{\ln 2} \frac{(\ln(x-1))^2}{2} \Big|_2^3$

$$= \frac{1}{\ln 2} [(\ln(3-1))^2 - (\ln(2-1))^2] = \frac{1}{\ln 2} [(\ln 2)^2 - (\ln 1)^2]$$

$$= \frac{(\ln 2)^2}{\ln 2} = \ln 2$$

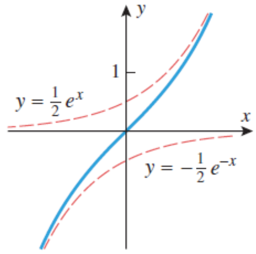
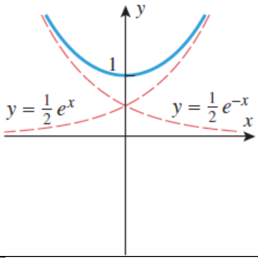
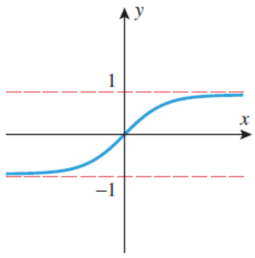
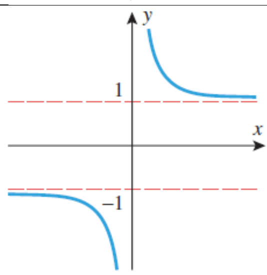
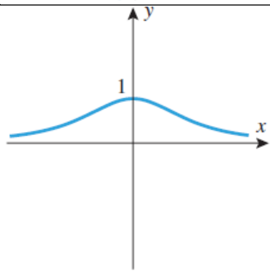
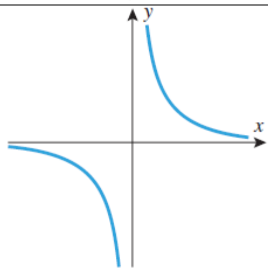
Exercise

1. Rewrite the expression as a single logarithm:
   i. $2\ln(x+1)+\frac{1}{3}\ln x-\ln(\cos x)$
   ii. $4\log 2-\log 3+\log 16$
   iii. $\frac{1}{2}\log x-3\log(\sin 2x)+2$

2. Solve for x without using a calculating utility
   i. $\log_{10}(1+x)=3$
   ii. $\log_{10}\sqrt{x}=-1$
   iii. $\ln\frac{1}{x}=-2$
   iv. $\ln\frac{1}{x}+\ln 2x^3=\ln\ 3$
   v. $3^x=2$
   vi. $5^{-2x}=3$
   vii. $xe^{-x}+2e^{-x}=0$
   viii. $e^{-2x}-3e^{-x}=-2$. [Hint: Rewrite the equation as a quartatic equation in $u=e^{-x}$]

3. Find the following integrals
   i. $\int\frac{2x^2+1}{x^3+x}dx$
   ii. $\int\frac{1}{x\ln x}dx$
   iii. $\int\frac{1}{\sqrt{x}(1+\sqrt{x})}dx$

Hyperbolic Functions

#	Function	Domain	Range	Graph
1	$\sinh x = \dfrac{e^x - e^{-x}}{2}$	$\mathbb{R}$	$\mathbb{R}$	
2	$\cosh x = \dfrac{e^x + e^{-x}}{2}$	$\mathbb{R}$	$[1, \infty]$	
3	$\tanh x = \dfrac{\sinh x}{\cosh x}$	$\mathbb{R}$	$(-1,1)$	
4	$\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{1}{\tanh x}$	$\mathbb{R} - \{0\}$	$\|y\| > 1$	
5	$\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x - e^{-x}}$	$\mathbb{R}$	$(0,1]$	
6	$\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x + e^{-x}}$	$\mathbb{R} - \{0\}$	$\mathbb{R} - \{0\}$	

☐ Properties of the hyperbolic functions

1) $\sinh 0 = \frac{e^0 - e^0}{2} = 0$

2) $\cosh 0 = \frac{e^0 + e^0}{2} = \frac{1+1}{2} = 1$

3) $\sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = \frac{e^{-x} - e^{x}}{2} = -\sinh x$ (odd function)

4) $\cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{-x} + e^{x}}{2} = \cosh x$ (even function)

$\cosh x + \sinh x = e^x$	$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
$\cosh x - \sinh x = e^{-x}$	$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$
$\cosh^2 x - \sinh^2 x = 1$	$\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$
$1 - \tanh^2 x = \operatorname{sech}^2 x$	$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$
$\coth^2 x - 1 = \operatorname{csch}^2 x$	$\sinh 2x = 2 \sinh x \cosh x$
$\cosh(-x) = \cosh x$	$\cosh 2x = \cosh^2 x + \sinh^2 x$
$\sinh(-x) = -\sinh x$	$\cosh 2x = 2\sinh^2 x + 1 = 2\cosh^2 x - 1$

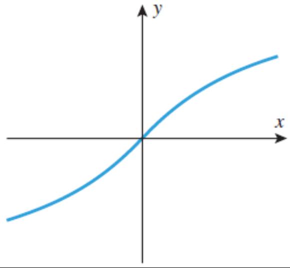
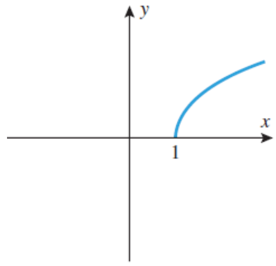
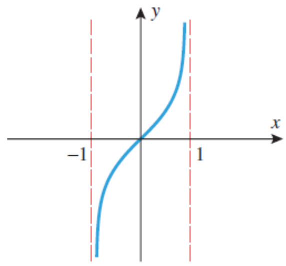
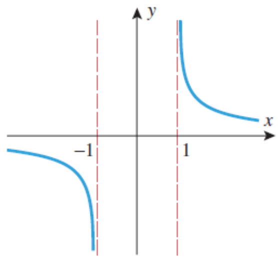
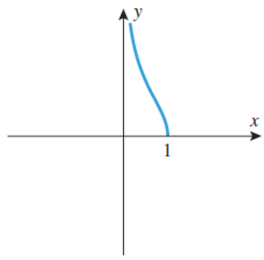
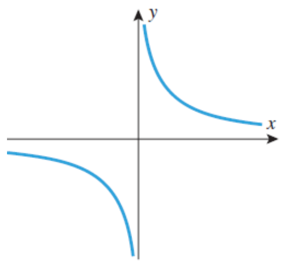
☐ Derivative and Integral of the Hyperbolic Functions

$\frac{d}{dx}[\sinh u] = \cosh u \frac{du}{dx}$	$\int \cosh u \, du = \sinh u + C$
$\frac{d}{dx}[\cosh u] = \sinh u \frac{du}{dx}$	$\int \sinh u \, du = \cosh u + C$
$\frac{d}{dx}[\tanh u] = \operatorname{sech}^2 u \frac{du}{dx}$	$\int \operatorname{sech}^2 u \, du = \tanh u + C$
$\frac{d}{dx}[\coth u] = -\operatorname{csch}^2 u \frac{du}{dx}$	$\int \operatorname{csch}^2 u \, du = -\coth u + C$
$\frac{d}{dx}[\operatorname{sech} u] = -\operatorname{sech} u \tanh u \frac{du}{dx}$	$\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$
$\frac{d}{dx}[\operatorname{csch} u] = -\operatorname{csch} u \coth u \frac{du}{dx}$	$\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$

Examples:

1) $y = \cosh x^3 \Rightarrow y' = \sinh x^3\,(3x^2)$
2) $y = \ln(\tanh x) \Rightarrow y' = \frac{\operatorname{sech}^2 x}{\tanh x}$
3) $y = e^{\sinh x} \Rightarrow y' = e^{\sinh x}\cosh x$
4) Find the integral: $\int \sinh^5 x \cosh x\, dx$
$$\int \sinh^5 x \cosh x\, dx = \frac{\sinh^6 x}{6} + C$$
5) Find the integral: $\int \tanh x\, dx$
$$\int \tanh x\, dx = \int \frac{\sinh x}{\cosh x} dx = \ln|\cosh x| + C$$
6) Find the integral: $\int \frac{\cosh(\ln x)}{x} dx = \sinh(\ln x) + C$

Inverse of Hyperbolic Functions

#	Function	Domain	Range	Graph
1	$y = \sinh^{-1} x$ iff $x = \sinh y$	$\mathbb{R}$	$\mathbb{R}$	
2	$y = \cosh^{-1} x$ iff $x = \cosh y$	$[1, \infty]$	$\mathbb{R}^+$	
3	$\mathrm{y} = \tanh^{-1} x$ iff $x = \tanh y$	$(-1,1)$	$\mathbb{R}$	
4	$\mathrm{y} = \coth^{-1} x$ iff $x = \coth y$	$\lvert x \rvert > 1$	$\mathbb{R} - \{0\}$	
5	$\mathrm{y} = \mathrm{sech}^{-1} x$ iff $x = \mathrm{sech}\, y$	$(0,1]$	$[0, \infty)$	
6	$\mathrm{y} = \mathrm{csch}^{-1} x$ iff $x = \mathrm{csch}\, y$	$\mathbb{R} - \{0\}$	$\mathbb{R} - \{0\}$	

☐ Derivative and Integral of the Inverse Hyperb. Fun.

$$\frac{d}{dx}(\sinh^{-1} u) = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \quad u > 1$$

$$\frac{d}{dx}(\tanh^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| < 1$$

$$\frac{d}{dx}(\coth^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| > 1$$

$$\frac{d}{dx}(\text{sech}^{-1} u) = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \quad 0 < u < 1$$

$$\frac{d}{dx}(\text{csch}^{-1} u) = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}, \quad u \neq 0$$

$$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C \;\; or \;\; \ln(u + \sqrt{u^2+a^2}\,) + C$$

$$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C \;\; or \;\; \ln(u + \sqrt{u^2-a^2}\,) + C, \quad u > a$$

$$\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C, & |u| < a \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C, & |u| > a \end{cases} \quad or \quad \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C, \quad |u| \neq a$$

$$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\text{sech}^{-1}\left|\frac{u}{a}\right| + C \;\; or \;\; -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-u^2}}{|u|}\right) + C, \quad 0 < |u| < a$$

$$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\text{csch}^{-1}\left|\frac{u}{a}\right| + C \;\; or \;\; -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2+u^2}}{|u|}\right) + C, \quad u \neq 0$$

Examples:

1) Find y' for the function $y = \cosh^{-1}(\sec x)\,, 0 < x < \frac{\pi}{4}$

$$y' = \frac{1}{\sqrt{\sec^2 x - 1}}\frac{d(\sec x)}{dx} = \frac{1}{\sqrt{\tan^2 x}}(\sec x \tan x) = \sec x$$

2) Find $\int \frac{dx}{\sqrt{4+x^2}}$

$$\int \frac{dx}{\sqrt{4+x\hat{}2}} = \int \frac{dx}{\sqrt{2^2+x^2}} = \sinh^{-1}(\frac{x}{2}) + C$$

3) Find $\int \frac{dx}{9-x^2}$

$$\int \frac{dx}{9-x^2} = \int \frac{dx}{3^2-x^2} = \frac{1}{(2)(3)}\ln\left|\frac{3+x}{3-x}\right| + C = \frac{1}{6}\ln\left|\frac{3+x}{3-x}\right| + C$$

☐ Integration Form of Inverse Trigonometric Functions

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

Examples:

1) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
2) $\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{\sqrt{3-x^2}} = \sin^{-1}(\frac{x}{3}) + C$
3) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$

Exercise

Find the following:

1) $\int \frac{dx}{5-4x^2}$
2) $\int \frac{dx}{1+x^2}$
3) $\int \frac{e^x}{1+e^{2x}} dx$

Integration by Parts

The formula for integration by parts is given by

$$\int u\, dv = u\, v - \int v\, du$$

And for definite integrals the formula is given by

$$\int_a^b u\, dv = u\, v|_a^b - \int_a^b v\, du$$

Example: Evaluate $\int x \cos x \; dx$

Solution: Let $u = x$ and $dv = \cos x \; dx$

$du = dx$ and $v = \int \cos x \; dx = \sin x$

$$\int x \cos x \; dx = u\, v - \int v\, du$$

$$= x \sin x - \int \sin x \; dx = x \sin x + \cos x + C$$

Example: Evaluate $\int x\, e^x\, dx$

Solution: Let $u = x$ and $dv = e^x\, dx$

$du = dx$ and $v = \int e^x\, dx = e^x$

$$\int x\, e^x\, dx = u\, v - \int v\, du$$

$$= x\, e^x - \int e^x dx = xe^x - e^x + C$$

Example: Evaluate $\int \ln x \; dx$

Solution: Let $u = \ln x$ and $dv = \; dx$

$du = \frac{1}{x} dx$ and $v = \int \; dx = x$

$$\int \ln x \; dx = u\, v - \int v\, du$$

$$= x \ln x - \int x \frac{dx}{x} = x \ln x - x + C$$

Example: Evaluate $\int x^2 e^{-x}\, dx$

Solution: Let $u = x^2$ and $dv = e^{-x}\, dx$

$du = 2x\, dx$ and $v = \int e^{-x}\, dx = -e^{-x}$

$$\int x^2 e^{-x}\, dx = x^2(-e^{-x}) - \int -e^{-x}\, 2x\, dx = -x^2 e^{-x} + 2\int x e^{-x} dx$$

Now let $u = x$ and $dv = e^{-x} dx$

$du = dx$ and $v = -e^{-x}$

$\therefore \int x^2 e^{-x}\, dx = -x^2 e^{-x} + 2x(-e^{-x}) - 2\int -e^{-x}\, dx$

$$= -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} = -(x^2 + 2x + 2)e^{-x} + C$$

Example: Evaluate $\int e^x \cos x\, dx$

Solution: Let $u = \cos x$ and $dv = e^x dx$

$du = -\sin x\, dx$ and $v = e^x$

$$\int e^x \cos x\, dx = e^x \cos x + \boxed{\int \sin x\, e^x dx}$$

Let $u = \sin x$ and $dv = e^x dx$

$du = \cos x\, dx$ and $v = e^x$

Now, $\boxed{\int \sin x\, e^x dx = e^x \sin x - \int \cos x\, e^x dx}$

$$\boxed{\int e^x \cos x\, dx} = e^x \cos x + e^x \sin x \boxed{- \int \cos x\, e^x dx}$$

$$2\int e^x \cos x\, dx = e^x \cos x + e^x \sin x$$

$$\int e^x \cos x\, dx = \frac{e^x}{2}(\cos x + \sin x) + C$$

Example: Evaluate $\int_0^1 \tan^{-1} x\, dx$

Solution: Let $u = \tan^{-1} x$ and $dv = dx$

$du = \frac{1}{1+x^2} dx$ and $v = x$

$$\int_0^1 \tan^{-1} x\, dx = x \tan^{-1} x|_0^1 - \int_0^1 \frac{x}{1+x^2} dx$$

Now,

$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx = \frac{1}{2}\ln(1+x^2)\big|_0^1 = \frac{1}{2}\ln 2$$

$$\therefore \int_0^1 \tan^{-1} x\,dx = \left(\frac{\pi}{4} - 0\right) - \frac{1}{2}\ln 2 = \frac{\pi}{4} - \ln\sqrt{2}$$

Exercise

Evaluate: $\int e^x \sin x \ dx$

☐ A Tabular Method for Repeated Integration by Parts

For integrals of the form

$$\int p(x)f(x)\,dx$$

Step 1. Differentiate $p(x)$ repeatedly until you obtain 0, and list the results in the first column.

Step 2. Integrate $f(x)$ repeatedly and list the results in the second column.

Step 3. Draw an arrow from each entry in the first column to the entry that is one row down in the second column.

Step 4. Label the arrows with alternating $+$ and $-$ signs, starting with a $+$.

Step 5. For each arrow, form the product of the expressions at its tip and tail and then multiply that product by $+1$ or -1 in accordance with the sign on the arrow. Add the results to obtain the value of the integral.

Example: Evaluate $\int (x^2 - x) \cos x \, dx$

Solution:

Repeated Differentiation		Repeated Integration
$x^2 - x$	+	$\cos x$
$2x - 1$	-	$\sin x$
2	+	$-\cos x$
0		$-\sin x$

$$\int (x^2 - x) \cos x \, dx = (x^2 - x) \sin x + (2x - 1) \cos x - 2 \sin x + C$$

$$= (x^2 - x - 2) \sin x + (2x - 1) \cos x + C$$

Example: Evaluate $\int x^2 \sqrt{x - 1} dx$

Solution:

Repeated Differentiation		Repeated Integration
x^2	+	$(x-1)^{1/2}$
$2x$	-	$\frac{2}{3}(x-1)^{3/2}$
2	+	$\frac{4}{15}(x-1)^{5/2}$
0		$\frac{8}{105}(x-1)^{7/2}$

$$\int x^2 \sqrt{x-1} dx = \frac{2}{3} x^2 (x-1)^{\frac{3}{2}} - \frac{8}{15} x (x-1)^{5/2} + \frac{16}{105} (x-1)^{7/2} + C$$

Integration of Trigonometric Functions

1

Integrals of the form

$$\int \sin^n x\, dx \quad \text{and} \quad \int \cos^n x\, dx$$

where n is positive integer.

☐ When n is Even

If n is even, use the following

$$\cos^2 x = \tfrac{1}{2}(1+\cos 2x) \quad \text{or} \quad \sin^2 x = \tfrac{1}{2}(1-\cos 2x)$$

to reduce the power of $\cos x$ and $\sin x$

Example: Evaluate $\int \sin^2 x\, dx$

$n = 2$ is even, we shall use $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$$\int \sin^2 x\, dx = \int \frac{1}{2}(1-\cos 2x)\, dx = \frac{1}{2}\left[\int dx - \int \cos 2x\, dx\right]$$
$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x + C\right] = \frac{1}{2}x - \frac{1}{4}\sin 2x + C_1$$

Example: Evaluate $\int \cos^2 x\, dx$

$n = 2$ is even, we shall use $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$$\int \cos^2 x\, dx = \int \tfrac{1}{2}(1+\cos 2x)\, dx = \tfrac{1}{2}\left[\int dx + \int \cos 2x\, dx\right]$$
$$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x + C\right] = \frac{1}{2}x + \frac{1}{4}\sin 2x + C_1$$

Example: Evaluate $\int \sin^4 x\, dx$

$n = 4$ is even, we shall use $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$$\int \sin^4 x\,dx = \int (\sin^2 x)^2 = \int \left[\tfrac{1}{2}(1-\cos 2x)\right]^2 dx = \tfrac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)dx = \tfrac{1}{4}\left[\int dx - \int 2\cos 2x\,dx + \int \cos^2 2x\,dx\right]$$

$$\because \int \cos^2 2x\,dx = \int \frac{1}{2}(1+\cos 4x)\,dx = \frac{1}{2}\left[\int dx + \int \cos 4x\,dx\right]$$

$$= \frac{1}{2}\left[x + \frac{1}{4}\sin 4x + C\right] = \frac{1}{2}x + \frac{1}{8}\sin 4x + C_1$$

$$\therefore \int \sin^4 x\,dx = \frac{1}{4}\left[x - \sin 2x + \frac{1}{2}x + \frac{1}{8}\sin 4x + C_1\right]$$

$$= \frac{3}{4}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C_2$$

☐ When n is Odd

If n is odd, we shall follow the following procedure

1) Split the odd power to (1)+even power.
2) Use $\cos^2 x = 1 - \sin^2 x$ or $\sin^2 x = 1 - \cos^2 x$

Example: Evaluate $\int \cos^5 x\,dx$

$n = 5$ is odd power of $\cos x$, so split 5 to (1 + 4) and we use

$$\cos^2 x = 1 - \sin^2 x$$

$$\int \cos^5 x\,dx = \int \cos^4 x \cos x\,dx = \int (\cos^2 x)^2 \cos x\,dx$$

$$= \int (1-\sin^2 x)^2 \cos x\,dx = \int (1 - 2\sin^2 x + \sin^4 x)\cos x\,dx$$

$$= \int \cos x\,dx - 2\int \sin^2 x \cos x\,dx + \int \sin^4 x \cos x\,dx$$

$$= \sin x - 2\frac{\sin^3 x}{3} + \frac{\sin^5 x}{5} + C$$

Exercise

Evaluate:

1) $\int \cos^4 x\,dx$
2) $\int \cos^3 x\,dx$
3) $\int \sin^3 x\,dx$

2

Integrals of the form

$$\int \sin^m x \cos^n x \, dx$$

where n and m are positive integers.

☐ When n and m are Even

If both powers n and m are even integers, then use the following

$$\cos^2 x = \tfrac{1}{2}(1+\cos 2x) \quad \text{or} \quad \sin^2 x = \tfrac{1}{2}(1-\cos 2x)$$

Example: Evaluate $\int \sin^2 x \cos^2 x \, dx$

Here, both $n = m = 2$, are even integers

$$\int \sin^2 x \cos^2 x \, dx = \int \left(\frac{1}{2}(1-\cos 2x)\right)\left(\frac{1}{2}(1+\cos 2x)\right) dx$$

$$= \frac{1}{4}\int (1-\cos 2x)(1+\cos 2x)\, dx = \int (1+\cos 2x - \cos 2x - \cos^2 2x) dx$$

$$= \frac{1}{4}\int (1-\cos^2 2x) dx = \frac{1}{4}\left[x - \int \cos^2 2x \, dx\right] = \frac{1}{4}\left[x - \frac{1}{2}\int (1+\cos 4x) dx\right]$$

$$= \frac{1}{4}\left[x - \frac{1}{2}(x + \frac{1}{4}\sin 4x) + C\right] = \frac{1}{8}x - \frac{1}{32}\sin 4x + C_1$$

☐ When n or m is Odd

If <u>one</u> of the powers n or m is odd integer, then we shall follow the following procedure

1) Split the odd power to (1)+even power.
2) Use $\cos^2 x = 1 - \sin^2 x$ or $\sin^2 x = 1 - \cos^2 x$

Example: Evaluate $\int \sin^2 x \cos^5 x \, dx$

Here, $n = 5$ is an odd power of $\cos x$, so we split 5 to (4+1) and use

$$\cos^2 x = 1 - \sin^2 x$$

$$\int \sin^2 x \cos^5 x\, dx = \int \sin^2 x \cos^4 x \cos x\, dx$$
$$= \int \sin^2 x\,(1-\sin^2 x)^2 \cos x\, dx$$
$$= \int \sin^2 x\,(1 - 2\sin^2 x + \sin^4 x) \cos x\, dx$$
$$= \int \sin^2 x \cos x\, dx - 2\int \sin^4 x \cos x\, dx + \int \sin^6 x \cos x\, dx$$
$$= \frac{\sin^3 x}{3} - 2\frac{\sin^5 x}{5} + \frac{\sin^7 x}{7} + C$$

INTEGRATING PRODUCTS OF SINES AND COSINES

$\int \sin^m x \cos^n x\, dx$	PROCEDURE	RELEVANT IDENTITIES
n odd	• Split off a factor of $\cos x$. • Apply the relevant identity. • Make the substitution $u = \sin x$.	$\cos^2 x = 1 - \sin^2 x$
m odd	• Split off a factor of $\sin x$. • Apply the relevant identity. • Make the substitution $u = \cos x$.	$\sin^2 x = 1 - \cos^2 x$
$\begin{cases} m \text{ even} \\ n \text{ even} \end{cases}$	• Use the relevant identities to reduce the powers on $\sin x$ and $\cos x$.	$\begin{cases} \sin^2 x = \frac{1}{2}(1-\cos 2x) \\ \cos^2 x = \frac{1}{2}(1+\cos 2x) \end{cases}$

Exercise

Evaluate:

1) $\int \sin^4 x \cos^5 x\, dx$
2) $\int \sin^4 x \cos^4 x\, dx$

Integration of product of $\sin x$ and $\cos x$ with <u>different</u> arguments. In this case we use one of the following relations:

$$\cos ax \cos bx = \frac{1}{2}(\cos(a+b)x + \cos(a-b)x)$$

$$\sin ax \cos bx = \frac{1}{2}(\sin(a+b)x + \sin(a-b)x)$$

$$\sin ax \sin bx = \frac{1}{2}(\cos(a-b)x - \cos(a+b)x)$$

Example: Evaluate $\int \sin 5x \cos 4x \; dx$

$$\int \sin 5x \cos 4x \; dx = \frac{1}{2}\int (\sin(5+4)x + \sin(5-4)x)dx$$

$$= \frac{1}{2}\int (\sin 9x + \sin x)dx = \frac{1}{2}\left(-\frac{\cos 9x}{9} - \cos x\right) + C = -\frac{\cos 9x}{18} - \frac{\cos x}{2} + C$$

Example: Evaluate $\int \cos 5x \cos 7x \; dx$

$$\int \cos 5x \cos 7x \; dx = \frac{1}{2}\int (\cos(5+7)x + \cos(5-7)x)dx$$

$$= \frac{1}{2}\int (\cos 12x + \cos(-2x))dx = \frac{1}{2}\int (\cos 12x + \cos 2x)dx$$

$$= \frac{1}{2}\left[\frac{\sin 12x}{12} + \frac{\sin 2x}{2}\right] + C$$

$$= \frac{\sin 12x}{24} + \frac{\sin 2x}{4} + C$$

Even function

$$f(-x) = f(x)$$

$$\cos(-2x) = \cos 2x$$

INTEGRATING RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

Recall that a rational function is a ratio of two polynomials.

$$H(x) = \frac{P(x)}{Q(x)}, \text{ where } Q(x) \neq 0$$

In this section we will give a general method for integrating rational functions that is based on the idea of decomposing a rational function into a sum of simple rational functions that can be integrated by the methods studied in earlier sections.

☐ Finding the Form of a Partial Fraction Decomposition

The first step in finding the form of the partial fraction decomposition of a proper rational function $\frac{P(x)}{Q(x)}$ is to factor $Q(x)$ completely into linear and irreducible quadratic factors, and then collect all repeated factors so that $Q(x)$ is expressed as a product of distinct factors of the form

$$(ax + b)^m \text{ and } (ax^2 + bx + c)^m$$

☐ Linear Factors

If all of the factors of $Q(x)$ are linear, then the partial fraction decomposition of $\frac{P(x)}{Q(x)}$ can be determined by using the following rule:

LINEAR FACTOR RULE For each factor of the form $(ax + b)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_m}{(ax+b)^m}$$

where $A_1, A_2, \ldots, A_m$ are constants to be determined. In the case where $m = 1$, only the first term in the sum appears.

Example: Evaluate $\int \frac{dx}{x^2+x-2}$

$$\frac{1}{x^2 + x - 2} = \frac{1}{(x-1)(x+2)}$$

$$\frac{1}{(x-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)}$$

where A and B are constants to be determined.

$$\frac{1}{(x-1)(x+2)}=\frac{A(x+2)+B(x-1)}{(x-1)(x+2)}=\frac{(A+B)x+2A-B}{(x-1)(x+2)}$$

Now,

$$(A+B)x+2A-B=1$$

So,

$$A+B=0\Rightarrow A=-B$$

and

$$2A-B=1\Rightarrow -2B-B=1\Rightarrow -3B=1\Rightarrow B=-\frac{1}{3}\Rightarrow A=\frac{1}{3}$$

$$\therefore \int \frac{dx}{x^2+x-2}=\int \frac{\frac{1}{3}}{(x-1)}dx+\int \frac{-\frac{1}{3}}{(x+2)}dx$$

$$=\frac{1}{3}\ln|x-1|-\frac{1}{3}\ln|x+2|+C=\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|+C$$

Example: Evaluate $\int \frac{x}{x^2-2x+1}dx$

$$\int \frac{x}{x^2-2x+1}dx=\int \frac{x}{(x-1)(x-1)}dx=\int \frac{x}{(x-1)^2}dx$$

Now,

$$\frac{x}{(x-1)^2}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}=\frac{A(x-1)+B}{(x-1)^2}=\frac{Ax-A+B}{(x-1)^2}$$

$$=\frac{Ax+(B-A)}{(x-1)^2}$$

$A=1$ and $B-A=0\Rightarrow B=A=1$

$$\therefore \frac{x}{(x-1)^2}=\frac{1}{(x-1)}+\frac{1}{(x-1)^2}$$

$$\Rightarrow \int \frac{x}{x^2-2x+1}dx=\int \frac{dx}{(x-1)}+\int \frac{dx}{(x-1)^2}=\ln|x-1|+\frac{(x-1)^{-1}}{-1}+C$$

$$=\ln|x-1|-\frac{1}{x-1}+C$$

Example: Evaluate $\int \frac{2x+4}{x^3-2x^2}dx$

$$\frac{2x+4}{x^3-2x^2}=\frac{2x+4}{x^2(x-2)}$$

Now,

$$\frac{2x+4}{x^2(x-2)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2}$$

$$\frac{2x+4}{x^2(x-2)}=\frac{Ax(x-2)+B(x-2)+Cx^2}{x^2(x-2)}=\frac{Ax^2-2Ax+Bx-2B+Cx^2}{x^2(x-2)}$$

$$=\frac{(A+C)x^2+(B-2A)x-2B}{x^2(x-2)}$$

So,

$$A+C=0\Rightarrow A=-C$$

$$-2B = 4 \Longrightarrow B = -2$$
$$B - 2A = 2 \Rightarrow -2 - 2A = 2 \Rightarrow -2A = 4 \Longrightarrow A = -2 \Longrightarrow C = 2$$

$$\therefore \int \frac{2x+4}{x^2(x-2)}dx = \int \frac{-2}{x}dx + \int \frac{-2}{x^2}dx + \int \frac{2}{x-2}dx$$
$$= -2\ln|x| + \frac{2}{x} + 2\ln|x-2| + C = 2\ln\left|\frac{x-2}{x}\right| + \frac{2}{x} + C$$

☐ Quadratic Factors

If some of the factors of $Q(x)$ are irreducible quadratics, then the contribution of those factors to the partial fraction decomposition of $\frac{P(x)}{Q(x)}$ can be determined from the following rule:

QUADRATIC FACTOR RULE For each factor of the form $(ax^2+bx+c)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_mx+B_m}{(ax^2+bx+c)^m}$$

where $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m$ are constants to be determined. In the case where $m = 1$, only the first term in the sum appears.

Example: Evaluate $\int \frac{x^2+x-2}{3x^3-x^2+3x-1}dx$

$$3x^3 - x^2 + 3x - 1 = x^2(3x-1) + (3x-1) = (3x-1)(x^2+1)$$

$$\frac{x^2+x-2}{3x^3-x^2+3x-1} = \frac{A}{3x-1} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)(3x-1)}{(3x-1)(x^2+1)}$$

$$x^2 + x - 2 = (A+3B)x^2 + (-B+3C)x + (A-C)$$

$$A + 3B = 1$$
$$-B + 3C = 1$$
$$A - C = -2$$

Solving these three equations, we get

$$A = -\frac{7}{5}, \quad B = \frac{4}{5}, \quad C = \frac{3}{5}$$

$$\int \frac{x^2+x-2}{3x^3-x^2+3x-1}dx = \int \frac{x^2+x-2}{(3x-1)(x^2+1)}dx$$
$$= \int \frac{-\frac{7}{5}}{3x-1}dx + \int \frac{\frac{4}{5}x+\frac{3}{5}}{x^2+1}dx$$
$$= -\frac{7}{5}\int \frac{dx}{3x-1} + \frac{4}{5}\int \frac{x}{x^2+1}dx + \frac{3}{5}\int \frac{dx}{x^2+1}$$
$$= -\frac{7}{5}\ln|3x-1| + \frac{2}{5}\ln|x^2+1| + \frac{3}{5}\tan^{-1}x + C$$

Example: Evaluate $\int \frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}dx$

$$\frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2} = \frac{A}{(x+2)} + \frac{Bx+C}{x^2+3} + \frac{Dx+E}{(x^2+3)^2}$$

$$3x^4+4x^3+16x^2+20x+9$$
$$= A(x^2+3)^2 + (Bx+C)(x^2+3)(x+2) + (Dx+E)(x+2)$$

$$= (A+B)x^4 + (2B+C)x^3 + (6A+3B+2C+D)x^2$$
$$+ (6B+3C+2D+E)x + (9A+6C+2E)$$

So,

$$A+B=3$$
$$2B+C=4$$
$$6A+3B+2C+D=16$$
$$6B+3C+2D+E=20$$
$$9A+6C+2E=9$$

Solving these equations gives

$$A=1, B=2, C=0, D=4, E=0$$

$$\therefore \int \frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}dx$$
$$= \int \frac{1}{(x+2)}dx + \int \frac{2x}{x^2+3}dx + \int \frac{4x}{(x^2+3)^2}dx$$

$$= \ln|x+2| + \ln(x^2+3) - \frac{2}{x^2+3} + C$$

☐ Integrating Improper Rational Functions

Although the method of partial fractions only applies to proper rational functions, an improper rational function can be integrated by performing a long division and expressing the function as the quotient plus the remainder over the divisor. The remainder over the divisor will be a proper rational function, which can then be decomposed into partial fractions.

$$\frac{P(x)}{Q(x)} = polynomial + \frac{P_1(x)}{Q(x)}$$

This idea is illustrated in the following example.

Example: Evaluate $\int \frac{x^3}{x^2+2x+1} dx$

$$\begin{array}{r|l} & x-2 \\ \hline x^2+2x+1 & x^3 \\ & x^3+2x^2+x \\ & \text{---------------------} \\ & \quad -2x^2-x \\ & \quad -2x^2-4x-2 \\ & \text{---------------------} \\ & \qquad\quad 3x+2 \end{array}$$

$$\frac{x^3}{x^2+2x+1} = x-2+\frac{3x+2}{x^2+2x+1}$$

$$\int \frac{x^3}{x^2+2x+1} dx = \int (x-2)dx + \int \frac{3x+2}{x^2+2x+1} dx$$

$$= \frac{x^2}{2} - 2x + \int \frac{3x+2}{x^2+2x+1} dx$$

Since,

$$\frac{3x+2}{x^2+2x+1} = \frac{3x+2}{(x+1)(x+1)} = \frac{3x+2}{(x+1)^2} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2}$$

$$= \frac{A(x+1)+B}{(x+1)^2} = \frac{Ax+(A+B)}{(x+1)^2}$$

It is clear that

$$A = 3 \text{ and } B = -1$$

$$\therefore \int \frac{3x+2}{x^2+2x+1} dx = \int \frac{3}{(x+1)} dx - \int \frac{1}{(x+1)^2} dx$$

$$= 3\ln|x+1| + \frac{1}{x+1} + C$$

Finally,

$$\int \frac{x^3}{x^2+2x+1} dx = \frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1} + C$$

Example: Evaluate $\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}dx$

$$\begin{array}{r|l} & 3x^2+1 \\ \hline x^2+x-2 & 3x^4+3x^3-5x^2+x-1 \\ & 3x^4+3x^3-6x^2 \\ & \text{-------------------------------} \\ & \qquad\qquad x^2+x-1 \\ & \qquad\qquad x^2+x-2 \\ & \text{-------------------------------} \\ & \qquad\qquad\qquad\qquad 1 \end{array}$$

$$\frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2} = (3x^2+1) + \frac{1}{x^2+x-2}$$

and hence

$$\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}dx = \int (3x^2+1)dx + \int \frac{1}{x^2+x-2}dx$$

1) $\int (3x^2+1)dx = x^3+x+C_1$

2) $\int \frac{1}{x^2+x-2}dx$

$$\frac{1}{x^2+x-2} = \frac{1}{(x-1)(x+2)}$$

$$\frac{1}{(x-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)}$$

where A and B are constants to be determined.

$$\frac{1}{(x-1)(x+2)} = \frac{A(x+2)+B(x-1)}{(x-1)(x+2)} = \frac{(A+B)x+2A-B}{(x-1)(x+2)}$$

Now,

$$(A+B)x+2A-B = 1$$

So,

$$A+B = 0 \Rightarrow A = -B$$

and

$$2A-B = 1 \Rightarrow -2B-B = 1 \Rightarrow -3B = 1 \Rightarrow B = -\frac{1}{3} \Rightarrow A = \frac{1}{3}$$

$$\therefore \int \frac{dx}{x^2+x-2} = \int \frac{\frac{1}{3}}{(x-1)}dx + \int \frac{-\frac{1}{3}}{(x+2)}dx$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C_2 = \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C_2$$

Finally,

$$\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}dx = x^3+x+\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C$$

☐ Concluding Remarks

There are some cases in which the method of partial fractions is inappropriate. For example, it would be inefficient to use partial fractions to perform the integration

$$\int \frac{3x^2 + 2}{x^3 + 2x - 8}dx = \ln|x^3 + 2x - 8| + C$$

since the substitution $u = x^3 + 2x - 8$ is more direct. Similarly, the integration

$$\int \frac{2x - 1}{x^2 + 1}dx = \int \frac{2x}{x^2 + 1}dx - \int \frac{1}{x^2 + 1}dx = \ln(x^2 + 1) - \tan^{-1} x + C$$

requires only a little algebra since the integrand is already in partial fraction form.

Exercise

1) Evaluate:
   a. $\int \frac{dx}{x^2-3x-4}$
   b. $\int \frac{5x-5}{3x^2-8x-3}dx$
   c. $\int \frac{dx}{x^2-6x-7}$
   d. $\int \frac{x^4+6x^3+10x^2+x}{x^2+6x+10}dx$
   e. $\int \frac{x^3+x^2+x+2}{(x^2+1)(x^2+2)}dx$
   f. $\int \frac{e^t}{e^{2t}-4}dt$
   g. $\int \frac{e^{3x}}{e^{2x}+4}dx$

2) Show that: $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$