Solutions of Differential Equations

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what is Ordinary Differential Equations

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SOLUTIONS OF DIFFERENTIAL EQUATIONS

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Ordinary Differential Equation

Differential equation. A differential equation is any equation which contains derivatives, either ordinary derivatives or partial derivatives.

$$F(x,y,y',y'',\ldots,y^{(n)})=0$$

Where x is called the independent variable and y is the dependent.

Here are a few more examples of differential equations.

$$ay''+by'+cy=g(t) \tag{5}$$

$$\sin(y)\frac{d^2y}{dx^2}=(1-y)\frac{dy}{dx}+y^2e^{-5y} \tag{6}$$

$$y^{(4)}+10y'''-4y'+2y=\cos(t) \tag{7}$$

$$\alpha^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t} \tag{8}$$

$$a^2u_{xx}=u_{tt} \tag{9}$$

$$\frac{\partial^3 u}{\partial^2 x\partial t}=1+\frac{\partial u}{\partial y} \tag{10}$$

Order

The order of a differential equation is the largest derivative present in the differential equation.

Examples: In the differential equations listed above(5), (6), (8), and (9) are second order differential equations, (10) is a third order differential equation and (7) is a fourth order differential equation.

Ordinary and Partial Differential Equations

Definition A differential equation is called an **ordinary differential equation**, abbreviated by **ode,** if it has ordinary derivatives in it

$$F(x,y,y',y'',\ldots,y^{(n)})=0$$

Definition a differential equation is called a **partial differential equation,** abbreviated by **pde,** if it has differential derivatives in it. In the differential

Example: equations above (5) - (7) are ode's and (8) - (10) are pde's.

A linear differential equation is any differential equation that can be written in the following form.

$$a_n(t)\,y^{(n)}(t)+a_{n-1}(t)\,y^{(n-1)}(t)+\cdots+a_1(t)\,y'(t)+a_0(t)\,y(t)=g(t) \qquad \text{...(11)}$$

The important thing to note about linear differential equations is that there are no products of the function, $y(t)$, and its derivatives and neither the function or its derivatives occur to any power other than the first power. The coefficients $a_0(t), \ldots, a_n(t)$ and $g(t)$ can be zero or non-zero functions, constant or non-constant functions, linear or non-linear functions. Only the function, $y(t)$, and its derivatives are used in determining if a differential equation is linear.

If a differential equation cannot be written in the form, (11) then it is called a **non-linear** differential equation.

Examples In (5) - (7) above only (6) is non-linear, the other two are linear differential equations.

Definition A **solution** to a differential equation on an interval $\alpha < t < \beta$ is any function $y=y(t)$

which satisfies the differential equation in question on the interval

Example Show that $y(x) = x^{-\frac{3}{2}}$

is a solution to $4x^2y'' + 12xy' + 3y = 0$ for $x > 0$.

Solution We'll need the first and second derivative to do this.

$$y'(x) = -\frac{3}{2}x^{-\frac{5}{2}} \qquad y''(x) = \frac{15}{4}x^{-\frac{7}{2}}$$

Put these function into the differential equation.

$$4x^2\left(\frac{15}{4}x^{-\frac{7}{2}}\right) + 12x\left(-\frac{3}{2}x^{-\frac{5}{2}}\right) + 3\left(x^{-\frac{3}{2}}\right) = 0$$

$$15x^{-\frac{3}{2}} - 18x^{-\frac{3}{2}} + 3x^{-\frac{3}{2}} = 0$$

$$0 = 0$$

So, $y(x) = x^{-\frac{5}{2}}$ does satisfy the differential equation and hence is a solution.

Initial Condition(s) are a condition, or set of conditions, on the solution that will allow us to determine which solution that we are after. Initial conditions (often abbreviated i.c.'s) are of the form,

$$y(t_0) = y_0 \qquad \text{and/or} \qquad y^{(k)}(t_0) = y_k$$

So, in other words, initial conditions are values of the solution and/or its derivative(s) at specific points.

Note The ***number*** of initial conditions that are required for a given differential equation will depend upon the ***order*** of the differential equation as we will see.

Example $y(x) = x^{-\frac{3}{2}}$ is a solution to

$4x^2y'' + 12xy' + 3y = 0$, $y(4) = \frac{1}{8}$, and $y'(4) = -\frac{3}{64}$.

Solution As we saw in previous example the function is a solution and we can then note that

$$y(4) = 4^{-\frac{5}{2}} = \frac{1}{\left(\sqrt{4}\right)^3} = \frac{1}{8}$$

$$y'(4) = -\frac{3}{2}4^{-\frac{5}{2}} = -\frac{3}{2}\frac{1}{\left(\sqrt{4}\right)^5} = -\frac{3}{64}$$

and so this solution also meets the initial conditions of $y(4) = \frac{1}{8}$ and $y'(4) = -\frac{3}{64}$

Definition An **Initial Value Problem** (or **IVP**) is a differential equation along with an appropriate number of initial conditions.

Example The following is an IVP.

$$4x^2 y'' + 12xy' + 3y = 0 \qquad y(4) = \frac{1}{8}, \quad y'(4) = -\frac{3}{64}$$

Example Here's another IVP.

$$2ty' + 4y = 3 \qquad y(1) = -4$$

Definition The **general solution** to a differential equation is the most general form that the solution can take and doesn't take any initial conditions into account i.e contains **a constants** same as the order of DE.

Example $y(t) = (3/4) + (c/t^2)$ is the general solution to

$$2ty' + 4y = 3$$

Definition The **particular solution** to a differential equation is the specific solution that not only satisfies the differential equation, but also satisfies the given initial condition(s).

Example 6 What is the particular solution to the following IVP?

$$2ty' + 4y = 3 \qquad y(1) = -4$$

Solution This is actually easier to do than it might at first appear. From the previous example we already know (well that is provided you believe my solution to this example…) that all solutions to the differential equation are of the form.

$$y(t) = \frac{3}{4} + \frac{c}{t^2}$$

All that we need to do is determine the value of c that will give us the solution that we're after. To find this all we need do is use our initial condition as follows.

$$-4 = y(1) = \frac{3}{4} + \frac{c}{1^2} \qquad \Rightarrow \qquad c = -4 - \frac{3}{4} = -\frac{19}{4}$$

So, the actual solution to the IVP is.

$$y(t) = \frac{3}{4} - \frac{19}{4t^2}$$

Thank You

Any questions?

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Separable ODEs

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Separable ODEs

A separable differential equation is any differential equation that we can write in the following form.

$$N(y)\frac{dy}{dx} = M(x)$$

To solve this differential equation we first integrate both sides with respect to *x* to get,

$$\int N(y)\frac{dy}{dx}dx = \int M(x)dx$$

Simply, we integrate both sides as following:

$$\int N(y)dy = \int M(x)dx$$

01

Example

Solve the following differential equation

$$\frac{dy}{dx} = 6y^2x \qquad y(1) = \frac{1}{25}$$

$$y^{-2}dy = 6x\,dx$$

$$\int y^{-2}dy = \int 6x\,dx$$

$$-\frac{1}{y} = 3x^2 + c$$

We apply the initial condition and find the value of *c*.

$$-\frac{1}{1/25} = 3(1)^2 + c \qquad c = -28$$

$$\therefore \qquad -\frac{1}{y} = 3x^2 - 28$$

$$y(x) = \frac{1}{28 - 3x^2}$$

02

Example

Solve the following IVP

$$y' = \frac{3x^2 + 4x - 4}{2y - 4} \qquad y(1) = 3$$

$$(2y-4)\,dy = \left(3x^2 + 4x - 4\right)dx$$

$$\int (2y-4)\,dy = \int \left(3x^2 + 4x - 4\right)dx$$

$$y^2 - 4y = x^3 + 2x^2 - 4x + c$$

let's apply the initial condition at this point to determine the value of *c*.

$$(3)^2 - 4(3) = (1)^3 + 2(1)^2 - 4(1) + c \qquad c = -2$$

$$y^2 - 4y - \left(x^3 + 2x^2 - 4x - 2\right) = 0$$

So, upon using the quadratic formula on this we get.

$$y(x) = \frac{4 \pm \sqrt{16 - 4(1)\left(-\left(x^3 + 2x^2 - 4x - 2\right)\right)}}{2}$$

02

Example

$$y(x) = \frac{4 \pm 2\sqrt{4 + \left(x^3 + 2x^2 - 4x - 2\right)}}{2}$$
$$= 2 \pm \sqrt{x^3 + 2x^2 - 4x + 2}$$

We are almost there. Notice that we've actually got two solutions here (the "∓") and we only want a single solution. In fact, only one of the signs can be correct. So, to figure out which one is correct we can reapply the initial condition to this. Only one of the signs will give the correct value so we can use this to figure out which one of the signs is correct. Plugging $x = 1$ into the solution gives.

$$3 = y(1) = 2 \pm \sqrt{1 + 2 - 4 + 2} = 2 \pm 1 = 3, 1$$

In this case it looks like the "+" is the correct sign for our solution. So, the explicit solution for our differential equation is.

$$y(x) = 2 + \sqrt{x^3 + 2x^2 - 4x + 2}$$

03

Example

Solve the following IVP

$$y' = \mathbf{e}^{-y}(2x-4) \qquad y(5)=0$$

$$\mathbf{e}^{y}\,dy = (2x-4)\,dx$$

$$\int \mathbf{e}^{y}\,dy = \int (2x-4)\,dx$$

$$\mathbf{e}^{y} = x^2 - 4x + c$$

Applying the initial condition gives

$$1 = 25 - 20 + c \qquad c = -4$$

This then gives an implicit solution of.

$$\mathbf{e}^{y} = x^2 - 4x - 4$$

$$y(x) = \ln\left(x^2 - 4x - 4\right)$$

04

Example

Solve the following IVP

$$\frac{dr}{d\theta} = \frac{r^2}{\theta} \qquad r(1) = 2$$

$$\frac{1}{r^2}dr = \frac{1}{\theta}d\theta$$

$$\int \frac{1}{r^2}dr = \int \frac{1}{\theta}d\theta$$

$$-\frac{1}{r} = \ln|\theta| + c$$

Now, apply the initial condition to find *c*.

$$-\frac{1}{2} = \ln(1) + c \qquad c = -\frac{1}{2}$$

So, the implicit solution is then,

$$-\frac{1}{r} = \ln|\theta| - \frac{1}{2}$$

Solving for *r* gets us our explicit solution.

$$r = \frac{1}{\frac{1}{2} - \ln|\theta|}$$

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Homogeneous ODEs

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Definition

A function $f(x, y)$ is said to be homogeneous function of order n if

$$f(tx, ty) = t^n f(x, y)$$

01

Example

$$f(x,y) = x^2 + y^2 \ln \frac{y}{x}$$

$$f(tx,ty) = t^2x^2 + t^2y^2 \ln \frac{ty}{tx}$$

$$= t^2x^2 + t^2y^2 \ln \frac{y}{x}$$

$$= t^2\left(x^2 + y^2 \ln \frac{y}{x}\right) = t^2 f(x,y)$$

02

Example

$$f(x,y) = e^{\frac{y}{x}} + \tan\left(\frac{y}{x}\right)$$

$$f(tx,ty) = e^{\frac{ty}{tx}} + \tan\left(\frac{ty}{tx}\right) = t^0 f(x,y)$$

Definition

An equation of the form

$$P(x,y)dx + Q(x,y)dy = 0$$

is said to be homogeneous equation if the functions $P(x,y)$ and $Q(x,y)$ are homogeneous and of the same order.

The homogeneous equation can be rewritten in the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

03

Example

$$\frac{dy}{dx} = \frac{y^2 + 2xy}{x^2}$$

$$= \left(\frac{y}{x}\right)^2 + 2\frac{y}{x}$$

$$= F\left(\frac{y}{x}\right)$$

04

Example

$$\frac{dy}{dx} = \ln x - \ln y + \frac{x+y}{x-y}$$

$$= \ln \frac{x}{y} + \frac{1+(y/x)}{1-(y/x)}$$

$$= \ln \frac{1}{(y/x)} + \frac{1+(y/x)}{1-(y/x)}$$

$$= F\left(\frac{y}{x}\right)$$

we can solve the homogeneous equation using the following substitution

$$y = xv \qquad \text{i.e.} \quad v = \frac{y}{x} \qquad \text{---} \ (1)$$

so that

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \Rightarrow \frac{dy}{dx} = F(v) \qquad \text{---} \ (2)$$

from (1) we have

$$\frac{dy}{dx} = x\frac{dv}{dx} + v \qquad \text{---} \ (3)$$

put (3) in (2) we get

$$x\frac{dv}{dx} + v = F(v)$$

$$\Rightarrow \frac{dx}{x} = \frac{dv}{F(v) - v}$$ which is separable ODE

05

Example

Solve $\frac{dy}{dx} = \frac{y^2+2xy}{x^2}$

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + 2\frac{y}{x}$$

let $y = xv \Rightarrow v = \frac{y}{x}$

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$\Rightarrow x\frac{dv}{dx} + v = v^2 + 2v$$

$$\Rightarrow x\frac{dv}{dx} = v^2 + v$$

$$\Rightarrow \frac{dx}{x} = \frac{dv}{v(v+1)} = \left(\frac{1}{v} - \frac{1}{v+1}\right)dv$$

05

Example

$$\ln x + c_1 = \ln v - \ln(v+1)$$

$$\ln x + c_1 = \ln \frac{v}{v+1}$$

$$cx = \frac{v}{v+1}$$

now, Subsitute $v = y/x$

$$\Rightarrow cx = \frac{y/x}{\frac{y}{x}+1} \Rightarrow cx = \frac{y}{y+x} \Rightarrow cxy + cx^2 = y$$

$$\Rightarrow y(cx-1) = -cx^2$$

$$\therefore y = \frac{cx^2}{1-cx}$$

Equation of the form

$$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$$

Consider the equation

$$(a_1x+b_1y+c_1)dx+(a_2x+b_2y+c_2)dy=0 \quad \text{---④}$$

in which the Coefficients of dx and dy are linear.

The following Cases arise

1- if $c_1=c_2=0 \Rightarrow$ it is homogeneous equation.

2- if the two lines

$$a_1x+b_1y+c_1=0$$

$$a_2x+b_2y+c_2=0$$

are non parallel lines. then equation ④ can be solved Using the following subsitutions

$$x=x_1+h \quad , \quad y=y_1+k$$

06

Example

Solve $(2x-y+1)dx+(x+y)dy=0$

let $x=x_1+h$, $y=y_1+k$

$\Rightarrow dx=dx_1 \qquad dy=dy_1$

$2(x_1+h)-(y_1+k)+1=0 \Rightarrow 2x_1-y_1+2h-k+1=0$

$x_1+h+y_1+k=0 \Rightarrow x_1+y_1+h+k=0$

to make it homogeneous the following two equations must be satisfied

$2h-k+1=0$

$h+k=0 \Rightarrow \boxed{h=-k}$

$2h+h+1=0 \Rightarrow \boxed{h=-\frac{1}{3}} \Rightarrow \boxed{k=\frac{1}{3}}$

$\therefore \; x=x_1-\frac{1}{3} \qquad y=y_1+\frac{1}{3}$

06

Example

Now $(2x_1 - y_1)dx_1 + (x_1 + y_1)dy_1 = 0$

$$\frac{dy_1}{dx_1} = \frac{y_1 - 2x_1}{x_1 + y_1} = \frac{\frac{y_1}{x_1} - 2}{1 + \frac{y_1}{x_1}} = F\left(\frac{y_1}{x_1}\right) = F(v)$$

let $v = \frac{y_1}{x_1} \Rightarrow x_1 \frac{dv}{dx_1} + v = F(v) = \frac{v-2}{1+v}$

$$x_1 \frac{dv}{dx_1} = \frac{v-2}{1+v} - v \Rightarrow x_1 \frac{dv}{dx_1} = \frac{v - 2 - v - v^2}{1+v}$$

$$\Rightarrow x_1 \frac{dv}{dx_1} = \frac{-v^2 - 2}{1+v} \Rightarrow x_1 \frac{dv}{dx_1} = \frac{-(v^2+2)}{v+1}$$

06

Example

$$\Rightarrow \frac{dx_1}{x_1} = -\frac{v+1}{v^2+2}dv \Rightarrow \frac{dx_1}{x_1} = -\left[\frac{v\,dv}{v^2+2} + \frac{dv}{v^2+2}\right]$$

$$\Rightarrow \frac{dx_1}{x_1} = -\left[\frac{1}{2}\frac{2v\,dv}{v^2+2} + \frac{1}{2}\frac{dv}{\left(\frac{v}{\sqrt{2}}\right)^2+1}\right]$$

$$\Rightarrow \ln x_1 + c = -\frac{1}{2}\ln(v^2+2) - \frac{1}{\sqrt{2}}\tan^{-1}\frac{v}{\sqrt{2}}$$

Put $v = \frac{y_1}{x_1}$ $\Rightarrow \ln x_1 + c = -\frac{1}{2}\ln\left(\left(\frac{y_1}{x_1}\right)^2+2\right) - \frac{1}{\sqrt{2}}\tan^{-1}\frac{y_1/x_1}{\sqrt{2}}$

Substitute x_1 and y_1

$$\Rightarrow \ln\left(x+\frac{1}{3}\right) + c = -\frac{1}{2}\ln\left(\frac{(y-\frac{1}{3})^2}{(x+\frac{1}{3})^2}+2\right) - \frac{1}{\sqrt{2}}\tan^{-1}\frac{\frac{y-1/3}{x+1/3}}{\sqrt{2}}$$

Thank You

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Exact ODEs

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let M, N, M_y, N_x be continuous function in a rectangular region R such that $\alpha < x < \beta$, $\gamma < y < \delta$, then the equation

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is an excat equation in R iff

$$M_y(x,y) = N_x(x,y) \qquad \forall x,y \text{ in } R$$

and its solution $\psi(x,y)=c$ satisfies

$$\psi_x(x,y) = M(x,y) \text{ and } \psi_y(x,y) = N(x,y)$$

01

Example

Solve

$$2xy^3 + 3x^2y^2\frac{dy}{dx} = 0$$

$M(x,y) = 2xy^3 \Rightarrow M_y = 6xy^2$
$N(x,y) = 3x^2y^2 \Rightarrow N_x = 6xy^2$
$\} \Rightarrow$ Exact

$\psi_x = M = 2xy^3$

$\psi_y = N = 3x^2y^2$

method 1

$\psi_x = M = 2xy^3 \xrightarrow[\text{w.r.t } x]{\text{integration}} \psi = x^2y^3 + f(y)$

Diff w.r.t y

$\psi_y = 3x^2y^3 + f'(y)$

But $\psi_y = N = 3x^2y^2$

$\Rightarrow$ By Comparison $f'(y) = 0 \Rightarrow f(y) = k$

$\therefore \psi(x,y) = x^2y^3 + k = c$

$$\Rightarrow \boxed{y = c_1 x^{-2/3}}$$

01

Example

$\psi_x = M = 2xy^3$

$\psi_y = N = 3x^2y^2$

method 2

$\psi_y = N = 3x^2y^2 \xrightarrow[\text{w.r.t. } y]{\text{integration}} \psi = x^2y^3 + g(x)$

$\psi_x = 2xy^3 + g'(x) \xleftarrow{\text{Diff. w.r.t } x}$

But $\psi_x = M = 2xy^3$

$\Rightarrow$ By Comparison $g'(x) = 0 \Rightarrow g(x) = h$

so $\psi(x,y) = x^2y^3 + h = c$

$$\Rightarrow \boxed{y = c_1 x^{-2/3}}$$

01
Example

$\psi_x = M = 2xy^3$

$\psi_y = N = 3x^2y^2$

method 3

$$\psi_x = M = 2xy^3 \xrightarrow[\text{w.r.t. } x]{\text{integ.}} \psi = x^2y^3 + f(y)$$

$$\psi_y = N = 3x^2y^2 \xrightarrow[\text{w.r.t } y]{\text{integ.}} \psi = x^2y^3 + g(x)$$

By Comparison $f(y) = g(x) \Rightarrow f(y) = g(x) = k$

$$\therefore \psi(x,y) = x^2y^3 + k = c$$

$$\Rightarrow \boxed{y = c_1 x^{-2/3}}$$

02 Example

Solve

$$(y\cos x + 2xe^y) + (\sin x + x^2e^y + 2)y' = 0$$

$$M = y\cos x + 2xe^y \longrightarrow M_y = \cos x + 2xe^y$$

$$N = \sin x + x^2e^y + 2 \longrightarrow N_x = \cos x + 2xe^y$$

exact

$$\psi_x = M = y\cos x + 2xe^y \quad \cdots (1)$$

$$\psi_y = N = \sin x + x^2e^y + 2 \quad \cdots (2)$$

integrate (1) w.r.t. x

$$\psi = y\sin x + x^2e^y + f(y) \quad \cdots (3)$$

Differ. (3) w.r.t. y

$$\psi_y = \sin x + x^2e^y + f'(y) \equiv N = \sin x + x^2e^y + 2$$

02

Example

$$\therefore f'(y) = 2 \Rightarrow f(y) = 2y$$

$$\psi(x,y) = y\sin x + x^2 e^y + 2y$$

$\Rightarrow$ The Solution of the equation is

$$\boxed{y\sin x + x^2 e^y + 2y = c}$$

Integrating Factor

A multiplying factor which will convert an inexact DE into exact one is called integrating factor.

$$(y^2+y)dx - x\,dy = 0$$

$M(x,y) = y^2+y \longrightarrow M_y = 2y+1$

$N(x,y) = -x \longrightarrow N_x = 0$

$\Big]\longrightarrow$ not exact

multiply both side by the I.F. y^{-2}

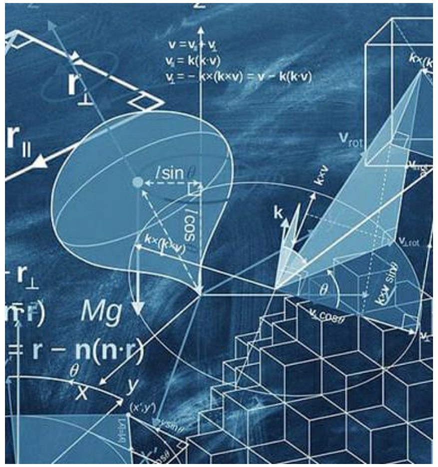
$$\Rightarrow \left(1+\frac{1}{y}\right)dx - \frac{x}{y^2}dy = 0$$

$M(x,y) = 1+\frac{1}{y} \longrightarrow M_y = -\frac{1}{y^2}$

$N(x,y) = -\frac{x}{y^2} \longrightarrow N_x = -\frac{1}{y^2}$

$\Big]$ exact ODE

Integrating Factors



We are looking for $u(x, y)$ such that

$$\frac{\partial\left(u \cdot M(x,y)\right)}{\partial y} = \frac{\partial\left(u \cdot N(x,y)\right)}{\partial x}$$

Special Cases

- $u(x, y) = u(x)$
- $u(x, y) = u(y)$

Case I

If $\frac{M_y - N_x}{N} = F(x)$ then

$$I.F. = u(x) = e^{\int F(x)dx}$$

Case II

If $\frac{N_x - M_y}{M} = G(y)$ then

$$I.F. = u(y) = e^{\int G(y)dy}$$

01 Example

Solve $(e^x - \sin y)\,dx + \cos y\,dy = 0$

$M = e^x - \sin y \longrightarrow M_y = -\cos y$

$N = \cos y \longrightarrow N_x = 0$

} not exact

$$\frac{M_y - N_x}{N} = \frac{-\cos y - 0}{\cos y} = -1 = F(x)$$

$$\therefore\ u(x) = e^{-\int dx} = e^{-x}$$

multiply both sides with $u(x) = e^{-x}$

$$\Rightarrow (1 - e^{-x}\sin y)\,dx + e^{-x}\cos y\,dy = 0$$

01

Example

$$M = 1 - e^{-x}\sin y \longrightarrow M_y = -e^{-x}\cos y$$

$$N = e^{-x}\cos y \longrightarrow N_x = -e^{-x}\cos y$$

$\Big\}$ ∴ exact

$$\therefore \psi_x = M = 1 - e^{-x}\sin y \xrightarrow[\text{w.r.t. } x]{\text{integration}} \psi = x + e^{-x}\sin y + f(y)$$

$$\psi_y = e^{-x}\cos y + f'(y) \xleftarrow[\text{w.r.t. } y]{\text{diff.}}$$

$$\equiv N = e^{-x}\cos y$$

$$\therefore f'(y) = 0 \Rightarrow f(y) = k$$

$$\therefore \boxed{\psi(x,y) = x + e^{-x}\sin y = c}$$

02

Example

Solve

$$xy\,dx + (1+x^2)\,dy = 0$$

$M = xy \longrightarrow M_y = x$

$N = 1+x^2 \longrightarrow N_x = 2x$ } not exact

$$\frac{N_x - M_y}{M} = \frac{2x - x}{xy} = \frac{x}{xy} = \frac{1}{y} = G(y)$$

$$\therefore u(y) = e^{\int \frac{dy}{y}} = e^{\ln y} = y$$

multiply both sides with y

$$xy^2dx + (y + x^2y)\,dy = 0$$

$M = xy^2 \longrightarrow M_y = 2xy$

$N = y + x^2y \longrightarrow N_x = 2xy$ } exact

02

Example

$$\psi_x = M = xy^2 \xrightarrow[\text{w.r.t } x]{\text{integration}} \psi = \frac{x^2y^2}{2} + f(y)$$

$$\psi_y = x^2y + f'(y) \xleftarrow{\text{diff. w.r.t. } y}$$

$$\equiv N = y + x^2y \Rightarrow f'(y) = y \Rightarrow f(y) = \frac{y^2}{2}$$

$$\therefore \boxed{\psi(x,y) = \frac{x^2y^2}{2} + \frac{y^2}{2} = c}$$

Thank You

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1st Order Linear ODEs

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DIFFERENTIAL EQUATIONS

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Department of Mathematics
College of Science - University of Baghdad

The general form of the 1st order ode is:

$$a(x)y' + b(x)y + c(x) = 0 \quad , \quad a(x) \neq 0 \qquad --- (1)$$

we can rewrite this equation to be:

$$\frac{dy}{dx} + p(x)y = Q(x) \qquad --- (2)$$

where $p(x) = \frac{b(x)}{a(x)}$ and $Q(x) = -\frac{c(x)}{a(x)}$

we can rewrite (2) in the following form:

$$(\underbrace{P(x)y - Q(x)}_{M})dx + \underbrace{1}_{N}dy = 0$$

Now,

$$\frac{M_y - N_x}{N} = \frac{P(x) - 0}{1} = p(x)$$

∴ The integrating factor is

$$\boxed{I.F. = e^{\int p(x)\,dx} := u(x)} \quad \text{------} \quad (3)$$

Multipling both sides by this factor

$$e^{\int p(x)dx}\frac{dy}{dx} + e^{\int p(x)dx} p(x) y = e^{\int p(x)dx} \varphi(x)$$

$$\Rightarrow \frac{d}{dx}\left(y e^{\int p(x)dx}\right) = e^{\int p(x)dx} \varphi(x)$$

Subsituting $u(x) = e^{\int p(x)dx}$

$$\frac{d}{dx}\left(u(x)\,y\right) = u(x)\,\varphi(x)$$

Integrating both Sides w.r.t. x

$$\boxed{y = \frac{\int u(x)\,\varphi(x)\,dx + c}{u(x)}} \quad \text{------} \quad (4)$$

01

Example

Solve $\frac{dy}{dx} + \frac{2y}{x} = 4x$

$P(x) = \frac{2}{x}$ $\qquad Q(x) = 4x$

So $u(x) = e^{\int \frac{2}{x} dx} = e^{2\ln x} = e^{\ln x^2} = x^2$

$$y = \frac{\int x^2 \cdot (4x) dx}{x^2}$$

$$= \frac{\int 4x^3 dx}{x^2}$$

$$= \frac{x^4 + c}{x^2} = x^2 + cx^{-2}$$

02

Example

Solve $\frac{dy}{dx} - 5y = 3e^{5x}$, where $y(0) = 8$

$P(x) = -5$ $\quad Q(x) = 3e^{5x}$

$u(x) = e^{-\int 5\,dx} = e^{-5x}$

$$y = \frac{3\int e^{-5x} e^{5x}\,dx}{e^{-5x}}$$

$$= \frac{3\int dx}{e^{-5x}} = \frac{3x + c}{e^{-5x}} = e^{5x}(3x + c)$$

Using the intital Condition

$$8 = e^{0}(0 + c) \Rightarrow c = 8$$

$$\therefore y = e^{5x}(3x + 8)$$

03

Example

Solve

$$\cos x\, y' + \sin x\, y = 2\cos^3 x \sin x - 1 \quad , \quad y\left(\frac{\pi}{4}\right) = 3\sqrt{2}$$

$$y' + \frac{\sin x}{\cos x} y = 2\cos^2 x \sin x - \frac{1}{\cos x}$$

$$y' + \tan x\, y = 2\cos^2 x \sin x - \sec x$$

$$p(x) = \tan x \qquad Q(x) = 2\cos^2 x \sin x - \sec x$$

$$u(x) = e^{\int \tan x\, dx} = e^{-\ln|\cos x|} = e^{\ln|\cos x|^{-1}} = \sec x$$

$$y = \frac{\int \sec x (2\cos^2 x \sin x - \sec x) dx}{\sec x}$$

$$= \frac{\int (2\cos x \sin x - \sec^2 x) dx}{\sec x}$$

03

Example

$$= \frac{\int (2\cos x \sin x - \sec^2 x)\,dx}{\sec x}$$

$$= \left(-\frac{1}{2}\cos(2x) - \tan x + c\right)\cos x$$

$$= -\frac{1}{2}\cos(2x)\cos x - \sin x + c\cos x$$

$\int 2\sin x \cos x$

$= \int \sin(2x)\,dx$

$= -\frac{1}{2}\cos(2x)$

$$y\left(\frac{\pi}{4}\right) = 3\sqrt{2} \Rightarrow 3\sqrt{2} = -\frac{1}{2}(0)\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + c\frac{1}{\sqrt{2}}$$

$$\Rightarrow 3\sqrt{2} = -\frac{1}{\sqrt{2}} + c\frac{1}{\sqrt{2}} \Rightarrow 3(2) = -1 + c \Rightarrow \boxed{c = 7}$$

$$\boxed{\therefore y = \cos x\left(-\frac{1}{2}\cos(2x) + 7\right) - \sin x}$$

Thank You

Any questions?

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Bernolli ODEs

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SOLUTIONS OF
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Bernolli ODEs

It takes the form

$$y' + p(x)y = q(x)y^n$$

if $n=0 \longrightarrow y' + p(x)y = q(x) \longrightarrow$ linear ode

if $n=1 \longrightarrow y' + p(x)y = q(x)y$

$$\Rightarrow y' + (p(x) - q(x))y = 0 \longrightarrow \text{linear ode}$$

otherwise divided both sides by y^n

$$\Rightarrow y^{-n}y' + p(x)y^{1-n} = q(x) \quad \text{.....} \quad (1)$$

and let $\boxed{v = y^{1-n}}$ $\Rightarrow \dfrac{dv}{dx} = (1-n)y^{-n}y'$

$$\Rightarrow y' = \frac{y^n}{1-n}\frac{dv}{dx} \quad \text{...} \quad (2)$$

Put (2) in (1)

$$y^{-n}\frac{y^n}{1-n}\frac{dv}{dx}+P(x)v=q(x)$$

$$\Rightarrow \frac{dv}{dx}+(1-n)P(x)v=(1-n)q(x)$$

which is linear ode in v with

$$P(x)=(1-n)p(x) \quad \& \quad Q(x)=(1-n)q(x)$$

01

Example

Solve $6y' - 2y = xy^4$

This is Bernolli ODE with $n=4$, So let

$$v = y^{1-4} = y^{-3} \Rightarrow \frac{dv}{dx} = -3y^{-4}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{3}y^4\frac{dv}{dx}$$

$$6y' - 2y = xy^4 \quad / \ y^4 \Rightarrow 6y^{-4}y' - 2y^{-3} = x$$

$$\Rightarrow 6y^{-4}y' - 2y^{-3} = x \Rightarrow 6y^{-4}\left(-\frac{1}{3}y^4\frac{dv}{dx}\right) - 2v = x$$

$$\Rightarrow -2\frac{dv}{dx} - 2v = x \Rightarrow \frac{dv}{dx} + v = -\frac{1}{2}x$$

which is linear in v with $P(x) = 1$, $Q(x) = -\frac{1}{2}x$

$$u(x) = e^{\int dx} = e^x$$

01

Example

$$\Rightarrow v = \frac{\int -\frac{x}{2} e^x dx}{e^x} = \frac{-\frac{1}{2}\int x e^x dx}{e^x}$$

let $u = x$

$du = dx$

$dv = e^x dx$

$v = e^x$

$$= \frac{-\frac{1}{2}\left[xe^x - \int e^x dx\right] + c}{e^x} = \frac{-\frac{1}{2}\left[xe^x - e^x\right] + c}{e^x}$$

$$= -\frac{1}{2}(x-1) + ce^{-x}$$

$$\Rightarrow y^{-3} = -\frac{1}{2}(x-1) + ce^{-x}$$

Thank You

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2nd order Linear Homogeneous ODEs

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SOLUTIONS OF DIFFERENTIAL EQUATIONS

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2nd order Linear Homogeneous ODEs

The general form of the linear Second order ode is

$$p(t)y'' + q(t)y' + r(t)y = g(t)$$

where $p(t) \neq 0$.

In the case where we assume Constant Coefficient we will use the following differential equation

$$ay'' + by' + cy = g(t)$$

where $a \neq 0$.

2nd order Linear Homogeneous ODEs

Definition when $g(t)=0$ we call the differential equation homogeneous, otherwise, we call it nonhomogeneous.

Remark If $y_1(t)$ and $y_2(t)$ are two solutions to a linear, homogeneous differential equation then so is

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

Now, let's assume that all solutions to

$$ay''+by'+cy=0$$

will be of the form

$$y(t)=e^{rt}$$

So,

$$y'(t)=re^{rt} \quad \text{and} \quad y''(t)=r^2e^{rt}$$

Substitude these in the differential equation

$$ar^2e^{rt}+bre^{rt}+ce^{rt}=0$$

$$(ar^2+br+c)e^{rt}=0$$

Since $e^{rt}\neq 0$, then

$$ar^2+br+c=0$$

This equation is called the characteristic equation

Since it is a quadratic equation, So, Solving the equation will give two values of r and we will have one of the following Cases:

1. Real, distint roots, $r_1 \neq r_2$.

In this case, the Solution will be

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

2. Double roots, $r_1 = r_2 = r$

In this case, the Solution will be

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}$$
$$= e^{rt}(c_1 + c_2 r)$$

3. Complex root, $r_{1,2} = \lambda \mp \mu i$

In this case, the Solution will be

$$y(t) = e^{\lambda t}(c_1 \cos(\mu t) + c_2 \sin(\mu t))$$

01

Example

Solve the following IVP

$$y'' + 3y' - 10y = 0 \qquad y(0) = 4 \ , \ y'(0) = -2$$

$$r^2 + 3r - 10 = 0$$

$$(r+5)(r-2) = 0$$

$\Rightarrow r_1 = -5$ and $r_2 = 2$

$$\therefore y(t) = c_1 e^{-5t} + c_2 e^{2t}$$

to apply the initial Conditions we have to find $y'(t)$

$$y'(t) = -5c_1 e^{-5t} + 2c_2 e^{2t}$$

substituting the IC.s

$$4 = y(0) = c_1 + c_2 \qquad \text{--- (1)}$$

$$-2 = y'(0) = -5c_1 + 2c_2 \qquad \text{--- (2)}$$

(1) $\Rightarrow$ $c_1 = 4 - c_2$ --- (3)

01

Example

Put (3) in (2)

$$-2 = -5(4 - c_2) + 2c_2$$

$$-2 = -20 + 5c_2 + 2c_2$$

$$18 = 7c_2 \Rightarrow \boxed{c_2 = \frac{18}{7}}$$

$$\Rightarrow c_1 = 4 - \frac{18}{7} \Rightarrow c_1 = \frac{28 - 18}{7}$$

$$\Rightarrow \boxed{c_1 = \frac{10}{7}}$$

The actual solution to the differential equation is

$$y(t) = \frac{10}{7} e^{-5t} + \frac{18}{7} e^{2t}$$

02
Example

Solve the following IVP

$$16y'' - 40y' + 25y = 0 \qquad y(0) = 3, \; y'(0) = -\frac{9}{4}$$

$$16r^2 - 40r + 25 = 0$$

$$(4r - 5)(4r - 5) = 0$$

$$(4r - 5)^2 = 0$$

$$\Rightarrow r_1 = r_2 = \frac{5}{4}$$

$$\therefore y(t) = c_1 e^{\frac{5}{4}t} + c_2 t e^{\frac{5}{4}t}$$

So,

$$y'(t) = \frac{5}{4} c_1 e^{\frac{5}{4}t} + c_2\left(\frac{5}{4} t e^{\frac{5}{4}t} + e^{\frac{5}{4}t}\right)$$

using the I.C.s

$$3 = y(0) = c_1$$

$$-\frac{9}{4} = y'(0) = \frac{5}{4} c_1 + c_2$$

$$\Rightarrow c_1 = 3 \text{ and } c_2 = -6$$

$$\therefore y(t) = 3e^{\frac{5}{4}t} - 6te^{\frac{5}{4}t}$$

03

Example

Solve the following IVP

$$4y'' + 24y' + 37y = 0 \qquad y(\pi) = 1, \quad y'(\pi) = 0$$

$$4r^2 + 24r + 37 = 0$$

$$r_{1,2} = \frac{-24 \mp \sqrt{(24)^2 - 4(4)(37)}}{2(4)}$$

$$= \frac{-24 \mp \sqrt{576 - 592}}{8}$$

$$= \frac{-24 \mp \sqrt{-16}}{8} = \frac{-24 \mp 4i}{8}$$

$$= -3 \mp \frac{1}{2} i$$

$$\Rightarrow y(t) = e^{-3t}\left(c_1 \cos\left(\tfrac{t}{2}\right) + c_2 \sin\left(\tfrac{t}{2}\right)\right)$$

03 Example

$$y(t) = e^{-3t}\left(c_1 \cos\left(\tfrac{t}{2}\right) + c_2 \sin\left(\tfrac{t}{2}\right)\right)$$

$$\Rightarrow \quad y'(t) = e^{-3t}\left(-\tfrac{c_1}{2}\sin\left(\tfrac{t}{2}\right) + \tfrac{c_2}{2}\cos\left(\tfrac{t}{2}\right)\right) + (-3e^{-3t})\left(c_1 \cos\left(\tfrac{t}{2}\right) + c_2 \sin\left(\tfrac{t}{2}\right)\right)$$

$$y(\pi) = 1 \quad \Rightarrow \quad 1 = e^{-3\pi}\left(c_1 \cos\left(\tfrac{\pi}{2}\right) + c_2 \sin\left(\tfrac{\pi}{2}\right)\right)$$

(zero)

$$1 = c_2 e^{-3\pi} \quad \Rightarrow \quad \boxed{c_2 = e^{3\pi}}$$

$$y'(\pi) = 0 \quad \Rightarrow \quad 0 = e^{-3\pi}\left(-\tfrac{c_1}{2}\sin\left(\tfrac{\pi}{2}\right) + \tfrac{c_2}{2}\cos\left(\tfrac{\pi}{2}\right)\right) - 3e^{-3\pi}\left(c_1 \cos\left(\tfrac{\pi}{2}\right) + c_2 \sin\left(\tfrac{\pi}{2}\right)\right)$$

(zero) (zero)

$$0 = e^{-3\pi}\left(-\tfrac{c_1}{2}\right) - 3e^{-3\pi}(c_2)$$

Put $c_2 = e^{3\pi}$

$$\Rightarrow \quad 0 = e^{-3\pi}\left(-\tfrac{c_1}{2}\right) - 3 \quad \Rightarrow \quad c_1 = -3 * \frac{2}{e^{-3\pi}}$$

$$\Rightarrow \quad \boxed{c_1 = -6e^{3\pi}}$$

03 Example

The acutual solution to the IVP is

$$y(t) = e^{-3t}\left(-6e^{3\pi}\cos\left(\frac{t}{2}\right) + e^{3\pi}\sin\left(\frac{t}{2}\right)\right)$$

$$\Rightarrow y(t) = e^{-3(t-\pi)}\left(-6\cos\left(\frac{t}{2}\right) + \sin\left(\frac{t}{2}\right)\right)$$

Thank You

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Linearly Dependent & Independent Functions

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SOLUTIONS OF DIFFERENTIAL EQUATIONS

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Linearly Dependent & Independent Functions

<u>Definition</u> Given two non-zero functions $f(x)$ and $g(x)$. If we can find non-zero constant c and k such that

$$c f(x) + k g(x) = 0 \qquad \forall x$$

then $f(x)$ and $g(x)$ are <u>linearly dependent</u>.

On the other hand if the only two constants that satisfy the equation above are $c=0$ and $k=0$ then $f(x)$ and $g(x)$ are <u>linearly independent</u>.

01

Example

Determine if the following sets of functions are linearly dependent or linearly independent

(a) $f(x) = 9\cos(2x)$ $g(x) = 2\cos^2(x) - 2\sin^2(x)$

(b) $f(t) = 2t^2$ $g(t) = t^4$

Solution: (a)

$$c\,(9\cos(2x)) + k\,(2\cos^2(x) - 2\sin^2(x)) = 0$$

$$9c\cos(2x) + 2k\cos(2x) = 0$$

$$(9c + 2k)\cos(2x) = 0$$

$$\Rightarrow \quad 9c + 2k = 0$$

$$\Rightarrow \quad k = -\frac{9}{2}c$$

we can find infinite number of pair (c, k), for example

$c = 1 \longrightarrow k = -\frac{9}{2}$

$c = \frac{2}{9} \longrightarrow k = -1$

$c = -2 \longrightarrow k = 9$

⋮

etc

∴ $f(x)$ and $g(x)$ are linearly dependent

01

Example

(b) $f(t) = 2t^2$, $g(t) = t^4$

$$2ct^2 + kt^4 = 0$$

In this case there isn't any quick and simple formula to write one of the functions in terms of the other. So, we'll start by noticing that if the original equation is true, then if we differentiate everything we get a new equation that must also be true.

$$2ct^2 + kt^4 = 0$$

$$4ct + 4kt^3 = 0$$

$$c = -kt^2 \quad \leftarrow \text{Solve w.r.t } c$$

Put this in the first equation

$$2(-kt^2)t^2 + kt^4 = 0$$

$$-kt^4 = 0$$

The only way that this will ever be zero for all t if $k = 0 \Rightarrow c = 0 \Rightarrow f(t)$ and $g(t)$ are linearly Independent

Definition The Wronskian of $f(t)$ and $g(t)$ is defined as follows

$$W(f,g)(t) = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} = f(t)g'(t) - f'(t)g(t)$$

Remark Given two functions $f(x)$ and $g(x)$ that are differentiable on some interval I.

(1) If $W(f,g)(x_0) \neq 0$ for some x_0 in I, then $f(x)$ and $g(x)$ are linearly independent on the interval I.

(2) If $f(x)$ and $g(x)$ are linearly dependent on I then $W(f,g)(x) = 0$ for all x in the interval I

Linearly Dependent & Independent Functions

$W(f,g)(x_0) \neq 0$ for <u>some</u> $x_0 \in I$ ⟶ f and g are Linearly independent

$f(x)$ and $g(x)$ linearly dependent ⟶ $W(f,g)(x) = 0$ $\forall x \in I$

Note that It <u>DOES NOT</u> say that if $W(f,g)(x) = 0$ then $f(x)$ and $g(x)$ are linearly dependent.

In fact it is possible for two linearly independent functions to have a zero Wronskian.

02

Example

Verify the remark above using the function in example 1.

(a) $f(x) = 9\cos(2x)$ $\quad g(x) = 2\cos^2(x) - 2\sin^2(x)$

$$W = \begin{vmatrix} 9\cos(2x) & 2\cos^2(x) - 2\sin^2(x) \\ -18\sin(2x) & -4\cos(x)\sin(x) - 4\sin(x)\cos(x) \end{vmatrix}$$

$$= \begin{vmatrix} 9\cos(2x) & 2\cos(2x) \\ -18\sin(2x) & -8\cos(x)\sin(x) \end{vmatrix}$$

$\cos^2 x - \sin^2 x = \cos 2x$

$$= \begin{vmatrix} 9\cos(2x) & 2\cos(2x) \\ -18\sin(2x) & -4\sin(2x) \end{vmatrix}$$

$2\cos(x)\sin(x) = \sin(2x)$

$$= -36\cos(2x)\sin(2x) - (-36\cos(2x)\sin(2x)) = 0$$

We know that f and g are linearly dependent, so we shold get zero.

02

Example

(b) $f(t) = 2t^2 \qquad g(t) = t^4$

$$W = \begin{vmatrix} 2t^2 & t^4 \\ 4t & 4t^3 \end{vmatrix} = 8t^5 - 4t^5 = 4t^5 \neq 0$$

The Wronskian is non-zero as we expected.

Thank You

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Method of Undetermined Coefficients

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Part I

9

SOLUTIONS OF DIFFERENTIAL EQUATIONS

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2^{nd} Order Linear Nonhomogeneous ODEs

The general form of this equation is

$$p(t)\,y'' + q(t)\,y' + r(t)\,y = g(t)$$

where $g(t) \neq 0$

The general Solution of this equation can be written as

$$y(t) = y_c(t) + y_p(t)$$

$y_c(t)$: The general solution of the homogeneous eq.

$y_p(t)$: A Particular Solution for the nonhomogeneous eq.

There are two Common method for finding y_p:

1) Undetermined Coefficients method
2) Variation of Parameters method

Undetermined Coefficients

Method of Undetermined Coefficients

If β is not a root of the characteristic equation.

$g(t)$	y_p guess
$ae^{\beta t}$	$Ae^{\beta t}$
$a\cos(\beta t)$	$A\cos(\beta t)+B\sin(\beta t)$
$b\sin(\beta t)$	$A\cos(\beta t)+B\sin(\beta t)$
$a\cos(\beta t)+b\sin(\beta t)$	$A\cos(\beta t)+B\sin(\beta t)$
n^{th} degree Polynomial	$A_n t^n + A_{n-1}t^{n-1} \dots + A_1 t + A_0$

01 Example

Determine a particular solution to

$$y'' - 4y' - 12y = 3e^{5t}$$

$$r^2 - 4r - 12 = 0 \Rightarrow (r-6)(r+2) = 0 \Rightarrow r_1 = -2, r_2 = 6$$

The Complimentary Solution is then

$$y_c(t) = c_1 e^{-2t} + c_2 e^{6t}$$

Since $\beta = 5$ is not root of the char. eq. then let

$$y_p(t) = A e^{5t}$$

$$\Rightarrow y_p' = 5Ae^{5t} \text{ and } y_p'' = 25Ae^{5t}$$

01

Example

Substitute these in the ode

$$25Ae^{5t} - 4(5Ae^{5t}) - 12Ae^{5t} = 3e^{5t}$$

$$(25A - 20A - 12A)e^{5t} = 3e^{5t}$$

$$-7Ae^{5t} = 3e^{5t}$$

$$\Rightarrow -7A = 3 \Rightarrow \boxed{A = -\frac{3}{7}}$$

$$\therefore y_p(t) = -\frac{3}{7}e^{5t}$$

$$y = y_c + y_p = c_1 e^{-2t} + c_2 e^{6t} - \frac{3}{7}e^{5t}$$

02

Example

Find a particular solution for the following ode

$$y'' - 4y' - 12y = \sin(2t)$$

$$y_c = c_1 e^{-2t} + c_2 e^{6t}$$ as we know from example1

let $y_p = A\sin 2t + B\cos 2t$

$$y_p' = 2A\cos 2t - 2B\sin 2t$$

$$y_p'' = -4A\sin 2t - 4B\cos 2t$$

$$-4A\sin 2t - 4B\cos 2t - 4(2A\cos 2t - 2B\sin 2t) - 12(A\sin 2t + B\cos 2t) = \sin 2t$$

$$(-4A + 8B - 12A)\sin 2t + (-4B - 8A - 12B)\cos 2t = \sin 2t$$

$$(-16A - 8B)\sin 2t + (-8A - 16B)\cos 2t = \sin 2t$$

02

Example

$$-16A + 8B = 1$$

$$-8A - 16B = 0$$

$$8A = -16B \rightarrow A = -2B$$

$$-16(-2B) + 8B = 1 \rightarrow B = \frac{1}{40}$$

$$A = -2\left(\frac{1}{40}\right) \rightarrow A = -\frac{1}{20}$$

$$\therefore y_p = -\frac{1}{20}\sin 2t + \frac{1}{40}\cos 2t$$

03

Example

Find a particular solution for the following ode

$$y'' - 4y' - 12y = 2t^3 - t + 3$$

let $y_p(t) = At^3 + Bt^2 + Ct + D$

$$y_p' = 3At^2 + 2Bt + C$$

$$y_p'' = 6At + 2B$$

Now,

$$6At + 2B - 4(3At^2 + 2Bt + C) - 12(At^3 + Bt^2 + Ct + D) = 2t^3 - t + 3$$

$$-12At^3 + (-12A - 12B)t^2 + (6A - 8B - 12C)t + (2B - 4C - 12D) = 2t^3 - t + 3$$

t^3: $-12A = 2 \longrightarrow A = -\frac{1}{6}$

t^2: $-12A - 12B = 0 \longrightarrow B = -A \longrightarrow B = \frac{1}{6}$

t^1: $6A - 8B - 12C = -1 \longrightarrow -1 - \frac{4}{3} - 12C = -1 \longrightarrow C = -\frac{1}{9}$

t^0: $2B - 4C - 12D = 3 \longrightarrow \frac{1}{3} + \frac{4}{9} - 12D = 3 \longrightarrow D = -\frac{5}{27}$

$\therefore y_p = -\frac{1}{6}t^3 + \frac{1}{6}t^2 - \frac{1}{9}t - \frac{5}{27}$

Thank You

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Method of Undetermined Coefficients

Part II

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10

SOLUTIONS OF DIFFERENTIAL EQUATIONS

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Method of Undetermined Coefficients

Remark 1 If β is a root of the characteristic equation which is repeated k-times, then multiply the $y_p(t)$ by t^k.

Remark 2 If $g(t)$ Contains an exponential, ignore it and write down the guess for the reminder, then take the exponential back on without any leading coefficient.

Remark 3 For products of polynomials and trig. functions you first write down the guess for just the polynomial and multiply that by the appropriate Cosine. Then add on a new gess for the polynomial with different coefficients and multiply that by the appropriate sine.

Remark 4 If $g(t)$ is of the form

$$P(t)e^{\alpha t}\cos(\beta t) + Q(t)e^{\alpha t}\sin(\beta t)$$

where $p(t)$ and $Q(t)$ are polynomials, then the following cases are possible

a) If the number $\alpha + i\beta$ is not a root of the charactristic equation then

$$y_p = U(t)e^{\alpha t}\cos(\beta t) + V(t)e^{\alpha t}\sin(\beta t)$$

where $U(t)$ and $V(t)$ are polynomials of degree equal to the higest degree of the polynomials $P(t)$ and $Q(t)$.

Method of Undetermined Coefficients

Remark 4 If $g(t)$ is of the form

$$P(t)e^{\alpha t}\cos(\beta t) + Q(t)e^{\alpha t}\sin(\beta t)$$

where $p(t)$ and $Q(t)$ are polynomials, then the following cases are possible

a) If the number $\alpha + i\beta$ is not a root of the charactristic equation then

$$y_p = U(t)e^{\alpha t}\cos(\beta t) + V(t)e^{\alpha t}\sin(\beta t)$$

where $U(t)$ and $V(t)$ are polynomials of degree equal to the higest degree of the polynomials $P(t)$ and $Q(t)$.

b) If the number $\alpha + i\beta$ is a root of the char. equation, then

$$y_p = t\left(U(t)e^{\alpha t}\cos(\beta t) + V(t)e^{\alpha t}\sin(\beta t)\right)$$

04 Example

Find the particular solution of

$$y'' - 4y' - 12y = te^{4t}$$

let $y_p(t) = e^{4t}(At+B)$

$$y_p' = e^{4t}(A) + 4e^{4t}(At+B) = e^{4t}(4At+A+4B)$$

$$y_p'' = 4Ae^{4t} + 4e^{4t}(A) + 16e^{4t}(At+B)$$

$$= e^{4t}(4A+4A+16At+16B)$$

$$= e^{4t}(16At+8A+16B)$$

$$e^{4t}(16At+8A+16B) - 4e^{4t}(4At+A+4B) - 12e^{4t}(At+B) = te^{4t}$$

$$e^{4t}(-12At+4A-12B) = te^{4t}$$

te^{4t}: $-12A = 1 \Rightarrow \boxed{A = -\frac{1}{12}}$

e^{4t}: $4A - 12B = 0$

$\Rightarrow 4A = 12B \Rightarrow 4\left(-\frac{1}{12}\right) = 12B$

$\Rightarrow -\frac{1}{3(12)} = B \Rightarrow \boxed{B = -\frac{1}{36}}$

$$\therefore\ y_p(t) = e^{4t}\left(-\frac{t}{12} - \frac{1}{36}\right) = -\frac{1}{36}(3t+1)e^{4t}$$

Remark 5 If $y_{p_1}(t)$ is a particular solution for

$$y'' + p(t)y' + q(t)y = g_1(t)$$

and $y_{p_2}(t)$ is a particular solution for

$$y'' + p(t)y' + q(t) = g_2(t)$$

then $y_{p_1}(t) + y_{p_2}(t)$ is a particular solution for

$$y'' + p(t)y' + q(t) = g_1(t) + g_2(t)$$

05

Example

Find the particular solution of

$$y'' - 4y' - 12y = 3e^{4t} + \sin(2t) + te^{4t}$$

by example 1, 2 and 4

$$y_p(t) = -\frac{3}{7}e^{5t} + \frac{1}{40}\cos(2t) - \frac{1}{20}\sin(2t) - \frac{1}{36}(3t+1)e^{4t}$$

y_{p_1} y_{p_2} y_{p_3}

06

Example

write down the form of the particular Solution to $y''+p(t)y'+q(t)y=g(t)$

$g(t)$	$y_p(t)$
$16e^{7t}\sin(10t)$	$e^{7t}(A\cos(10t)+B\sin(10t))$
$(9t^2-103t)\cos t$	$(At^2+Bt+C)\cos t+(Dt^2+Et+F)\sin t$
$-e^{-2t}(3-5t)\cos(9t)$	$e^{-2t}(At+B)\cos(9t)+e^{-2t}(Ct+D)\sin(9t)$
$4\cos(6t)-9\sin(6t)$	$A\cos(6t)+B\sin(6t)$
$-2\sin t+\sin(14t)-5\cos(14t)$	$A\cos t+B\sin t+C\cos(14t)+D\sin(14t)$
$e^{7t}+6$	$Ae^{7t}+B$
$6t^2-7\sin(3t)+9$	$At^2+Bt+C+D\cos(3t)+E\sin(3t)$

06

Example

write down the form of the particular Solution to $y''+p(t)y'+q(t)y=g(t)$

$g(t)$	$y_p(t)$
$10e^{t}-5te^{-8t}+2e^{-8t}$	$Ae^{t}+(Bt+C)e^{-8t}$
$t^2\cos t-5t\sin t$	$(At^2+Bt+C)\cos t+(Dt^2+Et+F)\sin t$
$5e^{-5t}+e^{-5t}\cos(6t)-\sin(6t)$	$Ae^{-5t}+e^{-5t}(B\cos(6t)+C\sin(6t))+D\cos(6t)+E\sin(6t))$

07

Example

Find the particular Solution of the following ode

$$y'' - 4y' - 12 = e^{6t}$$

$$r^2 - 4r - 12 = 0$$

$$(r-6)(r+2) = 0$$

$$r_1 = 6 \qquad r_2 = -2$$

$$y_c = c_1 e^{6t} + c_2 e^{-2t}$$

Ate^{6t}

let $\quad y_p(t) = Ate^{6t}$

$$y_p'(t) = 6Ate^{6t} + Ae^{6t}$$

$$y_p''(t) = 36Ate^{6t} + 6Ae^{6t} + 6Ae^{6t}$$

$$= 36Ate^{6t} + 12Ae^{6t}$$

$$(36Ate^{6t} + 12Ae^{6t}) - 4(6Ate^{6t} + Ae^{6t}) - 12Ate^{6t} = e^{6t}$$

$$8Ae^{6t} = e^{6t}$$

$$\Rightarrow 8A = 1 \quad \Rightarrow \boxed{A = \frac{1}{8}}$$

$$\therefore y_p = \frac{t}{8} e^{6t}$$

08

Example

write down the guess for the particular Solution to the given differential equation. Do not Find the Coefficients

a) $y'' + 3y' - 28y = 7t + e^{-7t} - 1$

b) $y'' - 100y = 9t^2 e^{10t} + \cos t - t\sin t$

c) $4y'' + y = e^{-2t}\sin\left(\frac{t}{2}\right) + 6t\cos\left(\frac{t}{2}\right)$

d) $4y'' + 16y' + 17y = e^{-2t}\sin\left(\frac{t}{2}\right) + 6t\cos\left(\frac{t}{2}\right)$

e) $y'' + 8y' + 16y = e^{-4t} + (t^2 + 5)e^{-4t}$

08

Example

a) $y'' + 3y' - 28y = 7t + e^{-7t} - 1$

$$r^2 + 3r - 28 = 0$$

$$(r+7)(r-4) = 0$$

$$r_1 = -7 \quad , r_2 = 4$$

$$y_c = c_1 e^{-7t} + c_2 e^{4t}$$

let $y_p = (At+B) + cte^{-7t}$

b) $y'' - 100y = 9t^2 e^{10t} + \cos t - t\sin t$

$$r^2 - 100 = 0$$

$$(r-10)(r+10) = 0$$

$$r_1 = 10 \ , \ r_2 = -10$$

$$y_c = c_1 e^{10t} + c_2 e^{-10t}$$

$$y_p = t(At^2+Bt+c)e^{10t} + (Dt+E)\cos t + (Ft+G)\sin t$$

08

Example

c) $4y'' + y = e^{-2t}\sin\left(\frac{t}{2}\right) + 6t\cos\left(\frac{t}{2}\right)$

$$4r^2 + 1 = 0 \Rightarrow r^2 = -\frac{1}{2}i$$

$$\Rightarrow y_c = c_1\cos\left(\frac{t}{2}\right) + c_2\sin\left(\frac{t}{2}\right)$$

let

$$y_p(t) = e^{-2t}\left(A\cos\left(\frac{t}{2}\right) + B\sin\left(\frac{t}{2}\right)\right) + t(Ct+D)\cos\left(\frac{t}{2}\right) + t(Et+F)\sin\left(\frac{t}{2}\right)$$

d) $4y'' + 16y' + 17y = e^{-2t}\sin\left(\frac{t}{2}\right) + 6t\cos\left(\frac{t}{2}\right)$

$$4r^2 + 16r + 17 = 0$$

$$r_{1,2} = \frac{-16 \mp \sqrt{16^2 - 4(4)17}}{2(4)} = \frac{-16 \mp 4i}{8}$$

$$= -2 \mp \frac{1}{2}i$$

$$y_c = e^{-2t}\left(c_1\cos\left(\frac{t}{2}\right) + c_2\sin\left(\frac{t}{2}\right)\right)$$

$$y_p(t) = te^{-2t}\left(A\cos\left(\frac{t}{2}\right) + B\sin\left(\frac{t}{2}\right)\right) + (Ct+D)\cos\left(\frac{t}{2}\right) + (Et+F)\sin\left(\frac{t}{2}\right)$$

08

Example

e) $y'' + 8y' + 16y = e^{-4t} + (t^2+5)e^{-4t} = e^{-4t}(t^2+6)$

$r^2 + 8r + 16 = 0$

$(r+4)(r+4) = 0$

$r_{1,2} = -4$

$y_c = c_1 e^{-4t} + c_2 t e^{-4t}$

$y_p(t) = t^2(At^2 + Bt + c)e^{-4t}$

k $\quad t^k$

2 $\quad t^2$

Disadvantages of the Undetermind Coefficients Method

1) It is only work for a fairly small class of $g(t)$.

2) It is generally only useful for Constant Coefficient differential equations.

Thank You

Any questions?

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Variation of Parameters

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11

SOLUTIONS OF
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Variation of Parameters

Consider the 2^{nd} order linear nonhomogeneous ode

$$p(t)y'' + q(t)y' + r(t)y = g(t)$$

if the Complimantry Solution of it is

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t)$$

Then we will assume that the Particular Solution is of the form

$$y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$$

Provided that

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = g(t)$$

Note that in this System we know the two Solutions and So the only two unkowns here are u_1' and u_2'. The system Can be put in matrix form

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

Variation of Parameters

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

which can be solve by Cramer's rule

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g(t) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = -\frac{y_2 g(t)}{W(y_1, y_2)}$$

and

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g(t) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 g(t)}{W(y_1, y_2)}$$

Recall that $y_1(t)$ and $y_2(t)$ are a fundamental set of Solutions and we know that the Wronskian won't be zero.

then $u_1(t) = \int u_1' \, dt$ and $u_2(t) = \int u_2' \, dt$

01

Example

Find a general Solution to the following equation

$$2y'' + 18y = 6\tan(3t)$$

Solution:

$$y'' + 9y = 3\tan(3t)$$

$$r^2 + 9 = 0 \Rightarrow r_{1,2} = \mp 3i$$

$$y_c = c_1 \cos 3t + c_2 \sin 3t$$

let

$$y_p = u_1 \cos 3t + u_2 \sin 3t$$

Such that

$$u_1' \cos 3t + u_2' \sin 3t = 0$$

$$-3u_1' \sin 3t + 3u_2' \cos 3t = 3\tan 3t$$

01

Example

$$u_1' = \frac{\begin{vmatrix} 0 & \sin 3t \\ 3\tan 3t & 3\cos 3t \end{vmatrix}}{\begin{vmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix}} = \frac{-3\sin 3t \tan 3t}{3(\cos^2 3t + \sin^2 3t)} = -\frac{\sin^2 3t}{\cos 3t}$$

$$u_2' = \frac{\begin{vmatrix} \cos 3t & 0 \\ -3\sin 3t & 3\tan 3t \end{vmatrix}}{3} = \frac{3\cos 3t \tan 3t}{3} = \sin 3t$$

$$u_1(t) = -\int \frac{\sin^2 3t}{\cos 3t} dt = -\int \frac{1-\cos^2 3t}{\cos 3t} dt = -\int [\sec 3t - \cos 3t] dt$$

$$= -\left[\frac{\ln|\sec 3t + \tan 3t|}{3} - \frac{\sin 3t}{3}\right] = -\frac{1}{3}\left[\ln|\sec 3t + \tan 3t| - \sin 3t\right]$$

$$u_2(t) = \int \sin 3t \, dt = -\frac{1}{3}\cos 3t$$

$$\therefore y_p = -\frac{1}{3}\left[\ln|\sec 3t + \tan 3t| - \sin 3t\right]\cos 3t - \frac{1}{3}\cos 3t \sin 3t$$

$$\Rightarrow y_p = -\frac{\cos 3t}{3}\left[\ln|\sec 3t + \tan 3t|\right]$$

$$y = y_c + y_p$$

02

Example

Find a general solution to the following ode

$$y'' - 2y' + y = \frac{e^t}{t^2+1}$$

$$r^2 - 2r + 1 = 0$$

$$(r-1)(r-1) = 0$$

$$r_1 = r_2 = 1$$

$$\therefore y_c = c_1 e^t + c_2 t e^t$$

Now, let

$$y_p = u_1 e^t + u_2 t e^t$$

Such that

$$u_1' e^t + u_2' t e^t = 0$$

$$u_1' e^t + u_2' (t e^t + e^t) = \frac{e^t}{t^2+1}$$

$$W(e^t, te^t) = \begin{vmatrix} e^t & te^t \\ e^t & te^t + e^t \end{vmatrix} = e^t(te^t + e^t) - e^t(te^t) = e^{2t}$$

02 Example

$$u_1' = \frac{\begin{vmatrix} 0 & te^t \\ \frac{e^t}{t^2+1} & te^t+e^t \end{vmatrix}}{e^{2t}} = \frac{-\frac{te^{2t}}{t^2+1}}{e^{2t}} = -\frac{t}{t^2+1}$$

$$u_2' = \frac{\begin{vmatrix} e^t & 0 \\ e^t & \frac{e^t}{t^2+1} \end{vmatrix}}{e^{2t}} = \frac{\frac{e^{2t}}{t^2+1}}{e^{2t}} = \frac{1}{t^2+1}$$

$$u_1 = \int \frac{-t}{t^2+1}\,dt = -\frac{1}{2}\ln(1+t^2)$$

$$u_2 = \int \frac{1}{t^2+1}\,dt = \tan^{-1} t$$

$$\therefore y_p = -\frac{1}{2}\ln(1+t^2)\cdot e^t + te^t \tan^{-1} t$$

$$\Rightarrow y = \underbrace{c_1 e^t + c_2 te^t}_{y_c} \underbrace{- \frac{e^t}{2}\ln(1+t^2) + te^t \tan^{-1} t}_{y_p}$$

Thank You

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Laplace Transforms

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12

SOLUTIONS OF
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Definition Suppose that $f(t)$ is a piecewise Continuous function. The Laplace transform of $f(t)$ is denoted by $\mathcal{L}f(t)$ and defined as

$$F(s) = \mathcal{L}f(t) = \int_0^\infty e^{-st} f(t)\,dt$$

01

Example

Compute $\mathcal{L}1$

$$\mathcal{L}1 = \int_0^\infty e^{-st}\,dt = \lim_{A\to\infty}\int_0^A e^{-st}\,dt = \lim_{A\to\infty}\left(-\frac{1}{s}e^{-st}\right)\Big|_0^A$$

$$= \lim_{A\to\infty}\left(-\frac{1}{s}\left(e^{-sA}-1\right)\right) = -\frac{1}{s}(-1) = \frac{1}{s} \quad \text{provided } s>0$$

$$\boxed{\mathcal{L}1 = \frac{1}{s},\ s>0}$$

02

Example

Compute $\mathcal{L}e^{at}$

$$\mathcal{L}e^{at} = \int_0^\infty e^{-st}e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = -\frac{1}{s-a}\left(e^{-(s-a)t}\right)_0^\infty$$

$$= -\frac{1}{s-a}(0-1) = \frac{1}{s-a} \qquad \text{provided } s-a>0 \Rightarrow s>a$$

$$\boxed{\mathcal{L}e^{at} = \frac{1}{s-a},\ s>a}$$

03

Example

Compute $\mathcal{L}t^n$

$$\mathcal{L}t^n = \int_0^\infty t^n e^{-st}\,dt$$

$$= \left(-\frac{1}{s} t^n e^{-st}\right)\Big|_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st}\,dt \quad (\to 0)$$

$$= \frac{n}{s}\mathcal{L}t^{n-1}$$

$u = t^n$
$du = nt^{n-1}dt$
$dv = e^{-st}dt$
$v = -\frac{1}{s}e^{-st}$

So we get a recursive relation

$$\mathcal{L}t^n = \frac{n}{s}\mathcal{L}t^{n-1} \quad \forall n$$

which means

$$\mathcal{L}t^{n-1} = \frac{n-1}{s}\mathcal{L}t^{n-2}, \quad \mathcal{L}t^{n-2} = \frac{n-2}{s}\mathcal{L}t^{n-3}, \quad \dots$$

03

Example

By induction, we get

$$\mathcal{L}t^n = \frac{n}{s}\mathcal{L}t^{n-1} = \frac{n}{s}\frac{n-1}{s}\mathcal{L}t^{n-2} = \frac{n}{s}\frac{n-1}{s}\frac{n-2}{s}\mathcal{L}t^{n-3}$$

$$= \cdots = \frac{n}{s}\frac{n-1}{s}\frac{n-2}{s}\cdots\frac{1}{s}\mathcal{L}1$$

$$= \frac{n!}{s^n}\frac{1}{s} = \frac{n!}{s^{n+1}}, \quad (s>0)$$

$$\boxed{\mathcal{L}t^n = \frac{n!}{s^{n+1}}, \quad s>0}$$

04

Example

Find the laplace transform of Sinat and Cosat

Here we will use Euler's formula

$$e^{iat} = \cos at + i \sin at$$

$$\mathcal{L} e^{iat} = \frac{1}{s - ia} \quad \frac{1(s+ia)}{(s-ia)(s+ia)} = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

Comparing the real and imaginary parts, we get

$$\boxed{\mathcal{L}\cos at = \frac{s}{s^2+a^2}} \quad \text{and} \quad \boxed{\mathcal{L}\sin at = \frac{a}{s^2+a^2}} \quad (s>0)$$

<u>Remark:</u> Given $f(t)$ and $g(t)$ then

$$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}f(t) + b\mathcal{L}g(t)$$

for any constants a and b.

i.e. Laplace transform is a linear operator.

05

Example

Find $\mathcal{L}f(t)$, where $f(t) = 6e^{-5t} + e^{3t} + 5t^3 - 9$

$$F(s) = \mathcal{L}f(t) = 6\mathcal{L}e^{-5t} + \mathcal{L}e^{3t} + 5\mathcal{L}t^3 - 9\mathcal{L}1$$

$$= 6\frac{1}{s-(-5)} + \frac{1}{s-3} + 5\frac{3!}{s^{3+1}} - 9\frac{1}{s}$$

$$= \frac{6}{s+5} + \frac{1}{s-3} + \frac{30}{s^4} - \frac{9}{s}$$

06

Example

Find $\mathcal{L}f(t)$, where $f(t) = 4\cos 4t - 9\sin 4t + 2\cos 10t$

$$F(s) = \mathcal{L}f(t) = 4\mathcal{L}\cos 4t - 9\mathcal{L}\sin 4t + 2\mathcal{L}\cos 10t$$

$$= 4\frac{s}{s^2+(4)^2} - 9\frac{4}{s^2+(4)^2} + 2\frac{s}{s^2+(10)^2}$$

$$= \frac{4s}{s^2+16} - \frac{36}{s^2+16} + \frac{2s}{s^2+100}$$

$$= \frac{4s-36}{s^2+16} + \frac{2s}{s^2+100}$$

Theorem If $\mathcal{L}f(t) = F(s)$, then

1- $\mathcal{L}e^{at}f(t) = F(s-a)$

2- $\mathcal{L}t^n f(t) = (-1)^n F^{(n)}(s)$

07

Example

Find $\mathcal{L}f(t)$, where $f(t) = 3\sinh 2t + 3\sin 2t$

$$F(s) = 3\mathcal{L}\sinh 2t + 3\mathcal{L}\sin 2t$$

$$= 3\frac{2}{s^2 - 2^2} + 3\frac{2}{s^2 + 2^2}$$

$$= \frac{6}{s^2 - 4} + \frac{6}{s^2 + 4}$$

08

Example

Find $\mathcal{L}f(t)$, where $f(t) = e^{3t} + \cos 6t - e^{3t}\cos 6t$

$$F(s) = \mathcal{L}e^{3t} + \mathcal{L}\cos 6t - \mathcal{L}e^{3t}\cos 6t$$

$$= \frac{1}{s-3} + \frac{s}{s^2+36} - \frac{s-3}{(s-3)^2+36}$$

09

Example

prove that $\mathcal{L}\, t\sin at = \dfrac{2as}{(s^2+a^2)^2}$

$$\mathcal{L}\sin at = \frac{a}{s^2+a^2}$$

by the above theorem $\mathcal{L}\, t\sin at = (-1)^1 \left(\dfrac{a}{s^2+a^2}\right)'$

$$= -\frac{(s^2+a^2)(0) - a(2s)}{(s^2+a^2)^2} = \frac{2as}{(s^2+a^2)^2}$$

H.W. Prove that $\mathcal{L}\, t\cos at = \dfrac{s^2-a^2}{(s^2+a^2)^2}$

10

Example

Find $\mathcal{L}\, t^2 \sin 2t$

we know that

$$\mathcal{L} \sin 2t = \frac{2}{s^2+4} \Rightarrow \mathcal{L}\, t^2 \sin 2t = (-1)^2 \frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right)$$

$$\mathcal{L}\, t^2 \sin 2t = \frac{d}{ds}\left(\frac{-2(2s)}{(s^2+4)^2}\right) = \frac{d}{ds}\left(\frac{-4s}{(s^2+4)^2}\right)$$

$$= \frac{(s^2+4)^2(-4) + 4s\,(2(s^2+4)(2s))}{(s^2+4)^4}$$

$$= \frac{(s^2+4)\,(\,-4(s^2+4)+16s^2)}{(s^2+4)^4}$$

$$= \frac{12s^2-16}{(s^2+4)^3}$$

11 Example

Find $\mathcal{L}g'(t)$

By Def.

$$\mathcal{L}g'(t) = \int_0^\infty g'(t)\,e^{-st}\,dt$$

let $u = e^{-st}$

$du = -s e^{-st}\,dt$

$dv = g'(t)\,dt$

$v = g(t)$

$$= \left[g(t)\,e^{-st}\right]_0^\infty + s\int_0^\infty g(t)\,e^{-st}\,dt$$

$$= -g(0) + s\,G(s)$$

$$= s\,G(s) - g(0)$$

12

Example

Find $\mathcal{L}\, t g'(t)$

$$\mathcal{L}\, t g'(t) = -\frac{d}{ds}\, \mathcal{L} g'(t)$$

$$= -\frac{d}{ds}\left[s G(s) - g(0) \right]$$

$$= -\left[s G'(s) + G(s) \right] = -s G'(s) - G(s).$$

13

Example

Find $\mathcal{L}\, t^4 e^{2t}$

$$\mathcal{L}\, t^4 e^{2t} = (-1)^4 \frac{d^4}{ds^4}\left(\frac{1}{s-2}\right) = \frac{d^3}{ds^3}\,\frac{-1}{(s-2)^2}$$

$$= \frac{d^2}{ds^2}\,\frac{(s-2)^2(0)+1(2(s-2))}{(s-2)^4} = \frac{d^2}{ds^2}\,\frac{2}{(s-2)^3}$$

$$= \frac{d}{ds}\,\frac{-2(3(s-2)^2)}{(s-2)^6} = \frac{d}{ds}\,\frac{-6}{(s-2)^4} = \frac{d}{ds}\,\frac{-3!}{(s-2)^4}$$

$$= \frac{-(-3!)(4(s-2)^3)}{(s-2)^8} = \frac{4!}{(s-2)^5}$$

13

Example

<u>OR</u>

Since $\mathcal{L}\, t^n e^{at} = \dfrac{n!}{(s-a)^{n+1}}$

$$\Rightarrow \mathcal{L}\, t^4 e^{2t} = \frac{4!}{(s-2)^5}$$

<u>OR</u>

Since $\mathcal{L}\, t^n = \dfrac{n!}{s^{n+1}}$

$$\mathcal{L}\, t^4 = \frac{4!}{s^5} \Rightarrow \mathcal{L}\, t^4 e^{2t} = \frac{4!}{(s-2)^5}$$

$\mathcal{L}\, f(t) e^{at} = F(s-a)$

Thank You

Any questions?

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