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Gamma Function

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MATHEMATICAL PHYSICS II

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The Factorial Functions

Let $\alpha > 0$, then

$$\int_0^\infty e^{-\alpha x}\,dx = -\frac{1}{\alpha} e^{-\alpha x}\Big|_0^\infty = \frac{1}{\alpha}$$

differentiate both Sides w.r.t. α

$$\int_0^\infty -x e^{-\alpha x}\,dx = -\frac{1}{\alpha^2}$$

$$\Rightarrow \int_0^\infty x e^{-\alpha x}\,dx = \frac{1}{\alpha^2}$$

differentiate agin

$$\Rightarrow \int_0^\infty x^2 e^{-\alpha x}\,dx = \frac{2}{\alpha^3}$$

and agin

$$\Rightarrow \int_0^\infty x^3 e^{-\alpha x}\,dx = \frac{3!}{\alpha^4}$$

In general $\displaystyle\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$

Put $\alpha = 1$ weget

$$\boxed{\int_0^\infty x^n e^{-x}\,dx = n!}$$ where $n = 1, 2, \ldots$

$$\boxed{\int_0^{\infty} x^n e^{-x}\,dx = n!}$$ where $n = 1, 2, \ldots$

we can find $0!$ by Putting $n=0$

$$\Rightarrow \int_0^{\infty} e^{-x}\,dx = 0!$$

$$-e^{-x}\Big|_0^{\infty} = 0!$$

$$-(0-1) = 0!$$

$$\boxed{1 = 0!}$$

Gamma Function

we can define the gamma function as

$$\Gamma P = \int_0^\infty x^{P-1} e^{-x}\, dx$$

Notice that

$$\Gamma P{+}1 = \int_0^\infty x^P e^{-x}\, dx$$

integrating by parts

$$= -e^{-x} \cancel{x^{P}} \Big|_0^\infty + P \int_0^\infty e^{-x} x^{P-1}\, dx$$

$$= P \int_0^\infty x^{P-1} e^{-x}\, dx$$

$$= P\,\Gamma P \quad \Rightarrow \quad \boxed{\Gamma P{+}1 = P\,\Gamma P}$$

from this we obtain

$$\Gamma 2 = 1\,\Gamma 1 = 1 \int_0^\infty e^{-x}\, dx = 1$$

$$\Gamma 3 = 2\,\Gamma 2 = 2 \cdot 1 = 2 = 2!$$

$$\Gamma 4 = 3\,\Gamma 3 = 3 \cdot 2 \cdot 1 = 6 = 3!$$

$$\vdots$$

$$\Gamma n = (n-1)!$$

Moreover,

$$\Gamma 2.5 = (1.5)\,\Gamma 1.5 = (1.5)(0.5)\,\Gamma 0.5$$

and to find the value of gamma of number between 0 and 1 we use

$$\Gamma P = \frac{1}{P}\,\Gamma P{+}1 \qquad 0<P<1$$

for example $\Gamma 0.5 = \frac{1}{0.5}\,\Gamma 1.5$

It is possible to extend the domain of Γp to negative values of p

$$\Gamma(p+1) = p\,\Gamma p \quad \text{----}(1)$$

$$\Rightarrow \Gamma p = \frac{\Gamma(p+1)}{p}$$

$$\Rightarrow \Gamma 0 = \frac{\Gamma 1}{0} \longrightarrow \infty$$

Similarly $\Gamma(-1) = \frac{\Gamma 0}{-1} = \frac{\Gamma 1}{(-1)(0)} \longrightarrow \infty$

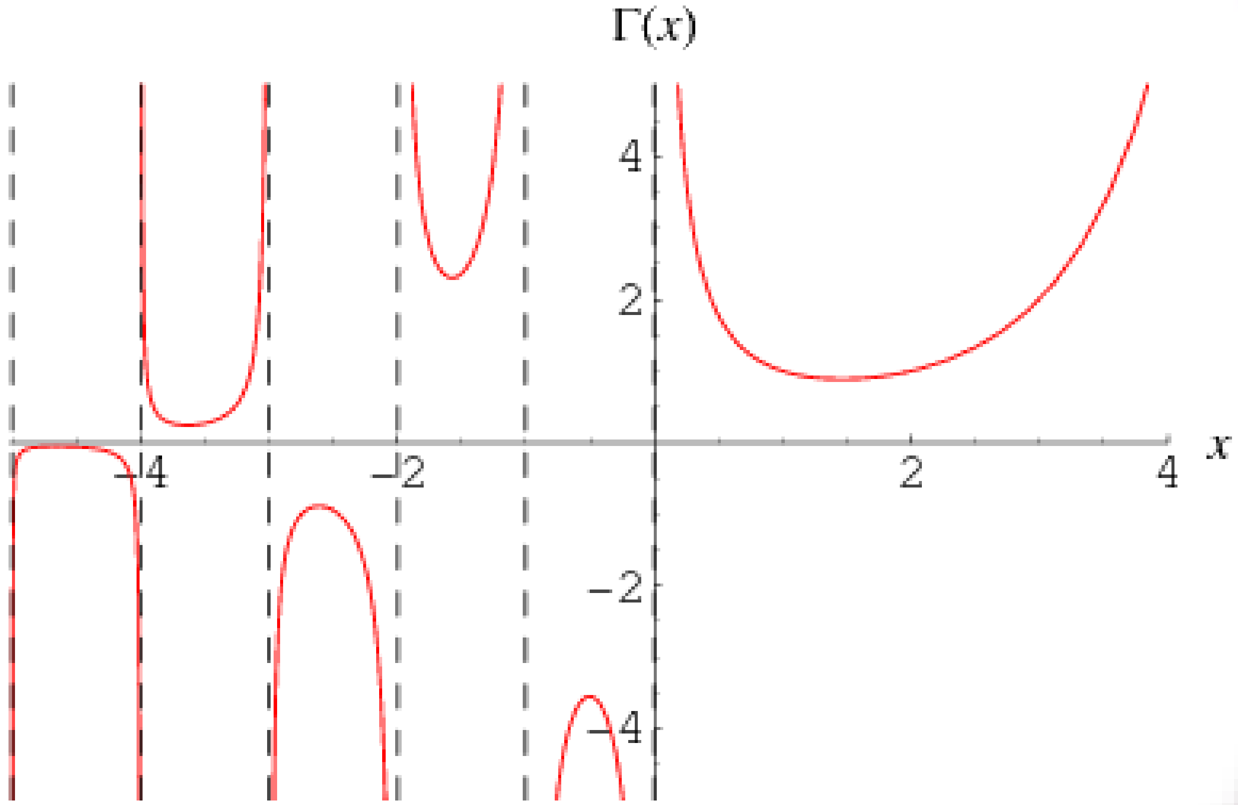
$$\Gamma(-2) = \frac{\Gamma(-1)}{-2} = \frac{\Gamma 1}{(-2)(-1)(0)} \longrightarrow \infty$$

For any other negative value of p, we can compute Γp using (1) until $\Gamma p+1$ has positive argument

Examples $\Gamma -\frac{3}{2} = \frac{\Gamma -\frac{1}{2}}{-\frac{3}{2}} = \frac{\Gamma \frac{1}{2}}{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)} = \frac{4}{3}\Gamma \frac{1}{2}$

$$\Gamma -\frac{5}{2} = \frac{\Gamma -\frac{3}{2}}{-\frac{5}{2}} = \frac{\Gamma -\frac{1}{2}}{\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)} = \frac{\Gamma \frac{1}{2}}{\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)} = -\frac{8}{15}\Gamma \frac{1}{2}$$

Hence, Γp is well defined for any $p \in \mathbb{R}$ except $x = 0, -1, -2, \ldots$


Γ(x)
4
2
−2
−4
−4
−2
2
4
x

$$\Gamma\tfrac{1}{2} = \int_0^\infty t^{-1/2} e^{-t}\,dt = \int_0^\infty \frac{1}{\sqrt{t}} e^{-t}\,dt$$

let $y^2 = t \Rightarrow 2y\,dy = dt$

$$\Rightarrow \Gamma\tfrac{1}{2} = \int_0^\infty \frac{1}{y} e^{-y^2} 2y\,dy = 2\int_0^\infty e^{-y^2}\,dy \quad \text{---} (1)$$

or we can say $\Gamma\tfrac{1}{2} = 2\int_0^\infty e^{-x^2}\,dx$ --- (2)

multiply (1) and (2) we get

$$\left(\Gamma\tfrac{1}{2}\right)^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

and this is an integral in the first quadrant

$$\left(\Gamma\tfrac{1}{2}\right)^2 = 4\int_{\theta=0}^{\pi/2} \int_{r=0}^\infty e^{-r^2} r\,dr\,d\theta = 4\,\frac{\pi}{2}\,\frac{e^{-r^2}}{-2}\Big|_0^\infty = \pi$$

$$\Rightarrow \boxed{\Gamma\tfrac{1}{2} = \sqrt{\pi}} \quad \square$$

Example

Example 1 Evaluate $\int_0^\infty x^3 e^{-x} dx = \Gamma 4 = 3! = 6$

Example 2 Evaluate $\int_0^\infty x^6 e^{-2x} dx$

let $2x = y \Rightarrow 2dx = dy \Rightarrow \frac{dy}{2} = dx$

$$\int_0^\infty \left(\frac{y}{2}\right)^6 e^{-y} \frac{dy}{2} = \frac{1}{2^7} \int_0^\infty y^6 e^{-y} dy = \frac{\Gamma 7}{2^7}$$

$$= \frac{6!}{2^7} = \frac{45}{8}$$

Example Evaluate $\int_0^\infty \sqrt{y}\, e^{-y^3} dy$

let $y^3 = x \Rightarrow y = x^{1/3}$

$\Rightarrow y^2 = x^{2/3}$

$3y^2 dy = dx$

$dy = \frac{dx}{3y^2}$

$dy = \frac{dx}{3x^{\frac{2}{3}}}$

$$\Rightarrow \int_0^\infty x^{1/6} e^{-x} \frac{dx}{3x^{2/3}} = \frac{1}{3} \int_0^\infty x^{-1/2} e^{-x} dx$$

$$= \frac{1}{3} \Gamma \frac{1}{2} = \frac{1}{3} \sqrt{\pi}$$

Examples

Example Evaluate $\int_0^1 \frac{dx}{\sqrt{-\ln x}}$

let $-\ln x = u \Rightarrow \ln x = -u \Rightarrow x = e^{-u}$

$dx = -e^{-u}\,du$

when $x = 1$, $u = 0$

when $x = 0$, $u = \infty$

$$\int_0^\infty u^{-1/2} e^{-u}\,du = \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

Example Evaluate $\int_0^\infty \sqrt{x}\, e^{-\sqrt[6]{x}}\,dx$, let $x = t^6 \Rightarrow dx = 6t^5\,dt$

$$\int_0^\infty t^3 e^{-t}\, 6t^5\,dt = 6\int_0^\infty t^8 e^{-t}\,dt = 6\,\Gamma 9 = (6)(8!)$$

Examples

Thank You

Any questions?

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Beta Function

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The Beta Function is a two-Parameter Composition of gamma functions that has been useful enough in application to gain its own name. Its definition is

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

If $x \geqslant 1$ and $y \geqslant 1$, this is a proper integral. If $x > 0$ and $y > 0$ and either or both $x < 1$ or $y < 1$, the integral is improper but Convergent

Properties of Gamma Function

$$\boxed{B(x,y) = B(y,x)}$$

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

use the tranformation $u = 1-t$

$\Rightarrow du = -dt$ and $t = 1-u$

$$\therefore B(x,y) = -\int_1^0 (1-u)^{x-1} u^{y-1}\,du$$

$$= \int_0^1 u^{y-1}(1-u)^{x-1}\,du = B(y,x) \quad \square$$

$$\boxed{B(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}\theta \cos^{2y-1}\theta \, d\theta}$$

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

use the transformation $t = \sin^2\theta$

$\Rightarrow dt = 2\sin\theta\cos\theta\, d\theta$

$$\therefore B(x,y) = \int_0^{\pi/2} (\sin^2\theta)^{x-1}(1-\sin^2\theta)^{y-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} (\sin^2\theta)^{x-1}(\cos^2\theta)^{y-1} \sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2x-2}\theta \cos^{2y-2}\theta \sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2x-1}\theta \cos^{2y-1}\theta\, d\theta \quad \square$$

$$\boxed{B(x,y) = \int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}}\, dt}$$

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$$

$$\text{Put } t = \frac{u}{1+u} \Rightarrow dt = \frac{(1+u)\,du - u\,du}{(1+u)^2} = \frac{du}{(1+u)^2}$$

$$\therefore B(x,y) = \int_0^{\infty} \left(\frac{u}{1+u}\right)^{x-1}\left(1 - \frac{u}{1+u}\right)^{y-1} \frac{du}{(1+u)^2}$$

$$= \int_0^\infty \left(\frac{u}{1+u}\right)^{x-1} \left(\frac{1+u-u}{1+u}\right)^{y-1} \frac{du}{(1+u)^2}$$

$$= \int_0^\infty u^{x-1} \cdot \left(\frac{1}{1+u}\right)^{x-1} \left(\frac{1}{1+u}\right)^{y-1} \left(\frac{1}{1+u}\right)^2 du$$

$$= \int_0^\infty u^{x-1} \cdot \left(\frac{1}{1+u}\right)^{x-1+y-1+2} du = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}} du$$

replace each u by t we get

$$B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}} dt \quad \square$$

$$\boxed{B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}}$$

$$\Gamma m = \int_0^\infty t^{m-1} e^{-t} dt \qquad \text{let } t = x^2 \Rightarrow dt = 2x\,dx$$

$$\Rightarrow \Gamma m = \int_0^\infty (x^2)^{m-1} e^{-x^2} \cdot 2x\,dx$$

$$= 2\int_0^\infty x^{2m-2+1} e^{-x^2} dx$$

$$= 2\int_0^\infty x^{2m-1} e^{-x^2} dx$$

Similarly,

$$\Gamma n = 2\int_0^\infty y^{2n-1} e^{-y^2} dy$$

$$\Rightarrow \Gamma m \Gamma n = 4\int_0^{\infty} e^{-x^2} x^{2m-1} dx \cdot \int_0^{\infty} e^{-y^2} y^{2n-1} dy$$

$$= 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dx\, dy$$

Put $x = r\cos\theta$, $y = r\sin\theta$

$$\Rightarrow \Gamma m \Gamma n = 4\int_0^{\infty}\int_0^{\pi/2} e^{-r^2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1} r\, d\theta\, dr$$

$$= 4\int_0^{\infty} e^{-r^2} r^{2m-1+2n-1+1} dr \cdot \int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta$$

$$= 4\int_0^\infty e^{-r^2} r^{2m+2n-1}\,dr \int_0^{\pi/2} \cos^{2m-1}\theta \,\sin^{2n-1}\theta \,d\theta$$

$$= \underline{\underline{(2)}}(2)\int_0^\infty e^{-r^2} r^{2(m+n)-1}\,dr \int_0^{\pi/2} \cos^{2m-1}\theta \,\sin^{2n-1}\theta \,d\theta$$

$$= \Gamma(m+n)\, B(m,n)$$

$$\therefore B(m,n) = \frac{\Gamma m \,\Gamma n}{\Gamma(m+n)}$$

Example $B(7,5) = \dfrac{\Gamma 7\,\Gamma 5}{\Gamma 7+5} = \dfrac{6!\,4!}{11!}$

$= \dfrac{\cancel{4}*\cancel{3}*\cancel{2}}{11*10*\cancel{9}*\cancel{8}*7} = \dfrac{1}{11*10*3*7} = \dfrac{1}{2310}$

Example prove that

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$$

we know that

$$B(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}\theta \cos^{2y-1}\theta \, d\theta$$

$$\therefore \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}B(x,y) = \frac{1}{2}B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2}\,\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}} \quad \square$$

Examples

Examples

Example find $\int_0^{\pi/2} \sin^5\theta \cos^3\theta \, d\theta$

$$= \frac{1}{2} \frac{\Gamma 3 \, \Gamma 2}{\Gamma 5} = \frac{1}{2} \frac{2!\,(1)}{4!} = \frac{1}{\cancel{2}} \frac{\cancel{2}}{24} = \frac{1}{24}$$

Example Find $\int_0^1 x^4 (1-x)^3 \, dx$

$$\int_0^1 x^4 (1-\cancel{x})^3 \, dx = \frac{\Gamma 5 \, \Gamma 4}{\Gamma 9} = \frac{4!\,3!}{8!} = \frac{1}{280}$$

Example $\displaystyle\int_0^2 \frac{x^2\,dx}{\sqrt{2-x}}$

let $x = 2t \Rightarrow dx = 2dt \Rightarrow dt = \frac{1}{2}dx$

$x \to 0 \ , \ t \to 0$

$x \to 2 \ , \ t \to 1$

$$I = \int_0^2 \frac{x^2\,dx}{\sqrt{2-x}} = 4\sqrt{2}\int_0^1 \frac{t^2}{\sqrt{1-t}}\,dt = 4\sqrt{2}\int_0^1 t^2 (1-t)^{-\frac{1}{2}}\,dt$$

$m-1 = 2 \Rightarrow m = 3 \quad , \quad n-1 = -\frac{1}{2} \Rightarrow n = \frac{1}{2}$

$$I = 4\sqrt{2}\,B\left(3, \tfrac{1}{2}\right) = 4\sqrt{2}\,\frac{\Gamma 3\,\Gamma\frac{1}{2}}{\Gamma\frac{7}{2}} = 4\sqrt{2}\,\frac{2!\,\cancel{\Gamma\frac{1}{2}}}{\frac{5}{2}\,\frac{3}{2}\,\frac{1}{2}\,\cancel{\Gamma\frac{1}{2}}} = \frac{64\sqrt{2}}{15}$$

Examples

Example $\int_0^a y^4\sqrt{a^2-y^2}\,dy$

let $y^2 = a^2x \Rightarrow 2y\,dy = a^2dx$

$$\Rightarrow \int_0^a y^4\sqrt{a^2-y^2}\,dy = \int_0^1 a^4x^2\sqrt{a^2-a^2x}\,\frac{a^2dx}{2a\sqrt{x}}$$

$$= a^6\int_0^1 x^{3/2}(1-x)^{1/2}dx = a^6\beta\left(\frac{5}{2},\frac{3}{2}\right) = a^6\frac{\Gamma\frac{5}{2}\Gamma\frac{3}{2}}{\Gamma 4}$$

$$= a^6\frac{\left(\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}\right)\left(\frac{1}{2}\Gamma\frac{1}{2}\right)}{3!}$$

$$= a^6\frac{3\pi}{8(3\times 2)} = \frac{a^6\pi}{16}$$

Examples

Example Evaluate $\int_0^{\pi/2} \sin^6\theta \, d\theta$

we already proved that

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$$

$\Rightarrow p = 6 \quad , q = 0$

$$\int_0^{\pi/2} \sin^6\theta \, d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(4)}$$

$$= \frac{1}{2} \frac{\left(\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right)\left(\Gamma\left(\frac{1}{2}\right)\right)}{3!} = \frac{5\pi}{32}$$

Examples

Thank You

Any questions?

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Series Solutions of ODEs

- Power Series Method

The power series method is the standard for solving linear ODEs with variable coefficients. It gives solutions in the form of power series. These series can be used for computing values, graphing curves, proving formulas, and exploring properties of solutions.

From calculus we remember that a power series (in powers of $x - x_0$) is an infinite series of the form

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

Here, x is a variable. $a_0, a_1, a_2, \ldots$ are constants, called the coefficients of the series. x_0 is a constant, called the center of the series. In particular, if $x_0 = 0$, we obtained a power series in powers of x

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots$$

we shall assume tha all variables and constants are real and m is positive integer (neither negative nor fractional

Example 1 Familiar power Series are the Maclaurin Series

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots \qquad (|x|<1, \text{ geometric Series})$$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

The basic idea of the power Series method for Solving differention equations is very Simple as we can See in the next example,

Example 2 Solve $y' - y = 0$

let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \cdots = \sum_{m=0} a_m x^m$

$$\Rightarrow y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_{m=1}^{\infty} m a_m x^{m-1}$$

Substitute in the original equation

$$(a_1 + 2a_2 x + 3a_3 x^2 + \cdots) - (a_0 + a_1 x + a_2 x^2 + \cdots) = 0$$

$$\Rightarrow (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \cdots = 0$$

Equating the coefficient of each power of x to zero, we have

$$a_1 - a_0 = 0, \quad 2a_2 - a_1 = 0, \quad 3a_3 - a_2 = 0 \ \cdots$$

$\Rightarrow a_1 = a_0 \ , \quad a_2 = \frac{a_1}{2} = \frac{a_0}{2!} \ , \ a_3 = \frac{a_2}{3} = \frac{a_0}{3!} \ , \dots$

$$\Rightarrow y = a_0 + a_1 x + a_2 x^2 + \cdots = a_0 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$
$$= a_0 e^x$$

<u>Example 3</u> Solve $y'' + y = 0$

$$\text{let } y = \sum_{m=0}^{\infty} a_m x^m$$

$$\Rightarrow y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$$

$$\Rightarrow y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

$$\therefore \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^m = 0$$

$$2(1) a_2 + 3(2) a_3 x + 4(3) a_4 x^2 + \cdots + a_0 + a_1 x + a_2 x^2 + \cdots = 0$$

$$(2a_2 + a_0) + (6a_3 + a_1) x + (12 a_4 + a_2) x^2 + \cdots = 0$$

$2a_2 + a_0 = 0 \Rightarrow a_2 = \frac{-a_0}{2!}$

$6a_3 + a_1 = 0 \Rightarrow a_3 = \frac{-a_1}{6} = \frac{-a_1}{3!}$

$12 a_4 + a_2 = 0 \Rightarrow a_4 = -\frac{a_2}{12} = \frac{a_2}{4!} \ , \dots$

a_0 and a_1 are arbitrary

$$y = a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 + \cdots$$

$$= a_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots\right) + a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$y = a_0 \cos x + a_1 \sin x$$

Example 4 Solve a Special Legender Equation

$$(1-x^2)y'' - 2xy' + 2y = 0$$

let $y = \sum_{m=0}^{\infty} a_m x^m \Rightarrow y' = \sum_{m=1}^{\infty} m a_m x^{m-1} \Rightarrow y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$

$$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + \cdots$$

$$-x^2 y'' = -2a_2 x^2 - 6a_3 x^3 - 12a_4 x^4 - \cdots$$

$$-2xy' = -2a_1 x - 4a_2 x^2 - 6a_3 x^3 - 8a_4 x^4 - \cdots$$

$$2y = 2a_0 + 2a_1 x + 2a_2 x^2 + 2a_3 x^3 + 2a_4 x^4 + \cdots$$

Now
- $2a_2 + 2a_0 = 0 \Rightarrow a_2 = -a_0$
- $6a_3 - 2a_1 + 2a_1 = 0 \Rightarrow a_3 = 0$
- $12a_4 - 2a_2 - 4a_2 + 2a_2 = 0$

  $\Rightarrow 12a_4 = 4a_2 \Rightarrow a_4 = \frac{a_2}{3} = -\frac{a_0}{3}$
- $20a_5 - 6a_3 - 6a_3 + 2a_3 = 0$

  $20a_5 - 10a_3 = 0$ but $a_3 = 0 \Rightarrow a_5 = 0$
- $30a_6 - 12a_4 - 8a_4 + 2a_4 = 0$

  $\Rightarrow 30a_6 = 18a_4 \Rightarrow a_6 = \frac{18}{30} a_4 = \frac{18}{30}\left(\frac{-a_0}{3}\right)$

  $\Rightarrow a_6 = -\frac{1}{5} a_0$

$$\therefore y = a_1 x + a_0\left(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots\right)$$

a_0 and a_1 are arbitrary

hence the general Solution Consists of two independent Solutions (x) and $\left(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots\right)$

Exercieses

Solve the following problems Using the Power Series method

1) $(x+1)y' - (x+2)y = 0$

Ans: $y = a_0(1 + 2x + \frac{3}{2}x^2 + \frac{2}{3}x^3 + \frac{5}{24}x^4 + \cdots)$

2) $(x+1)y' = 2y$

3) $(1-x^2)y' = y$

4) $(x-2)y' = 3y$

Theory of the Power Series Method

The nth partial sum of $\sum_{m=0}^{\infty} a_m(x-x_0)^m$ is

$$S_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n$$

where $n = 0, 1, \cdots$. If we omit the terms of S_n from the original Series, the remaning expression is

$$R_n(x) = a_{n+1}(x-x_0)^{n+1} + a_{n+2}(x-x_0)^{n+2} + \cdots$$

This expression is called the remainder of the Series after the term $a_n(x-x_0)^n$.

For example, in the case of geometric Series

$$1 + x + x^2 + \cdots + x^n + \cdots$$

we have

$$S_0 = 1 \qquad R_0 = x + x^2 + x^3 + \cdots$$

$$S_1 = 1 + x \qquad R_1 = x^2 + x^3 + x^4 + \cdots$$

$$S_2 = 1 + x + x^2 \qquad R_2 = x^3 + x^4 + x^5 + \cdots \text{ etc}$$

In this way we have now associated with $\sum_{m=0}^{\infty} a_m (x-x_0)^m$ the sequence of the partial sums $S_0(x)$, $S_1(x)$, $S_2(x)$, ---. If for some $x = x_1$ this sequence converges, say,

$$\lim_{n\to\infty} S_n(x_1) = S(x_1)$$

then the series is called <u>convergent</u> at $x = x_1$, the number $S(x_1)$ is called the <u>value</u> or sum of the series at x_1, and we write

$$S(x_1) = \sum_{m=0}^{\infty} a_m (x_1 - x_0)^m.$$

Then we have for every n,

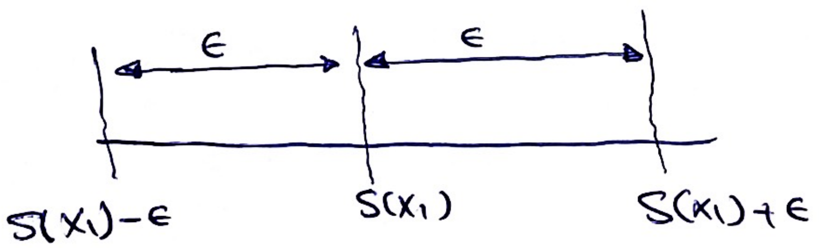
$$S(x_1) = S_n(x_1) + R_n(x_1)$$

If that sequence diverges at $x = x_1$, the series is called <u>divergent</u> at $x = x_1$.

In the case of convergence, for any positive ϵ there is an N (depending on ϵ) such that

$$|R_n(x_1)| = |S(x_1) - S_n(x_1)| < \epsilon \quad \text{for all } n > N$$

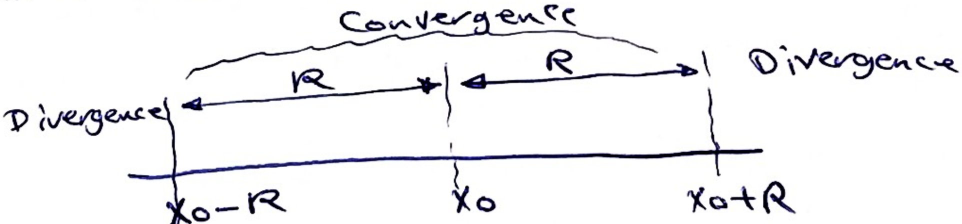
Geometrically, this means that all $S_n(x_1)$ with $n > N$ lie between $S(x_1) - \epsilon$ and $S(x_1) + \epsilon$. Partically, this means that in the case of convergence we can approximate the sum $S(x_1)$ of the series at x_1 by $S_n(x_1)$ as accurately as we please, by taking n large enough



where does a power Series Converge? Now if we choose $x = x_0$ in $\sum_{m=0}^{\infty} a_m (x - x_0)^m$, the Series reduces to the single term a_0 because the other terms are zero. Hence the series Converges at x_0. In some cases this may be the only Value of x for which the Series Converges. If there are other values of x for which the Series converges, these values form an interval, the Convergence interval. This interval may be finite as in the figure below, with midpoint x_0. Then the Series Converges for all x in the interior of the interval, that is, for all x for which

$$|x - x_0| < R$$

and diverges for $|x - x_0| > R$. The interval may also be infinite, that is, the series may converge for all x.



The quantity R is called the radius of Convergence. If the Series Converges for all x, we set $R = \infty$.

The radius of Convergence can be determined from the Coefficients of the Series by means of each of the formulas

ⓐ $R = 1 \Big/ \lim_{m \to \infty} \sqrt[m]{|a_m|}$ ⓑ $R = 1 \Big/ \lim_{m \to \infty} \left| \frac{a_{m+1}}{a_m} \right|$

Provided these limits exist and are not zero. If these limits are infinite, then the Series Converges only at x_0.

Example Convergence Radius $R = \infty, 1, 0$

For all three Series let $m \to \infty$

- $e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \cdots$

$$\left|\frac{a_{m+1}}{a_m}\right| = \frac{1/(m+1)!}{1/m!} = \frac{1}{m+1} \to 0 \quad , \quad R = \infty$$

- $\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots$

$$\left|\frac{a_{m+1}}{a_m}\right| = \frac{1}{1} = 1 \qquad R = 1$$

- $\sum_{m=0}^{\infty} m!\, x^m = 1 + x + 2x^2 + \cdots$

$$\left|\frac{a_{m+1}}{a_m}\right| = \frac{(m+1)!}{m!} = \frac{(m+1)\,m!}{m!} = m+1 \to \infty, \quad R = 0$$

Convergence for all x ($R=\infty$) is the best possible case, Convergence in some finite interval the usual, and Convergence only at the Center ($R=0$) is usless.

Theorem (Existence of Power Series Solutions)

Let $y'' + p(x)y' + q(x)y = r(x)$

If p, q and r are analytic at $x = x_0$, then every solution is analytic at $x = x_0$ and can thus be represented by a power Series in powers of $x - x_0$ with radius of Convergence $R > 0$

3

Series Solution of ODEs

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Legendre's Equation, Legendre Polynomials $P_n(x)$

Legendre's differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{---- (1)}$$

(n Constant)

is one of the most important ODEs in physics. It arises in numerous problems, particularly in boundary value porblems for spheres.

The equation involves a parameter n, whose value depends on the physical or engineering problem. So (1) is actually a whole family of ODEs. For $n=1$ we solved it in last section. Any solution of (1) is called a Legendre function. The study of these and other "higher" functions is called the theory of special functions.

Deviding (1) by $1-x^2$, we obtain the standard form needed in the perivious theorem and we see that the coefficients $\frac{-2x}{(1-x^2)}$ and $\frac{n(n+1)}{(1-x^2)}$ of the new equation are analytic at $x=0$, so that we may apply the power series method, substituting

$$y = \sum_{m=0}^{\infty} a_m x^m \quad \text{---- (2)}$$

and its derivatives into (1), and denoting the constant $n(n+1)$ simply by k, we obtain

$$(1-x^2)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - 2x\sum_{m=1}^{\infty} m a_m x^{m-1} + k\sum_{m=0}^{\infty} a_m x^m = 0$$

By writing the first expression as two separate series we have the equation

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m - \sum_{m=1}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} k a_m x^m = 0$$

To obtain the general power x^s in all four series, set $m-2=s$ (thus $m=s+2$) in the first series and simply write s instead of m in the other three series. This gives

$$\sum_{s=0}^{\infty}(s+2)(s+1)a_{s+2}x^s - \sum_{s=2}^{\infty} s(s-1)a_s x^s - \sum_{s=1}^{\infty} 2s a_s x^s + \sum_{s=0}^{\infty} k a_s x^s = 0$$

Now,

$$2\cdot 1 a_2 + 3\cdot 2 a_3 x^1 + \sum_{s=2}^{\infty}(s+2)(s+1)a_{s+2}x^s - \sum_{s=2}^{\infty} s(s-1)a_s x^s$$
$$-2\cdot 1 a_1 x - \sum_{s=2}^{\infty} 2s a_s x^s + k a_0 + k a_1 x + \sum_{s=2}^{\infty} k a_s x^s = 0$$

$$\Rightarrow 2\cdot 1 a_2 + k a_0 + (3\cdot 2 a_3 - 2\cdot 1 a_1 + k a_1)x +$$
$$\sum_{s=2}^{\infty}\left[(s+2)(s+1)a_{s+2} + (-s(s-1) - 2s + k)a_s\right]x^s = 0$$

$$\therefore \quad 2\cdot 1 a_2 + k a_0 = 0$$

$$2\cdot 1 a_2 + n(n+1) = 0$$

$$\Rightarrow a_2 = -\frac{n(n+1)}{2!} a_0 \qquad \text{---- (3a)}$$

and

$$3\cdot 2 a_3 + \left[-2\cdot 1 + n(n+1)\right] a_1 = 0$$

$$\Rightarrow a_3 = -\frac{-2+n^2+n}{3!} a_1 = -\frac{n^2+n-2}{3!} a_1$$

$$a_3 = \frac{-(n-1)(n+2)}{3!} a_1 \qquad ---- \qquad (3b)$$

and

$$(s+2)(s+1)a_{s+2} + \left[-s(s-1) - 2s + n(n+1)\right] a_s = 0 \quad ---- (3c)$$

$$\begin{aligned} & -s^2 + s - 2s + n^2 + n \\ &= -s^2 - s + n^2 + n \\ &= n - s + n^2 - s^2 \\ &= n - s + (n-s)(n+s) \\ &= (n-s)(n+s+1) \end{aligned}$$

$$\Rightarrow \boxed{a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)} a_s \qquad (s = 0, 1, 2, \ldots)} \quad (4)$$

This is called a recurrence relation. It gives each coefficient in terms of the second one preceding it, except for a_0 and a_1, which are left as arbitrary constants. We find successively

$$a_2 = -\frac{n(n+1)}{2!} a_0 \qquad\qquad a_3 = -\frac{(n-1)(n+2)}{3!} a_1$$

$$a_4 = -\frac{(n-2)(n+3)}{4\cdot 3} a_2 \qquad\qquad a_5 = -\frac{(n-3)(n+4)}{5\cdot 4} a_3$$

$$= \frac{(n-2)n(n+1)(n+3)}{4!} a_0 \qquad\qquad = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_1$$

and so on

By inserting these expressions for the Coefficients into (2) we obtain

$$y(x) = a_0 y_1(x) + a_1 y_2(x) \quad \text{----} \quad (5)$$

where

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)(n)(n+1)(n+3)}{4!}x^4 - \cdots \quad (6)$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \cdots \quad (7)$$

These Series Converges for $|x|<1$.

Since (6) Containes even powers of x only, while (7) Contains odd powers of x only, the ratio y_1/y_2 is not Constant, So that y_1 and y_2 not propotional and thus linearly independent Solutions. Hence (5) is a general Solution of (1) on the interval $-1<x<1$.

Note that $x = \mp 1$ are the points at which $1-x^2=0$, So that the Coefficients of the Standarized ODE are no longer analytic. So it should not Surprise you that we do not get a longer Convergence interval of (6) and (7), unless these Series terminate after finitly many powers. In that Case, the Series become polynomials.

Polynomial Solution. Legendre polynomials $P_n(x)$

The reduction of power series to polynomials is great advantage because then we have solutions for all x, without convergence restrictions.

For Legendre's equation this happens when the parameter n is a nonnegative integer because then the right side of (1) is zero for $s=n$.

$$a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)} a_s \quad (s=0,1,2,\ldots) \quad \ldots(1)$$

So that $a_{n+2}=0$, $a_{n+4}=0$, $a_{n+6}=0, \ldots$ Hence if n is even, $y_1(x)$ reduced to polynomial of degree n.

If n is odd, the same is true for $y_2(x)$.

These polynomials, multiplied by some constants, are called <u>Legendre polynomials</u> and are denoted by $P_n(x)$. The standard choice of such constants is done as follows. we choose the coefficients a_n of the highest power x^n as

$$a_n = \frac{(2n)!}{2^n (n!)^2} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} \quad (n \text{ +ve integer}) \quad (2)$$

(and $a_n = 1$ if $n=0$). Then we calutate the other coefficients from (1), solved for a_s in terms of a_{s+2}, that is

$$a_s = -\frac{(s+2)(s+1)}{(n-s)(n+s+1)} a_{s+2} \quad \cdots (3)$$

the choice (2) makes $p_n(1) = 1$ for every n; this motivates (2). From (3) with $s = n-2$ and (2) we obtain

$$a_{n-2} = -\frac{n(n-1)}{2(2n-1)} a_n = -\frac{n(n-1)}{2(2n-1)} \frac{(2n)!}{2^n (n!)^2}$$

Using $(2n)! = 2n(2n-1)(2n-2)!$ in the numerator and $n! = n(n-1)!$ and $n! = n(n-1)(n-2)!$ in the denominator, we obtain

$$a_{n-2} = -\frac{n(n-1)2n(2n-1)(2n-2)!}{2(2n-1)2^n n(n-1)! \, n(n-1)(n-2)!}$$

$n(n-1)2n(2n-1)$ cancels, so that we get

$$a_{n-2} = -\frac{(2n-2)!}{2^n (n-1)!(n-2)!}$$

Similarly,

$$a_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)} a_{n-2}$$

$$= \frac{(2n-4)!}{2^n 2! (n-2)! (n-4)!}$$

and so on, and in general, when $n-2m \geq 0$

$$a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n m! (n-m)! (n-2m)!} \quad \cdots (4)$$

The resulting Solution of Legendre's differential equation is called the Legendre polynomial of degree n and is denoted by $P_n(x)$

from (4) we obtain

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m}$$

$$= \frac{(2n)!}{2^n(n!)^2} x^n - \frac{(2n-2)!}{2^n 1!(n-1)!(n-2)!} x^{n-2} + - \cdots .$$

where $M = \frac{n}{2}$ or $\frac{n-1}{2}$, whichever is an integer.

The first few of these functions are

$P_0(x) = 1$

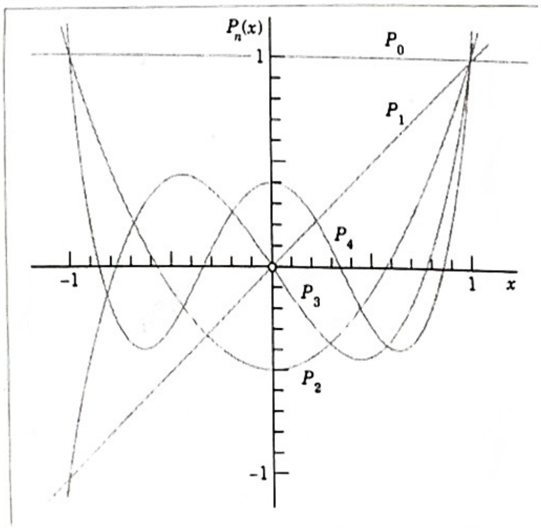
$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$P_5(x) = \frac{1}{8}(36x^5 - 70x^3 + 15x)$



Legendre Polynomials

4

Legendre's Equation

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Extended Power Series Method: Frobenius Method

Theorem 1: (Frobenius Method)

Let $b(x)$ and $c(x)$ be any functions that are analytic at $x=0$. Then the ODE

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0 \qquad \text{---(1)}$$

has at least one Solution that can be represented in the form

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r (a_0 + a_1 x + a_2 x^2 + \cdots) \quad (a_0 \neq 0) \qquad \text{---(2)}$$

where the exponent r may be any (real or complex) number (and r is chosen so that $a_0 \neq 0$).

The ODE (1) also has a second Solution (such that these two Solutions are linearly independent) that may be similar to (2) (with a different r and diffrent Coefficients) or may Contain a logarithmic term.

For example, Bessel's equation

$$y'' + \frac{1}{x} y' + \left(\frac{x^2 - \nu^2}{x^2}\right) y = 0 \qquad (\nu \text{ a parameter})$$

is of the form (1) with $b(x) = 1$ and $c(x) = x^2 - \nu^2$ analytic at $x=0$, so that the theorem applies. This ODE Could not be handled in full generality by the power Series method.

Regular and Singular Points

The following terms are practical and Commonly used. A regular point of the ODE

$$y'' + p(x)y' + q(x)y = 0$$

is a point x_0 at which the coefficients p and q are analytic. Similarly, a regular point of the ODE

$$\tilde{h}(x)y'' + \tilde{p}(x)y' + \tilde{q}(x)y = 0$$

is an x_0 at which $\tilde{h}, \tilde{p}, \tilde{q}$ are analytic and $\tilde{h}(x_0) \neq 0$ (so what we can divide by $\tilde{h}$ and get the previous standard form). Then the power series method can be applied. If x_0 is not a regular point, it is called a <u>singular point</u>.

<u>Example</u>

$$2x^2y'' + 3xy' - (x^2+1)y = 0$$

$$y'' + \frac{3x}{2x^2}y' - \frac{x^2+1}{2x^2}y = 0$$

$$y'' + P(x)y' + Q(x)y = 0$$

$$P(x) = \frac{3}{2x} \qquad Q(x) = -\frac{x^2+1}{2x^2}$$

If $P(0)$ or $Q(0)$ is undefined $\rightarrow$ $x=0$ is a singular point.

There are two types of singular point

① regular singular point (we can use Frob-meth.)

② Irregular singular point (we cannont use Frob. meth)

Now, how we can test the singular point. Rewrite $P(x)$ and $Q(x)$ as

$$P(x) = \frac{3}{2x} = \frac{\overset{p(x)}{(3/2)}}{x} \quad \text{and} \quad Q(x) = -\frac{x^2+1}{2x^2} = \frac{\overset{q(x)}{\left(-\frac{x^2+1}{2}\right)}}{x^2}$$

if $p(0)$ and $q(0)$ is defined we say $x=0$ is regular singular point $\Rightarrow$ we can use Frobenius method.

Summary

Suppose $y'' + P(x)y' + Q(x)y = 0$ has $P(0)$ or $Q(0)$ undefined (i.e. singularity)

Then define

$$p(x) = xP(x) \quad , \quad q(x) = x^2 Q(x)$$

If $p_0 = p(0)$, $q_0 = q(0)$ exist , then it is regular singular point and we have

$$r(r-1) + p_0 r + q_0 = 0 \qquad \text{(indicial equation)}$$

Indicial Equation, Indicating the Form of Solutions

we shall now explain the Frobenius method for solving (1). Multiplication of (1) by x^2 gives the more convention form

$$x^2y'' + xb(x)y' + c(x)y = 0 \qquad \text{---- (1')}$$

we first expand $b(x)$ and $c(x)$ in power series

$$b(x) = p_0 + p_1 x + p_2 x^2 + \cdots , \qquad c(x) = q_0 + q_1 x + q_2 x^2 + \cdots$$

or we do nothing if $b(x)$ and $c(x)$ are polynomials. Then we differentiate (2) term by term, finding

$$y'(x) = \sum_{m=0}^{\infty} (m+r)a_m x^{m+r-1} = x^{r-1}\left[ra_0 + (r+1)a_1 x + \cdots\right]$$

$$y''(x) = \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r-2}$$

$$= x^{r-2}\left[r(r-1)a_0 + (r+1)ra_1 x + \cdots\right]$$

By inserting all these series into (1') we obtain

$$x^r[r(r-1)a_0+\cdots]+(p_0+p_1x+\cdots)x^r(ra_0+\cdots)$$
$$+(q_0+q_1x+\cdots)x^r(a_0+a_1x+\cdots)=0 \quad ---(3)$$

we now equate the sum of the coefficients of each x^r, x^{r+1}, x^{r+2}, ... to zero. This yields a system of equations involving the unknown coefficients a_m. The smallest power is x^r and the corresponding equation is

$$[r(r-1)+p_0r+q_0]a_0=0$$

Since by assumption $a_0\neq 0$, the expression in the brackets $[\cdots]$ must be zero. This gives

$$r(r-1)+p_0r+q_0=0 \quad --- (4)$$

This important quadratic equation is called the <u>indicial equation</u> of the ODE (1)

The Frobenius method yields a basis of solutions. One of the two solutions will always be of the form (2), where r is a root of (4). The other solution will be of a form indicated by the indicial equation

<u>Theorem 2: (Frobenius method. Basis of Solutions. Three Cases)</u>

Suppose that the ODE (1) satisfies the assumptions in theorem 1. Let r_1 and r_2 be the roots of the indicial equation (4). Then we have the following three cases.

Case 1. Distinct Roots Not Differing by an Integer

A basis is

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots) \quad \text{---}(5)$$

and

$$y_2(x) = x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots) \quad \text{----}(6)$$

with Coefficients obtained successively from (3) with $r = r_1$ and $r = r_2$, respectively.

Case 2. Double Roots $r_1 = r_2 = r$. A basis is

$$y_1(x) = x^{r}(a_0 + a_1 x + a_2 x^2 + \cdots) \quad \text{---}(7)$$

$$\left[r = \tfrac{1}{2}(1 - p_0)\right]$$

(of the same general form as before) and

$$y_2(x) = y_1 \ln x + x^{r}(A_1 x + A_2 x^2 + \cdots) \quad (x > 0) \quad \text{---}(8)$$

Case 3. Roots Differing by an Integer. A basis is

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots) \quad \text{---}(9)$$

(of the same form as before) and

$$y_2(x) = k y_1(x) \ln x + x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots) \quad \text{---}(10)$$

where the roots are so denoted that $r_1 - r_2 > 0$ and k may turn out to be zero.

Example Solve $2x^2y'' + 3xy' - (x^2+1)y = 0$

Solution

$$2y'' + \frac{3x}{x^2}y' - \frac{x^2+1}{x^2}y = 0$$

$$2y'' + \frac{3}{x}y' - \frac{x^2+1}{x^2}y = 0$$

let $y = x^r \sum_{n=0}^{\infty} a_n x^n \quad (a_0 \neq 0)$

$$= \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$\Rightarrow y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$\Rightarrow y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

we have $2x^2y'' + 3xy' - (x^2+1)y = 0$

$$\Rightarrow 2x^2y'' + 3xy' - x^2y - y = 0$$

$$\Rightarrow 2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + 3x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$- x^2\sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} 2(n+r-1)(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r)a_n x^{n+r}$$

$$+ \sum_{n=0}^{\infty} -a_n x^{n+r+2} + \sum_{n=0}^{\infty} -a_n x^{n+r} = 0$$

Here put $n-2$ instead of n

$$\Rightarrow \sum_{n=0}^{\infty} 2(n+r-1)(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r)a_n x^{n+r} + \sum_{n=2}^{\infty} -a_{n-2} x^{n+r}$$

$$+ \sum_{n=0}^{\infty} -a_n x^{n+r} = 0$$

Now,

$$\left(2(r-1)ra_0 + 3ra_0 - a_0\right)x^r + \left(2r(r+1)a_1 + 3(r+1)a_1 - a_1\right)x^{1+r}$$
$$+ \sum_{n=2}^{\infty}\left(\left(2(n+r-1)(n+r) + 3(n+r) - 1\right)a_n - a_{n-2}\right)x^{n+r} = 0$$

$$\Rightarrow \quad 2(r-1)ra_0 + 3ra_0 - a_0 = 0$$

$$\Rightarrow \quad \left(2(r-1)r + 3r - 1\right)a_0 = 0 \qquad \text{but } a_0 \neq 0$$

$$\Rightarrow \quad 2(r-1)r + 3r - 1 = 0$$

$$\Rightarrow \quad 2r^2 - 2r + 3r - 1 = 0$$

$$\Rightarrow \quad 2r^2 + r - 1 = 0 \qquad \Rightarrow (2r-1)(r+1) = 0$$

$$\Rightarrow \quad r = \frac{1}{2} \text{ or } r = -1$$

Now, take $r = \frac{1}{2}$ then the coeff. of x^{r+1} gives

$$\left(2r(r+1) + 3(r+1) - 1\right)a_1 = 0$$

$$\Rightarrow \left(2r^2 + 2r + 3r + 3 - 1\right)a_1 = 0$$

$$\Rightarrow \left(2r^2 + 5r + 2\right)a_1 = 0$$

Since $2r^2 + 5r + 2 \neq 0 \Rightarrow \boxed{a_1 = 0}$

and we know that

$$\left(2(n+r-1)(n+r) + 3(n+r) - 1\right)a_n - a_{n-2} = 0$$

putting $r = \frac{1}{2}$ gives

$$\left(2\left(n-\tfrac{1}{2}\right)\left(n+\tfrac{1}{2}\right) + 3\left(n+\tfrac{1}{2}\right) - 1\right)a_n = a_{n-2}$$

$$\left(2n^2 - \frac{1}{2} + 3n + \frac{3}{2} - 1\right)a_n = a_{n-2}$$

$$\Rightarrow \quad a_n = \frac{1}{2n^2 + 3n}\, a_{n-2} \qquad (n \geq 2)$$

So, for $r=\frac{1}{2}$ we have

$$a_1 = 0 \quad , \quad a_n = \frac{1}{2n^2+3n} a_{n-2} = \frac{1}{n(2n+3)} a_{n-2}$$

that means

$$a_2 = \frac{1}{2\cdot 7} a_0 = \frac{1}{14} a_0 \qquad a_3 = \frac{1}{3\cdot 9} a_1 = 0$$

$$a_4 = \frac{1}{4(11)} a_2 = \frac{1}{44\cdot 40} a_0 \qquad a_5 = 0$$

$$a_6 = \frac{1}{6(15)} a_4 = \frac{1}{(90)(44)(14)} a_0 \qquad a_7 = 0$$

$$\therefore y = x^{1/2}\left(a_0 + \frac{1}{14} a_0 x^2 + \frac{1}{(44)40} a_0 x^4 + \frac{1}{(90)(44)(14)} a_0 x^6 + \cdots\right)$$

Example Solve $x^2y'' + 5xy' + (3-x)y = 0$

let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

Indicial eq. is $r(r-1) + P_0 r + q_0 = 0$

$$y'' + \frac{5x}{x^2}y' + \frac{3-x}{x^2}y = 0$$

$$P(x) = \frac{5x}{x^2} = \frac{5}{x} \quad , \quad Q(x) = \frac{3-x}{x^2}$$

$$P(x) = x\,P(x) = x\frac{5}{x} = 5 \quad , \quad q(x) = x^2 Q(x) = x^2 \frac{3-x}{x^2} = 3-x$$

$$P_0 = P(0) = 5 \qquad q_0 = q(0) = 3-0 = 3$$

$\therefore$ The indicial equation be

$$r(r-1) + 5r + 3 = 0$$

$$r^2 - r + 5r + 3 = 0$$

$$r^2 + 4r + 3 = 0$$

$$(r+3)(r+1) = 0$$

$$\Rightarrow \quad r = -3 \quad \text{or} \quad r = -1$$

For $r = -1 \Rightarrow y = \sum_{n=0}^{\infty} a_n x^{n-1}$

$$x^2 \sum_{n=0}^{\infty} (n-1)(n-2)a_n x^{n-3} + 5x \sum_{n=0}^{\infty} (n-1)a_n x^{n-2} + 3\sum_{n=0}^{\infty} a_n x^{n-1} - x\sum_{n=0}^{\infty} a_n x^{n-1} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n-1)(n-2)a_n x^{n-1} + \sum_{n=0}^{\infty} 5(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} 3a_n x^{n-1} + \sum_{n=0}^{\infty} -a_n x^{n} = 0$$

we put $n \to n-1$

$n-1 = 0$

$n = 1$

$$\sum_{n=1}^{\infty} -a_{n-1} x^{n-1}$$

$$\Rightarrow \underbrace{(-1)(-2)a_0x^{-1} + (-5)a_0x^{-1} + 3a_0x^{-1}}_{\text{Zero}} + \sum_{n=1}^{\infty}\Big[(n-1)(n-2)a_n$$

$$+5(n-1)a_n + 3a_n - a_{n-1}\Big]x^{n-1} = 0$$

$$\therefore\quad a_n\left[(n-1)(n-2) + 5(n-1) + 3\right] - a_{n-1} = 0$$

$$a_n\left[n^2 - 3n + 2 + 5n - 5 + 3\right] - a_{n-1} = 0$$

$$a_n(n^2 + 2n) = a_{n-1}$$

$$a_n = \frac{1}{n(n+2)}\,a_{n-1} \qquad \textit{for } n \geq 1$$

$$n=1 \rightarrow \quad a_1 = \frac{1}{1\cdot 3}a_0$$

$$n=2 \rightarrow \quad a_2 = \frac{1}{2\cdot 4}a_1 = \frac{1}{2\cdot 4}\,\frac{1}{1\cdot 3}\,a_0 = \frac{1}{(2\cdot 1)(4\cdot 3)}\,a_0$$

$$n=3 \rightarrow \quad a_3 = \frac{1}{3\cdot 5}a_2 = \frac{1}{(3\cdot 2\cdot 1)(5\cdot 4\cdot 3)}\,a_0 = \frac{2\cdot 1}{3!\;5!}\,a_0$$

and the nth term

$$a_n = \frac{2}{n!\,(n+2)!}\,a_0$$

$$\therefore\; y = \sum_{n=0}^{\infty}\frac{2}{n!\,(n+2)!}\,a_0\,x^{n-1} = a_0\sum_{n=0}^{\infty}\frac{2}{n!\,(n+2)!}\,x^{n-1}$$

H.W. Solve $x^2y'' + 5xy' + (x^2 - 5)y = 0$

Frobenius Method

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6

Bessel's Equation

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The Series Solution of Bessel’s Equation

One of the most important of all variable-coefficient differential equations is

$$x^2y'' + xy' + (\lambda^2x^2 - \nu^2)y = 0 \quad \cdots (1)$$

which is known as Bessel's equation of order ν with a parameter λ.

To simplify equation (1) we will use $\boxed{t = \lambda x}$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \lambda \frac{dy}{dt}$$

and $$\frac{d^2y}{dx^2} = \frac{dy/dx}{dt} \cdot \frac{dt}{dx} = \lambda^2 \frac{d^2y}{dt^2}$$

$$\Rightarrow \quad t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2)y = 0 \quad \cdots (2)$$

which is known simply as Bessel's equation of order ν.

$$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+(t^2-\nu^2)y=0$$

Now, from the last Section (Frobenius Method) we have

$$P(t)=\frac{1}{t} \quad \text{and} \quad Q(t)=\frac{t^2-\nu^2}{t^2}$$

$$\Rightarrow p=tP(t)=1 \qquad \text{and}\; q=t^2Q(t)=t^2-\nu^2$$

$$\Rightarrow p_0=p(0)=1 \qquad \text{and}\; q_0=q(0)=-\nu^2$$

using the indicial equation

$$r(r-1)+p_0r+q_0=0$$

$$\Rightarrow r^2-r+(1)r+(-\nu^2)=0$$

$$\Rightarrow r^2-\nu^2=0$$

$$\Rightarrow r_1=\nu \quad \text{and} \quad r_2=-\nu$$

it is clear that the origin is regular Singular point and here we can Use the Frobenius method

Let $y_\nu = \sum_{k=0}^{\infty} a_k t^{\nu+k}$

$\Rightarrow y'_\nu = \sum_{k=0}^{\infty} (\nu+k) a_k t^{\nu+k-1}$ & $y''_\nu = \sum_{k=0}^{\infty} (\nu+k)(\nu+k-1) a_k t^{\nu+k-2}$

where $r = \nu$ $(\nu \geq 0)$. Substituting these in the original equation

$$t^2 \sum_{k=0}^{\infty} (\nu+k)(\nu+k-1) a_k t^{\nu+k-2} + t \sum_{k=0}^{\infty} (\nu+k) a_k t^{\nu+k-1} + (t^2 - \nu^2) \sum_{k=0}^{\infty} a_k t^{\nu+k} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} a_k \left[(\nu+k)(\nu+k-1) + (\nu+k) - \nu^2 \right] t^{\nu+k} + \sum_{k=0}^{\infty} a_k t^{\nu+k+2} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} a_k k(2\nu+k) t^{\nu+k} + \sum_{k=0}^{\infty} a_k t^{\nu+k+2} = 0$$

$(\nu+k)(\nu+k-1) + (\nu+k) - \nu^2$
$= (\nu+k)(\nu+k-1+1) - \nu^2$
$= (\nu+k)^2 - \nu^2$
$= \nu^2 + 2\nu k + k^2 - \nu^2$
$= 2\nu k + k^2 = k(2\nu+k)$

here we Sub. $k-2$ instead of each k

$$\Rightarrow \sum_{k=0}^{\infty} a_k k(2v+k)t^{v+k} + \sum_{k=2}^{\infty} a_{k-2} t^{v+k} = 0$$

$$\Rightarrow a_1(2v+1)t^{v+1} + \sum_{k=2}^{\infty} a_k k(2v+k)t^{v+k} + \sum_{k=2}^{\infty} a_{k-2} t^{v+k} = 0$$

$$\Rightarrow a_1(2v+1)t^{v+1} + \sum_{k=2}^{\infty} [k(2v+k)a_k + a_{k-2}] t^{v+k} = 0$$

Now, $a_1(2v+1) = 0$ ---- (3)

and $k(2v+k)a_k + a_{k-2} = 0$ ---- (4)

from (3) and the restriction that $v \geq 0$ it is clear that $\boxed{a_1 = 0}$

and from ④ we get

$$a_k = -\frac{a_{k-2}}{k(2\nu+k)} \qquad k = 2, 3, 4, \dots \qquad \text{---} (5)$$

Since $a_1 = 0 \not\Rightarrow a_3 = a_5 = a_7 = \dots = a_{2m+1} = 0$

and

$$a_2 = -\frac{a_0}{2(2\nu+2)} = -\frac{a_0}{2^2 \cdot 1!\,(\nu+1)}$$

$$a_4 = -\frac{a_2}{4(2\nu+4)} = -\frac{a_2}{2^2 \cdot 2\,(\nu+2)} = \frac{a_0}{2^4\, 2!\,(\nu+2)(\nu+1)}$$

$$a_6 = -\frac{a_4}{6(2\nu+6)} = -\frac{a_4}{2^2 \cdot 3\,(\nu+3)} = -\frac{a_0}{2^6 \cdot 3!\,(\nu+3)(\nu+2)(\nu+1)}$$

in general,

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} \cdot m!\,(\nu+m)(\nu+m-1)\cdots(\nu+3)(\nu+2)(\nu+1)}, \quad m = 1, 2, 3 \dots$$

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} \cdot m!\,(\nu+m)(\nu+m-1)\cdots(\nu+3)(\nu+2)(\nu+1)}, \quad m=1,2,3\ldots$$

Now, a_{2m} is the coefficient of $t^{\nu+2m}$ in the series of y_ν.
Hence it would probably be convenient if a_{2m} contained the factor $2^{\nu+2m}$, so we will multiply a_{2m} by $\frac{2^\nu}{2^\nu}$

$$\Rightarrow a_{2m} = \frac{(-1)^m}{2^{2m+\nu} \cdot m!\,(\nu+m)(\nu+m-1)\cdots(\nu+3)(\nu+2)(\nu+1)}\,(2^\nu a_0) \quad \text{--- (6)}$$

Here we will simplify the denominator using the following note

$$a_{2m} = \frac{(-1)^m}{2^{2m+v} \cdot m! \boxed{(v+m)(v+m-1)\cdots(v+3)(v+2)(v+1)}} (2^v a_0) \quad \text{--⑥}$$

Note:

$$(v+m)! = (v+m)(v+m-1)(v+m-2)\cdots(v+m-(m-1))\cdot(v+m-m)\cdot$$

$$(v+m-(m+1))\cdots\cdots 2\cdot 1$$

$$= (v+m)(v+m-1)(v+m-2)\cdots\cdots(v+1)\underbrace{v(v-1)\cdots 2\cdot 1}_{v!}$$

$$= (v+m)(v+m-1)(v+m-2)\cdots\cdots(v+1)v!$$

Now $\dfrac{(v+m)!}{v!} = (v+m)(v+m-1)\cdots\cdots(v+1)$

and that means $\dfrac{\Gamma(v+m+1)}{\Gamma(v+1)} = (v+m)(v+m-1)\cdots\cdots(v+1)$

Subsituting this in ⑥

$$a_{2m} = \frac{(-1)^m}{2^{\nu+2m}\, m!\, \Gamma(\nu+m+1)} \left[2^{\nu}\, \Gamma(\nu+1)\, a_0\right] \quad \cdots ⑦$$

Since a_0 is an arbitrary Constant, we Can Choose it as

$$a_0 = \frac{1}{2^{\nu}\, \Gamma(\nu+1)}$$

$$\Rightarrow \quad a_{2m} = \frac{(-1)^m}{2^{\nu+2m}\, m!\, \Gamma(\nu+m+1)} \quad , \quad m = 0, 1, 2, \ldots$$

with a_k thus determined for even values of k and $a_k = 0$ for odd values of k.

For each ν $(\nu \geq 0)$, the function y_ν is Called a Bessel function of the first Kind of order ν and denoted by the Symbole $J_\nu(t)$

$$J_v(t)=\sum_{m=0}^{\infty}\frac{(-1)^m t^{v+2m}}{2^{v+2m}\, m!\, \Gamma(v+m+1)}$$

for $r=-v$ we can prove that

$$J_{-v}(t)=\sum_{m=0}^{\infty}\frac{(-1)^m t^{-v+2m}}{2^{-v+2m}\, m!\, \Gamma(-v+m+1)}$$

provided that gamma function appearing in the denominater is defined, and this is true for the non integer value of v. Moreover, since $J_{-v}(t)$ containes negative power of t while $J_v(t)$ does not, that means that $J_v(t)$ and $J_{-v}(t)$ are linearly independent and

$$\boxed{y(t)=c_1 J_v(t)+c_2 J_{-v}(t)}$$

v is non integer

If ν is an integer, say $\nu = n$, then $J_n(t)$ and $J_{-n}(t)$ are linearly dependent. (without prove) we will take

$$Y_\nu(t) = \frac{\cos\nu\pi\, J_\nu(t) - J_{-\nu}(t)}{\sin\nu\pi}$$

and the Solution In this Case will be as follows

$$y(x) = c_1 J_\nu(t) + c_2 Y_\nu(t)$$

Theorem: For all values ν, a Complete Solution of Bessel's equation of order ν with parameter λ,

$$x^2y''+xy'+(\lambda^2x^2-\nu^2)y=0$$

Can be written as

$$y(x)=c_1J_\nu(\lambda x)+c_2Y_\nu(\lambda x)$$

where

$$Y_\nu(x)=\frac{\cos\nu\pi\, J_\nu(\lambda x)-J_{-\nu}(\lambda x)}{\sin\nu\pi}$$

If ν is not an Integer, a Complete Solution Can also be written

$$y(x)=c_1J_\nu(\lambda x)+c_2J_{-\nu}(\lambda x)$$

we Called J_ν is Bessel's of first kind, Y_ν is Bessel's function of 2nd kind.

Theorem

Example Find the general Solution of

$$x^2y''+xy'+(64x^2-16)y=0$$

then find the Bessel's function of first kind.

Solution

$$x^2y''+xy'+(\lambda^2x^2-16)y=0$$

$$\Rightarrow \lambda^2=64 \Rightarrow \lambda=8$$

and $\nu^2=16 \Rightarrow \nu=4$

$\therefore \quad y = C_1 J_4(8x)+C_2 Y_4(8x)$

and $$J_4(8x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(k+5)}\left(\frac{8x}{2}\right)^{4+2k}$$

Examples

Example Find the general Solution of

$$x^2y''+xy'+(x^2-4)y=0$$

then find the Bessel's function of first kind

$$\lambda^2=1 \Rightarrow \lambda=1$$

$$\nu^2=4 \Rightarrow \nu=2$$

$$\therefore y=C_1 J_2(x)+C_2 Y_2(x)$$

and

$$J_2(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(k+3)}\left(\frac{x}{2}\right)^{2+2k}$$

Examples

Thank You

Any questions?

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7

Bessel's Functions

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Bessel's Functions

$$J_n = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

if $n=0$

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}$$

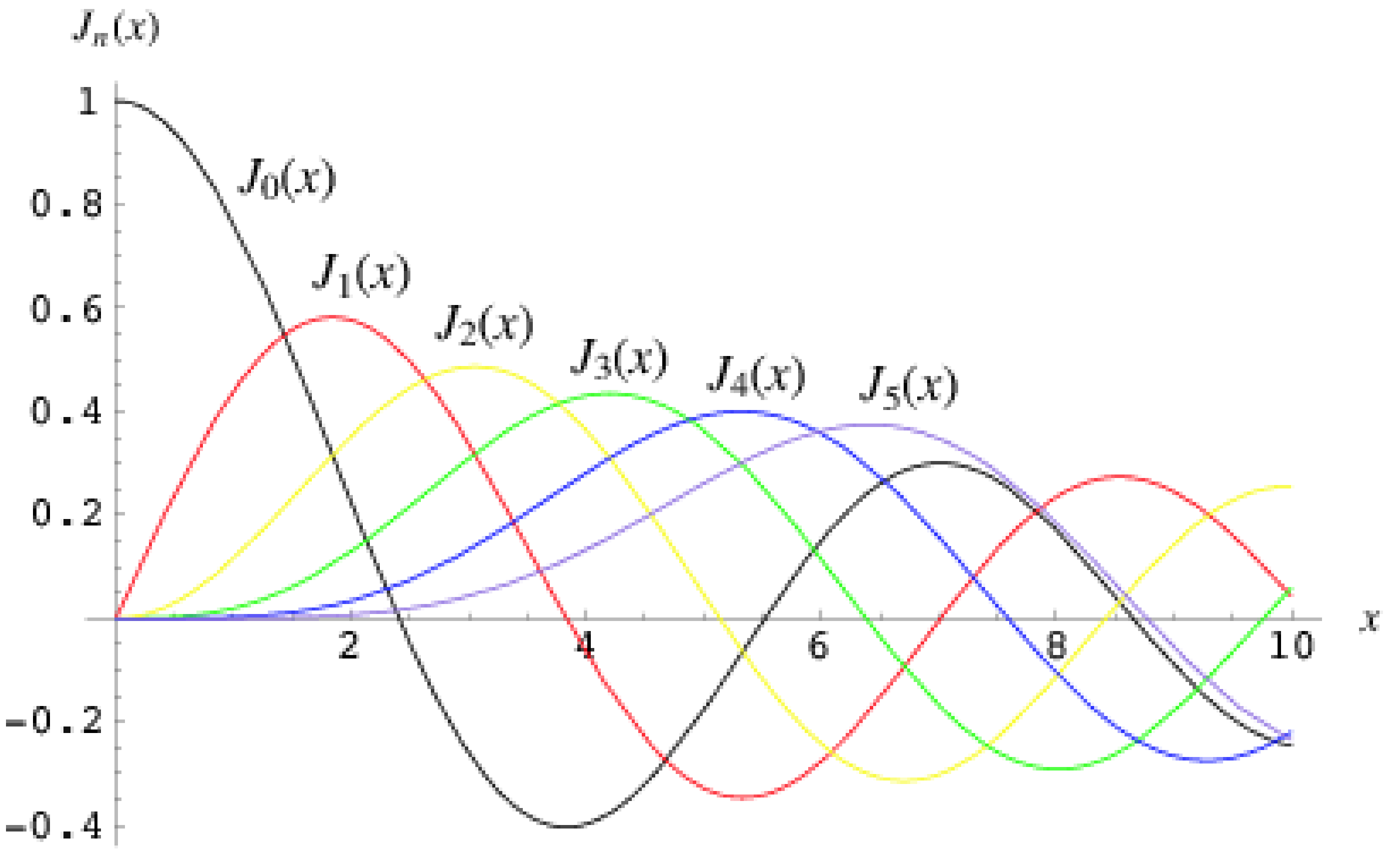
$$\Rightarrow J_0 = 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots$$

$$= 1 - \frac{x^2}{2^2 \cdot (1!)^2} + \frac{1}{2^4 \cdot (2!)^2} x^4 - \frac{x^6}{2^6 (3!)^2} + \cdots$$

if $n=1$

$$J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+2)} \left(\frac{x}{2}\right)^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+1)!} \left(\frac{x}{2}\right)^{2k+1}$$

$$\Rightarrow J_1(x) = \frac{x}{2} - \frac{x^3}{2^3\,1!\,2!} + \frac{x^5}{2^5\,2!\,3!} - \frac{x^7}{2^7\,3!\,4!} + \cdots$$


$J_n(x)$
1
0.8
0.6
0.4
0.2
-0.2
-0.4
$J_0(x)$
$J_1(x)$
$J_2(x)$
$J_3(x)$
$J_4(x)$
$J_5(x)$
2
4
6
8
10
x

Properties of Bessel's Functions

1) $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\,\sin x$

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2k}$$

$$\Rightarrow J_{\frac{1}{2}}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\frac{3}{2})}\left(\frac{x}{2}\right)^{\frac{1}{2}+2k} = \sqrt{x}\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+\frac{1}{2}}\,k!\,\Gamma(k+\frac{3}{2})} * \frac{\sqrt{2x}}{\sqrt{2x}}$$

$$= \sqrt{\frac{2}{x}}\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1}\,k!\,\Gamma(k+\frac{3}{2})} = \sqrt{\frac{2}{x}}\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^k\,2^{k+1}\,k!\,\Gamma(k+\frac{3}{2})}$$

The denominator can be written as a product AB, where

$$A = 2^k k! = 2^k\,\underbrace{(k(k-1)(k-2)(k-3)\cdots 2\cdot 1)}_{k\text{-terms}}$$

$$= 2k(2k-2)(2k-4)\cdots 4\cdot 2$$

$$B = 2^{k+1}\,\Gamma(k+\tfrac{3}{2}) = 2^{k+1}\,\underbrace{(k+\tfrac{1}{2})(k-\tfrac{1}{2})\cdots\tfrac{3}{2}\cdot\tfrac{1}{2}}_{k+1\text{ terms}}\,\Gamma(\tfrac{1}{2})$$

$$= (2k+1)(2k-1)\cdots 3\cdot 1\;\Gamma(\tfrac{1}{2})$$

$$= (2k+1)(2k-1)\cdots 3\cdot 1\cdot\sqrt{\pi}$$

$$A = 2^k k! = 2^k \underbrace{(k(k-1)(k-2)(k-3)\cdots 2\cdot 1)}_{k\text{-terms}}$$
$$= 2k(2k-2)(2k-4)\cdots\cdots 4\cdot 2$$

$$B = 2^{k+1}\Gamma\left(k+\tfrac{3}{2}\right) = 2^{k+1}\underbrace{\left(k+\tfrac{1}{2}\right)\left(k-\tfrac{1}{2}\right)\cdots\cdots\tfrac{3}{2}\cdot\tfrac{1}{2}}_{k+1\text{ terms}}\Gamma\left(\tfrac{1}{2}\right)$$
$$= (2k+1)(2k-1)\cdots\cdots 3\cdot 1\;\Gamma\left(\tfrac{1}{2}\right)$$
$$= (2k+1)(2k-1)\cdots 3\cdot 1\cdot\sqrt{\pi}$$

Now $AB = (2k+1)2k(2k-1)(2k-2)\cdots\cdots 3\cdot 2\cdot 1\sqrt{\pi}$
$$= (2k+1)!\sqrt{\pi}$$

Hence $J_{\frac{1}{2}}(x) = \sqrt{\dfrac{2}{x}}\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!\sqrt{\pi}} = \sqrt{\frac{2}{\pi x}}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$
$$= \sqrt{\frac{2}{\pi x}}\sin x$$

2) Similarly we can show that

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

3) For any integer n the Bessel function $J_n(x)$ and $J_{-n}(x)$ are linearly dependent, because

$$J_{-n}(x) = (-1)^n J_n(x) \qquad n = 1, 2, 3 \cdots$$

we know that,

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

$$\Rightarrow J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(-n+k+1)} \left(\frac{x}{2}\right)^{-n+2k}$$

The gamma functions in the Coefficients of the first n terms become infinite, the Coefficients become zero and the Summation Starts with $k=n$. Since in this Case

$$\Gamma(k-n+1) = (k-n)!$$

$$\therefore J_{-n}(x) = \sum_{k=n}^{\infty} \frac{(-1)^k x^{2k-n}}{2^{2k-n} k!\,(k-n)!}$$

$$\therefore\ J_{-n}(x) = \sum_{k=n}^{\infty} \frac{(-1)^k x^{2k-n}}{2^{2k-n} k!\,(k-n)!}$$

Put $k = n+s \Rightarrow s = k-n$

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{n+s} x^{2n+2s-n}}{2^{2n+2s-n}(n+s)!\,(n+s-n)!}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^{n+s} x^{2s+n}}{2^{2s+n} s!\,\Gamma(n+s+1)}$$

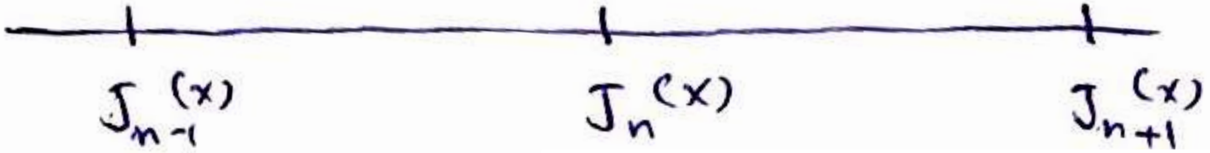
$$= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s+n}}{2^{2s+n} s!\,\Gamma(n+s+1)}$$

$$= (-1)^n J_n(x)$$

Recurrence Relations of Bessel’s Functions

$\boxed{\mathrm{I}}$ $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

$\boxed{\mathrm{II}}$ $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$



$\boxed{\text{I}}$ $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

$\boxed{\text{II}}$ $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

If we integrate both sides of I we get

$$\int x^n J_{n-1}(x)\,dx = x^n J_n(x) + c \quad \cdots \quad \boxed{\text{I}'}$$

and if we integrate both sides of II we get

$$\int x^{-n} J_{n+1}(x)\,dx = -x^{-n} J_n(x) + c \quad \cdots \quad \boxed{\text{II}'}$$

Example Evaluate $\int J_3(x)\,dx$

if we multiply and divide the integrand by x^2, we have

$$\int x^2 \left[x^{-2} J_3(x)\right] dx$$

$$\int x^{-n} J_{n+1}(x)\,dx = -x^{-n} J_n(x) + c \quad \cdots \quad \boxed{II'}$$

integrating by parts with

$u = x^2 \qquad dv = x^{-2} J_3(x)\,dx$

$du = 2x\,dx \qquad V = -x^{-2} J_2(x) \quad \text{--- using } \boxed{II'}_{n=2}$

$$\therefore \int J_3(x) = x^2\left(-x^{-2} J_2(x)\right) + \int \left(x^{-2} J_2(x)\right)(2x)\,dx$$

$$= -J_2(x) + 2\int x^{-1} J_2(x)\,dx$$

$$= -J_2(x) - 2x^{-1} J_1(x) + c \quad \text{--- using } \boxed{II'}_{n=1}$$

Example Evaluate $\int J_{-3}(x)\,dx$

$\because J_{-3}(x) = (-1)^3 J_3(x) \Rightarrow J_{-3}(x) = -J_3(x)$

$\Rightarrow J_{-3}(x) = J_2(x) + 2x^{-1} J_1(x) + c$ (By the previous example)

Example Evaluate $\int x^4 J_1(x)\,dx$

$$\int x^4 J_1(x)\,dx = \int x^2 (x^2 J_1)\,dx$$

$$\int x^n J_{n-1}(x)\,dx = x^n J_n(x) + c \quad \cdots \quad \boxed{I'}$$

$u = x^2$ $\qquad dv = x^2 J_1(x)\,dx$

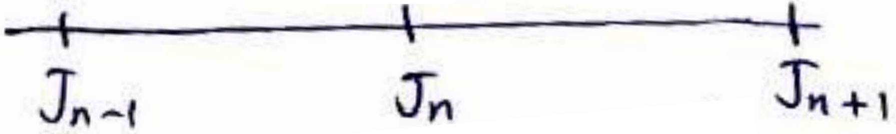
$du = 2x\,dx$ $\qquad v = x^2 J_2(x)$ $\qquad$ using $\boxed{I'}_{n=2}$

$$= x^2 (x^2 J_2(x)) - \int (x^2 J_2(x))(2x)\,dx$$

$$= x^4 J_2(x) - 2\int x^3 J_2(x)\,dx$$

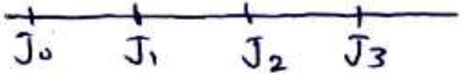
$$= x^4 J_2(x) - 2x^3 J_3(x)\,dx$$ $\qquad$ using $\boxed{I'}_{n=3}$

$\boxed{\text{III}}$ $$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

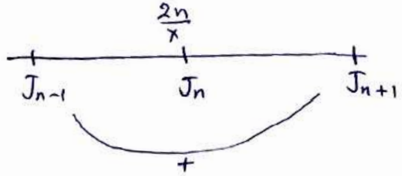


Example: Express $J_3(x)$ interms of $J_0(x)$ and $J_1(x)$

Using $\boxed{\text{III}}$ we get



$$J_{n+1} = \frac{2n}{x} J_n - J_{n-1}$$



Put $n=2$

$$J_3 = \frac{4}{x} J_2 - J_1 \quad \text{--- } (1)$$

Similarly put $n=1$

$$J_2 = \frac{2}{x} J_1 - J_0 \quad \text{--- } (2)$$

Put (2) in (1)

$$J_3 = \frac{4}{x}\left[\frac{2}{x} J_1 - J_0\right] - J_1 = \frac{8}{x} J_1 - \frac{4}{x} J_0 - J_1$$

$$= \frac{8-x}{x} J_1 - \frac{4}{x} J_0$$

Example: Express $J_{-\frac{5}{2}}$ interms of sin and Cos

Using $\boxed{\text{III}}$ weget

$$J_{n-1} = \frac{2n}{x} J_n - J_{n+1}$$

if $n-1 = -\frac{5}{2} \longrightarrow n = -\frac{3}{2}$

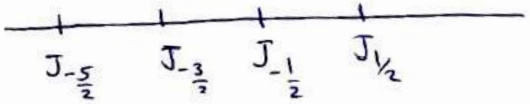
$$\therefore J_{-\frac{5}{2}} = \frac{2\left(\frac{-3}{2}\right)}{x} J_{-\frac{3}{2}} - J_{-\frac{1}{2}}$$

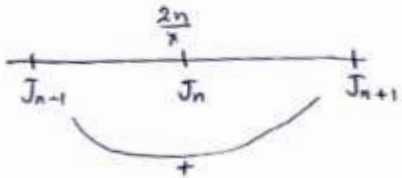
$$J_{-\frac{5}{2}} = -\frac{3}{x} \boxed{J_{-\frac{3}{2}}} - J_{-\frac{1}{2}} \quad ---- \quad (1)$$

if $n-1 = -\frac{3}{2} \longrightarrow n = -\frac{1}{2}$

$$\therefore J_{-\frac{3}{2}} = \frac{2\left(-\frac{1}{2}\right)}{x} J_{-\frac{1}{2}} - J_{\frac{1}{2}}$$

$$\boxed{J_{-\frac{3}{2}} = -\frac{1}{x} J_{-\frac{1}{2}} - J_{\frac{1}{2}}} \quad --- \quad (2)$$





$\boxed{\text{III}}$ $\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$

Put ② in ①

$$J_{-\frac{5}{2}} = -\frac{3}{x}\left[-\frac{1}{x}J_{-\frac{1}{2}} - J_{\frac{1}{2}}\right] - J_{-\frac{1}{2}}$$

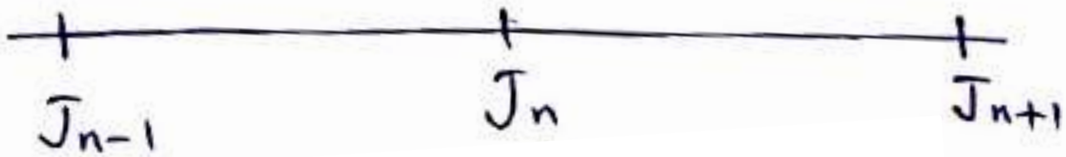
$$= \frac{3}{x^2}J_{-\frac{1}{2}} + \frac{3}{x}J_{\frac{1}{2}} - J_{-\frac{1}{2}}$$

$$= \frac{3-x^2}{x^2}J_{-\frac{1}{2}} + \frac{3}{x}J_{\frac{1}{2}}$$

$$= \frac{3-x^2}{x^2}\sqrt{\frac{2}{\pi x}}\cos x + \frac{3}{x}\sqrt{\frac{2}{\pi x}}\sin x$$

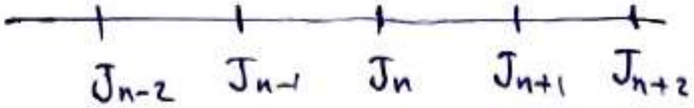
$\boxed{\mathrm{IV}}$

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

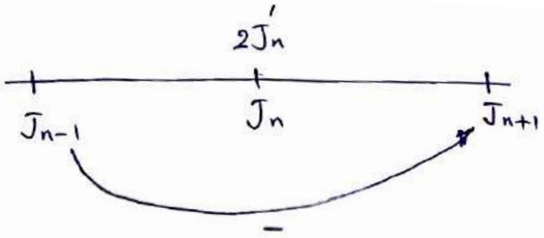


Example Prove that

$$J_n''(x) = \frac{1}{4}\left[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)\right]$$



Using $\boxed{IV}$ we have



$$J_n' = \frac{1}{2}\left[J_{n-1} - J_{n+1}\right]$$

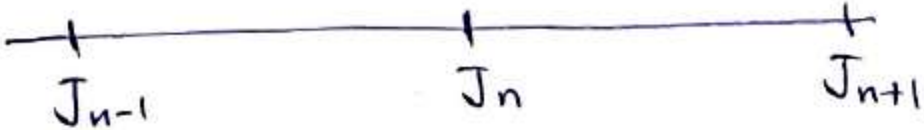
$$J_n'' = \frac{1}{2}\left[J_{n-1}' - J_{n+1}'\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}(J_{n-2} - J_n) - \frac{1}{2}(J_n - J_{n+2})\right]$$

$$= \frac{1}{4}\left[J_{n-2} - 2J_n + J_{n+2}\right]$$

$\boxed{\text{V}}$ $\quad x J_n' = n J_n - x J_{n+1}$

$\boxed{\text{VI}}$ $\quad x J_n' = x J_{n-1} - n J_n$



Example prove that $J_n''(x) = \left[\frac{n(n+1)}{x^2} - 1\right] J_n(x) - \frac{J_{n-1}(x)}{x}$

we know that $J_n(x)$ is a Solution of

$$x^2 y'' + x y' + (x^2 - n^2) y = 0$$

So $J_n(x)$ Satisfies this equation

$$x^2 J_n'' + x J_n' + (x^2 - n^2) J_n = 0$$

$$\boxed{\text{VI}} \quad x J_n' = x J_{n-1} - n J_n$$

$$\Rightarrow J_n'' = \frac{n^2 - x^2}{x^2} J_n - \frac{1}{x} \boxed{J_n'} \quad \text{---} \; (1)$$

From $\boxed{\text{VI}}$ we have

$$\boxed{J_n' = J_{n-1} - \frac{n}{x} J_n} \quad \text{----} \; (2)$$

Put ② in ①

$$J_n'' = \frac{n^2 - x^2}{x^2} J_n - \frac{1}{x}\left[J_{n-1} - \frac{n}{x} J_n\right]$$

$$= \left(\frac{n^2}{x^2} - 1\right) J_n - \frac{1}{x} J_{n-1} + \frac{n}{x^2} J_n$$

$$= \left(\frac{n^2}{x^2} + \frac{n}{x^2} - 1\right) J_n - \frac{1}{x} J_{n-1}$$

$$= \left(\frac{n(n+1)}{x^2} - 1\right) J_n - \frac{1}{x} J_{n-1} \quad \square$$

Thank You

Any questions?

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8

Examples on Bessel's Functions

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MATHEMATICAL PHYSICS II

Master Degree Class
Department of Astronomy and Space
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Example 1 Show that $J_{-\frac{1}{2}}(x) = J_{\frac{1}{2}}(x) \cot x$

Sol. we know that

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x \quad \text{and} \quad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

$$\therefore \quad \frac{J_{-\frac{1}{2}}(x)}{J_{\frac{1}{2}}(x)} = \frac{\cancel{\sqrt{\frac{2}{\pi x}}} \cos x}{\cancel{\sqrt{\frac{2}{\pi x}}} \sin x} = \cot x$$

$$\Rightarrow J_{-\frac{1}{2}}(x) = J_{\frac{1}{2}}(x) \cot x \quad \blacksquare$$

Example 2 Show that $J_{-\frac{3}{2}} = -\sqrt{\frac{2}{\pi x}}\left(\sin x + \frac{\cos x}{x}\right)$

Sol.

$\frac{2n}{x} \leftarrow n = -\frac{1}{2}$

$J_{-\frac{3}{2}} \quad J_{-\frac{1}{2}} \quad J_{\frac{1}{2}}$

$+$

$$J_{-\frac{3}{2}}(x) = -\frac{1}{x} J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x)$$

$$= -\frac{1}{x}\sqrt{\frac{2}{\pi x}}\cos x - \sqrt{\frac{2}{\pi x}}\sin x$$

$$= -\sqrt{\frac{2}{\pi x}}\left(\sin x + \frac{\cos x}{x}\right) \blacksquare$$

Example 3 prove that $J_0'(x) = -J_1(x)$

Sol. we know

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

take $n = 0$

$$\frac{d}{dx}\left[x^{0} J_0(x)\right] = -x^{0} J_1(x)$$

$$\Rightarrow \quad J_0'(x) = -J_1(x) \quad \blacksquare$$

Example 4 Prove that $\frac{d}{dx}[x J_1(x)] = x J_0(x)$

Sol. we know that

x^n $\quad \frac{d}{dx} x^n$

$J_{n-1} \quad J_n \quad J_{n+1}$

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

Put $n=1$

$$\frac{d}{dx}\left[x J_1(x)\right] = x J_0(x) \quad \blacksquare$$

Example 5 Show that $J_0''(x) = \frac{1}{2}[J_2(x) - J_0(x)]$

Sol. we know

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

Put $n=0$

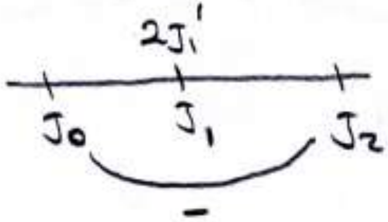
$$\Rightarrow \frac{d}{dx}\left[J_0(x)\right] = -J_1(x)$$

$$\Rightarrow \quad J_0'(x) = -J_1(x)$$

Differentiating with respect to x, we will have

$$J_0''(x) = -J_1'(x)$$

$$J_0''(x) = -\frac{1}{2}\left[J_0(x) - J_2(x)\right] \Leftarrow$$

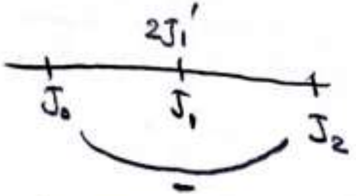


$$\Rightarrow \quad J_0''(x) = \frac{1}{2}\left[J_2(x) - J_0(x)\right] \blacksquare$$

Example 6 Show that $J_1''(x) = -J_1(x) + \frac{1}{x} J_2(x)$

Sol. we know

$$J_1'(x) = \frac{1}{2}\left[J_0(x) - J_2(x)\right]$$

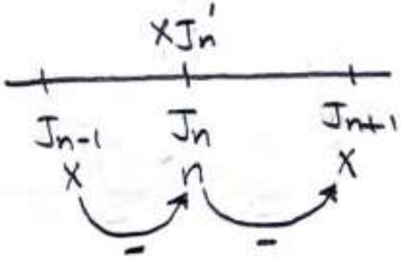


Differentiate both sides with respect to x

$$J_1''(x) = \frac{1}{2}\left[J_0'(x) - J_2'(x)\right] \quad \text{----} \ (1)$$

we know that $J_0'(x) = -J_1(x)$ ---- (2)

and $xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x)$



$$\Rightarrow J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)$$

Put $n=2$, we get

$$J_2'(x) = J_1(x) - \frac{2}{x} J_2(x) \quad \text{----} \ (3)$$

Put (2) and (3) in (1)

$$J_1''(x) = \frac{1}{2}\left[-J_1(x) - J_1(x) + \frac{2}{x} J_2(x)\right]$$

$$= \frac{1}{2}\left[-2J_1(x) + \frac{2}{x} J_2(x)\right]$$

$$= -J_1(x) + \frac{1}{x} J_2(x) \quad \blacksquare$$

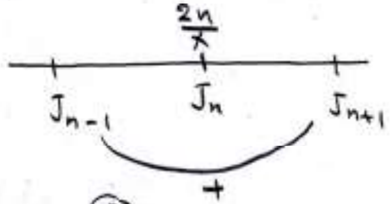
Example 7 prove that $\frac{d}{dx}\left[J_n^2(x)\right] = \frac{x}{2n}\left[J_{n-1}^2(x) - J_{n+1}^2(x)\right]$

Sol.

$$\text{LHS} = 2\,J_n(x)\,J_n'(x) \quad \text{.....} \ (1)$$

we know that

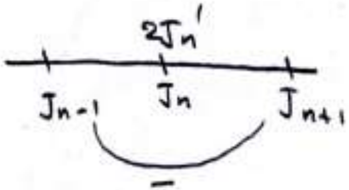
$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$



$$\Rightarrow J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right] \quad \text{.....} \ (2)$$

and

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right] \quad \text{..} \ (3)$$



Put (2) and (3) in (1)

$$\text{LHS} = 2\left[\frac{x}{2n}\left(J_{n-1}(x) + J_{n+1}(x)\right)\right]\frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

$$= \frac{x}{2n}\left[J_{n-1}^2(x) - \cancel{J_{n-1}(x)J_{n+1}(x)} + \cancel{J_{n+1}(x)J_{n-1}(x)} - J_{n+1}^2(x)\right]$$

$$= \frac{x}{2n}\left[J_{n-1}^2(x) - J_{n+1}^2(x)\right] \quad ■$$

Example 8 Prove that $\int J_0(x)\, J_1(x)\, dx = -\frac{1}{2}\left[J_0(x)\right]^2$

Sol. we know that

$$J_0'(x) = -J_1(x)$$

$$\Rightarrow \quad J_1(x) = -J_0'(x)$$

$$\therefore \int J_0(x)\, J_1(x)\, dx = -\int J_0(x)\, J_0'(x)\, dx$$

$$= -\frac{1}{2} J_0^2(x) \quad \blacksquare$$

Example: Prove that $\int_0^r x J_0(ax)\,dx = \frac{r}{a} J_1(ar)$

Sol. Let $ax = t \Rightarrow a\,dx = dt$

$\hookrightarrow x = \frac{t}{a}$ $\qquad \hookrightarrow dx = \frac{dt}{a}$

when $x = 0 \longrightarrow t = 0$

when $x = r \longrightarrow t = ar$

$$\therefore \int_0^r x J_0(ax)\,dx = \int_0^{ar} \frac{t}{a} J_0(t) \frac{dt}{a}$$

$$= \frac{1}{a^2} \int_0^{ar} t J_0(t)\,dt$$

$$\int x^n J_{n-1}(x)\,dx = x^n J_n(x)$$

$$= \frac{1}{a^2} \Big[t J_1(t) \Big]_0^{ar} = \frac{1}{a^2} \Big[ar J_1(ar) \Big] = \frac{r}{a} J_1(ar) \blacksquare$$

Equations Reducible to Bessel's Equations

In differential Calculus, we come across such differential equations which can be easily reduced to Bessel's equation and thus can be solved by the means of Bessel's function.

Recall that the differential equation of the form

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(k^2x^2-n^2)y=0$$

has the following solution

$$y=c_1J_n(kx)+c_2J_{-n}(kx)\ ,\ n \text{ is non-integer}$$

$$y=c_1J_n(kx)+c_2Y_n(kx)\ ,\ n \text{ is integer}$$

Example 1 Solve $x\frac{d^2y}{dx^2} + a\frac{dy}{dx} + k^2xy = 0$

Sol.

Putting $y = x^n z \Rightarrow \frac{dy}{dx} = x^n \frac{dz}{dx} + n x^{n-1} z$ ... (1)

and $\frac{d^2y}{dx^2} = x^n \frac{d^2z}{dx^2} + n x^{n-1} \frac{dz}{dx} + n x^{n-1} \frac{dz}{dx} + n(n-1) x^{n-2} z$

$\Rightarrow \frac{d^2y}{dx^2} = x^n \frac{d^2z}{dx^2} + 2n x^{n-1} \frac{dz}{dx} + n(n-1) x^{n-2} z$ ... (2)

Sub. (1) and (2) in the differential equation

$$x\left[x^n \frac{d^2z}{dx^2} + 2n x^{n-1} \frac{dz}{dx} + n(n-1) x^{n-2} z\right] + a\left[x^n \frac{dz}{dx} + n x^{n-1} z\right]$$

$$+ k^2 x (x^n z) = 0$$

$$\Rightarrow x^{n+1}\frac{d^2z}{dx^2}+2nx^n\frac{dz}{dx}+n(n-1)x^{n-1}z+ax^n\frac{dz}{dx}+anx^{n-1}z$$
$$+k^2x^{n+1}z=0$$

$$\Rightarrow x^{n+1}\frac{d^2z}{dx^2}+(2n+a)x^n\frac{dz}{dx}+(k^2x^2+n^2-n+an)x^{n-1}z=0$$

$$\Rightarrow x^{n+1}\frac{d^2z}{dx^2}+(2n+a)x^n\frac{dz}{dx}+(k^2x^2+\underbrace{n^2+(a-1)n}_{})x^{n-1}z=0$$

Dividing throughout by x^{n-1} and putting $2n+a=1$

$$\therefore\ a-1=2n \Rightarrow 1-a=-2n$$

$$\Rightarrow n^2+(a-1)n = n^2+(-2n)n$$
$$=n^2-2n^2$$
$$=-n^2$$

$$x^2\frac{d^2z}{dx^2}+x\frac{dz}{dx}+(k^2x^2-n^2)z=0$$

which is a Bessel's equation in z

The Solution is

$$Z = C_1 J_n(kx) + C_2 J_{-n}(kx)\ , \quad n \text{ is non-integer}$$

or $$Z = C_1 J_n(kx) + C_2 Y_n(kx), \quad n \text{ is integer}$$

$$\Rightarrow \quad y = x^n\left[C_1 J_n(kx) + C_2 J_{-n}(kx)\right] \quad n \text{ is non-integer}$$

or $$y = x^n\left[C_1 J_n(kx) + C_2 Y_n(kx)\right] \quad n \text{ is integer}$$

Example 2   Solve $x\dfrac{d^2y}{dx^2} + c\dfrac{dy}{dx} + k^2x^r y = 0$

Sol.

Putting $x = t^m \Rightarrow \dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx}$

$\hookrightarrow t = x^{1/m}$

$$\frac{dy}{dx} = \frac{dy}{dt}\left(\frac{1}{m} x^{\frac{1}{m}-1}\right)$$

$$= \frac{dy}{dt}\left(\frac{1}{m} x^{\frac{1}{m}} x^{-1}\right)$$

$$= \frac{dy}{dt}\left(\frac{1}{m} t\, t^{-m}\right)$$

$$= \frac{1}{m} t^{1-m} \frac{dy}{dt}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{1}{m} t^{1-m} \frac{dy}{dt}\right) \cdot \frac{1}{m} t^{1-m}$$

$$= \frac{1}{m^2} t^{2-2m} \frac{d^2y}{dt^2} + \frac{1-m}{m^2} t^{1-2m} \frac{dy}{dt}$$

Put the two above equations in the differential equatron

$$t^m\left[\frac{1}{m^2} t^{2-2m} \frac{d^2y}{dt^2} + \frac{1-m}{m^2} t^{1-2m} \frac{dy}{dt}\right] + c\left[\frac{1}{m} t^{1-m} \frac{dy}{dt}\right] + k^2 t^{mr} y = 0$$

$$\frac{1}{m^2} t^{2-m} \frac{d^2y}{dt^2} + \frac{1-m+cm}{m^2} t^{1-m} \frac{dy}{dt} + k^2 t^{mr} y = 0$$

Multipling throught by $\frac{m^2}{t^{1-m}}$, we get

$$t \frac{d^2y}{dt^2} + (1-m+cm) \frac{dy}{dt} + k^2 m^2 t^{mr+m-1} y = 0$$

let us set $mr+m-1 = 1 \Rightarrow m(r+1) = 2$

$$\Rightarrow m = \frac{2}{r+1}$$

and $a = 1 - m + cm = 1 - \frac{2}{r+1} + \frac{2c}{r+1}$

$$= \frac{r+1-2+2c}{r+1} = \frac{r+2c-1}{r+1}$$

$$\therefore\ t\frac{d^2y}{dx^2} + a\frac{dy}{dt} + (km)^2 t y = 0$$

which is similar to Example 1

Hence its solution is

$$y = t^n\left[c_1 J_n(km\,t) + c_2 J_{-n}(km\,t)\right],\ n \text{ non-integer}$$

$$y = t^n\left[c_1 J_n(kmt) + c_2 Y_n(kmt)\right],\ n \text{ integer}$$

$$\Rightarrow\ y = x^{n/m}\left[c_1 J_n(km\,x^{1/m}) + c_2 J_{-n}(km\,x^{1/m})\right]\ n \text{ noninteger}$$

$$y = x^{n/m}\left[c_1 J_n(km\,x^{1/m}) + c_2 Y_n(km\,x^{1/m})\right]\ n \text{ integer}$$

Thank You

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