**Definition 2.11:** Let (G, $*$) be a group and (H, $*$), (K, $*$) be two subgroups of G, then the product of H and K is the set:

$$H*K=\{h*k : h \in H, k \in K\}$$

**Notes 2.12:**

**(1)** H$*$H is write $H^2$

**(2)** If H = {a}, then H$*$K = a$*$K . If K = {b}, then H$*$K = H$*$b .

**(3)** $H \cup K \subseteq H*K$ .

**Theorem 2.13:** Let (G, $*$) be a group and (H, $*$), (K, $*$) are two subgroups of (G, $*$), then

**(1)** $H*K \neq \phi \wedge H*K \subseteq G$

**(2)** $H \subseteq H*K$ and $K \subseteq H*K$

**(3)** (H$*$K, $*$) is a subgroup of (G, $*$) iff H$*$K = K$*$H

**(4)** If (G, $*$) is commutative group, then (H$*$K, $*$) is a subgroup of (G,$*$).

**Proof:**

**(1)** $\because e \in H \ \wedge e \in K \Rightarrow e * e = e \in H*K$

$\therefore \ H*K \neq \phi$

And let $x \in H*K \Longrightarrow x = a * b \ni a \in H \subseteq G$ and $b \in K \subseteq G$

$\Longrightarrow a \in G \ \wedge \ b \in G$

$\Longrightarrow a * b = x \in G$

$\therefore \ H*K \subseteq G$

**(2)** Let $x \in H \Longrightarrow x = x * e \in H*K$

$\Longrightarrow x \in H*K$

$\therefore H \subseteq H*k$

Similarly $K \subseteq H*K$

**(3)** ($\Rightarrow$) suppose (H$*$K, $*$) is a subgroup of (G, $*$) To prove, H$*$K = K$*$H

(i.e.) $H*K \subseteq K*H \ \wedge \ K*H \subseteq H*K$

Let $x \in H*K \Longrightarrow x = a*b \ni a \in H \wedge b \in K$

Since H$*$K is subgroup of G $\Longrightarrow x^{-1} \in H*K$

Let $x^{-1} = c * d \ni c \in H \ \wedge \ d \in K$

$x = (x^{-1})^{-1} = (c*d)^{-1} = d^{-1}*c^{-1} \ni d^{-1} \in K \ \wedge \ c^{-1} \in H$

$\therefore \ x = d^{-1}*c^{-1} \in K*H$

$\therefore \ H*K \subseteq K*H.$

$K*H \subseteq H*K$ **(Home Work)**

($\Leftarrow$) Let H∗K = K∗H To prove, ( H∗K, ∗) is subgroup of(G, ∗)

$H*K \neq \phi$ and $H*K \subseteq G$ (by 1)

Let x, y ∈ H∗K To prove, $x*y^{-1} \in H*K$

$x \in H*K \implies x = a*b \ni a \in H \wedge b \in K$

$y \in H*K \implies y = c*d \ni c \in H \wedge d \in K$

$x*y^{-1} = (a*b)*(c*d)^{-1}$

$= (a*b)*(d^{-1}*c^{-1})$

$=a*(\underbrace{b*d^{-1}}_{\in K})*\underbrace{c^{-1}}_{\in H}$

$\therefore (b*d^{-1})*c^{-1} \in K*H = H*K$

$\therefore (b*d^{-1})*c^{-1} \in H*K$

$\Rightarrow \exists\, p \in H, \ell \in K \ni (b*d^{-1})*c^{-1} = p*\ell$

$\therefore a*(b*d^{-1})*c^{-1} = \underbrace{a*p}_{\in H}*\underbrace{\ell}_{\in K} \in H*K$

$\therefore x*y^{-1} \in H*k$

$\therefore$ (H∗K, ∗) is subgroup of (G, ∗)

**(4)** If (G, ∗) is commutative group, then (H∗K, ∗) is subgroup of (G, ∗).

**<u>Proof</u>**: $H*K \neq \phi$ and $H*K \subseteq G$ (by 1)

Let x, y ∈ H∗K To prove, $x*y^{-1} \in H*K$

$x \in H*K \implies x = a*b \ni a \in H \wedge b \in K$

$y \in H*K \implies y = c*d \ni c \in H \wedge d \in K$

$x*y^{-1} = (a*b)*(c*d)^{-1}$

$= (a*b)*(c^{-1}*d^{-1})$ (since G is commutative)

$= a*(b*c^{-1})*d^{-1}$ (∗ is associative)

$= (a*c^{-1})*(b*d^{-1})$ (∗ is commutative and associative)

$\therefore x*y^{-1} \in H*K$

$\therefore$ (H∗K, ∗) is a subgroup of (G,∗)

**<u>Example 2.14:</u>** In $(Z_8, +_8)$, Let H = $\{\bar{0}, \bar{4}\}$ and K= $\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$.

Find $H +_8 K$

**<u>Solution:</u>** $H +_8 K = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$.

**Notes 2.15:** Let (H, $*$) and (K, $*$) are two subgroup of (G, $*$), then :

**(1)** $H*K \neq K*H$

**(2)** ($H*K$, $*$) need not be subgroup of (G,$*$). Give example **(Home Work)**

**Exercises (1):** Is (H, $*$) a subgroup of (G, $*$) each of the following:

**(1)** ($Z_8$, $+_8$), H = $\{\bar{0}, \bar{6}\}$. Find $H^2$.

**(2)** ($Z_4$, $+_4$), H = $\{\bar{0}, \bar{1}, \bar{2}\}$. Find $H^2$. **(Home Work 5)**

**Definition 2.16:** The center of a group (G,$*$) denoted by **Cent(G)** or **C(G)** is the set

$$C(G) = \{c \in G: c*x = x*c, \forall\, x \in G\}$$

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**Note 2.17:** Cent(G) $\neq \phi$, since $\exists$ e $\in$ G s.t. $e*x = x*e \ \ \forall\, x \in G \ \Rightarrow e \in$ Cent(G).

**Examples 2.18:**

**(1)** In a group (R\{0}, .)
Cent(R\{0}) = R\{0},since R\{0} with multiplication is commutative

**(2)** In a group ($S_3$, $\circ$), Cent($S_3$) = $\{f_1\}$
Since Cent($S_3$) = $\{f \in S_3: f\circ g = g\circ f \ \ \forall\, g \in S_3\} = \{f_1\}$

**Theorem 2.19:** Let (G, $*$) be a group. Then (Cent(G), $*$) is a subgroup of (G, $*$).

**Proof:** Cent(G) $\neq \phi$ (by note (2.17))

Cent(G) = $\{a \in G: x*a = a*x, \forall\, x \in G\} \subseteq G$

Let a, b $\in$ Cent(G) To prove, $a*b^{-1} \in$ Cent(G)

$a \in$ Cent(G) $\Rightarrow a*x = x*a, \forall\, x \in G$

$b \in$ Cent(G) $\Rightarrow b*x = x*b, \forall\, x \in G$

To prove, $(a*b^{-1})*x = x*(a*b^{-1}) \ \ \forall\, x \in G$

Now, $(a*b^{-1})*x = a*(b^{-1}*x)$

$= a*(x^{-1}*b)^{-1}$

$= a*(b*x^{-1})^{-1}$ (since b $\in$ Cent (G))