Example 1.21: Let $G = \{1, -1, i, -i\}$ be a set and "." be operation on G.
Is $(G, .)$ a group ? Comm. ?

Solution:

(1) Closure is true.

(2) Asso. Law is true

(3) 1 is an identity element.

(4) $1^{-1} = 1$, $-1^{-1} = -1, i^{-1} = -i, -i^{-1} = i$

(5) Comm .is true

$\therefore$ (G , .) is a comm.group.

.	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

Example 1.22: Let G = { $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, a, b $\in$ Z }. Show that (G, +) group?
Is (G, +) is a comm. group? **(Home Work).**

Solution:

(1) Closure: Let $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} \in$ G such that $a,b,c,d \in$ Z, then

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} a+c & 0 \\ 0 & b+d \end{bmatrix} \in G$$

Since $a+c \in$Z and $b+d \in$ Z Closure is true

(2) Asso. Law: **(Home Work).**

(3) Identity:

Since $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, then

0 $\begin{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}$ is an identity element of G

(4) Inverse: Let $a, b \in Z$, then $-a, -b \in Z$ and since

$$A + B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix} = \begin{bmatrix} a+(-a) & 0 \\ 0 & b+(-b) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ B = A^{-1} $\forall$ A $\in$ G $\exists$ B $\in$ G such that B = A^{-1}.

$\therefore$ (G , +) is a group.

Example 1.23:

Let **G** = $R \times R = \{(a, b) : a , b \in R, a \neq 0\}$ and $*$ be defined by

$(a , b) * (c , d) = (ac , bc + d)$

Prove that $(G , *)$ is group. Is $(G, *)$ Comm.?

Solution:

(1) Closure : Let $(a , b) , (c , d) \in G \Longrightarrow a \neq 0 ,\ c \neq 0 \Longrightarrow ac \neq 0$

$(a , b) * (c , d) = (ac , bc + d) \in G \quad ac \neq 0$

(2) Asso. : Let $(a , b) , (c , d) , (e , f) \in G$, we have

$(a , b) *[(c , d) * (e , f)] = (a , b) * (ce , de + f)$

$= (ace , bce + de + f) \ldots\ldots(1)$

$[(a , b) * (c , d)] * (e , f) = (ac , bc + d) * (e , f)$

$= (ace , (bc + d)e + f))$

$= (ace , bce + de + f) \ldots\ldots (2)$

$\therefore$ (1) = (2) , then asso. is true

(3) Idenitity : Let $(a , b) , (x , y) \in G$

$(a , b) * (x , y) = (x , y) * (a , b) = (a , b)$

$(a , b) * (x , y) = (ax , bx + y) = (a , b)$

$\therefore ax = a \Rightarrow x = 1$

and $bx + y = b \Rightarrow b + y = b \Rightarrow y = 0$

$\therefore (x, y) = (1,0)$

Also, $(x , y) * (a , b) = (xa , ya + b) = (a, b)$

$\therefore xa = a \Rightarrow x = 1$

$ya + b = b \Rightarrow ya = b - b \Rightarrow ya = 0 \Rightarrow y = 0$

$\therefore (x, y) = (1, 0)$

$\therefore (1,0)$ *is an identity element of G*

(4) Inverse: $Let\ (a , b), (c , d) \in G , a \neq 0 , c \neq 0$

$(a , b) * (c , d) = (c , d) * (a , b) = (1 , 0)$

$(c , d) * (a , b) = (1 , 0)$

$(ac , bc + d) = (1,0) \Rightarrow ac = 1 \Rightarrow c = \frac{1}{a}$

$bc + d = 0 \Rightarrow b\frac{1}{a} + d = 0 \Rightarrow d = -\frac{b}{a}$

$\therefore (c , d) = \left(\frac{1}{a} , \frac{-b}{a}\right)$ *is an inverse of G*

(5) Comm : G is not comm. , since Take (3,5) , (4,6)

$$\left.\begin{array}{l}(3, 5) * (4, 6) = (12, 26) \\ (4, 6) * (3, 5) = (12 , 23)\end{array}\right\} \Rightarrow \quad \text{G is not comm..}$$

Example 1.24: Let (G , *) be an arbitrary group. The set of the function from G in to G : $F_G = \{f_a : a \in G\}$, $f_a: G \to G$ s.t. $f_a(x) = a * x$, $x \in G$,
With the composition (F_G , o) is forms a group, prove that.

Solution:

(1) Closure: Let $f_a, f_b \in F_G$, a, b ∈ G

$(f_a \circ f_b)(x) = f_a(f_b(x)) = f_a(b * x)$
$= a * (b * x)$
$= (a * b) * x$, since G is a group.
$= f_{a*b}(x) \in F_G$, since $a*b \in G$

(2) Asso : Let $f_a, f_b, f_c \in F_G$, a, b, c ∈ G

$(f_a \circ f_b) \circ f_c = f_{a*b} \circ f_c = f_{(a*b)*c}$

since * is asso. on G

$= f_{a*(b*c)} = f_a \circ f_{b*c} = f_a \circ (f_a \circ f_c)$

(3) Identity : f_e is an identity of F_G, since

$f_a \circ f_e = f_{a*e} = f_{e*a} = f_e \circ f_a = f_a$

(4) Inverse : The inverse of f_a in F_G is f_a^{-1}, since

$f_a \circ f_a^{-1} = f_{a*a^{-1}} = f_{a^{-1}*a} = f_a^{-1} \circ f_a = f_e$

Also, if G is comm. group, then (F_G, o) is comm. group .

Exercises (3): Determine the systems (G, ∗) . Is (G,*) group? Is (G,*) comm. group? **(Home Work 3).**

[1] $(G = \{f_1, f_2, f_3, f_4, f_5, f_6\}, o)$, where

$f_1(x) = x$, $f_2(x) = \frac{1}{x}$, $f_3(x) = 1+x$, $f_4(x) = \frac{x+1}{x}$, $f_5(x) = \frac{x}{x+1}$, $f_6(x) = \frac{1}{1+x}$

[2] $G = \{(a, b) : a, b \in R , a \neq 0 , b \neq 0\}$ s.t.

$(a, b) * (c, d) = (ac, b+d)$

[3] $(G = \{ am : m \in Z \} , +)$

[4] $G = Q^+$, $a * b = \frac{ab}{5}$.

[5] G = Z, a ∗ b = a + b - 2

[6] Let $G = \{ \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$, $a \in Z \}$. Is (G, +) group?

[7] Let $= \{f_1, f_2, f_3, f_4\}$, where $f_i \ni i = 1, 2, 3, 4$, are mappings on $R\backslash\{0\} \ni$

$f_1(x) = x, f_2(x) = -x , f_3(x) = \frac{1}{x} , f_4(x) = -\frac{1}{x}$.

Show that $(G,)$ is a group. Is (G , ∘) Comm. ?