$= a*(x*b^{-1})$
$=(a*x)*b^{-1}$
$=(x*a)*b^{-1}$ (since $b \in$ Cent (G))
$= x*(a*b^{-1})$

$\therefore (a*b^{-1}) \in$ Cent(G)

$\therefore$ (Cent(G),$*$) is a subgroup of (G, $*$).

Theorem 2.20: Let (G, $*$) be a group. Then

Cent(G) = G $\Leftrightarrow$ G is a commutative group.

Proof: ($\Rightarrow$) $\forall a \in G \Rightarrow a \in$ Cent(G)

$\therefore a*x = x*a, \forall x \in G$

$\therefore a*x = x*a, \forall x, a \in G$

$\therefore$ G is commutative group

($\Leftarrow$) Suppose that G is commutative group.

To prove, Cent(G) = G , (i.e) To prove, Cent(G) $\subseteq$ G $\wedge$ G $\subseteq$ Cent(G)

By definition of Cent(G) we have Cent(G) $\subseteq$ G.

To prove, G $\subseteq$ Cent(G)

Let $x \in G$, G is commutative group $\Rightarrow x*a = a*x, \forall a \in G$

$\therefore x \in$ Cent(G) $\Rightarrow$ G $\subseteq$ Cent(G)

$\therefore$ Cent G = G

Cyclic Groups الزمر الدوارة أو (الزمر الدائرية)

Definition 2.21: Let (G, $*$) be a group and $a \in$ G,

(1) the **cyclic subgroup of G generated by *a*** is denoted by <a> and defined as

$$<a> = \{a^{k} : k \in Z\} = \{\ldots, a^{-1}, a^{0}, a^{1}, \ldots\}$$

الزمرة الجزئية الدائرية المتولدة بالعنصر $a \in$ G

Definition 2.22: A group (G, $*$) is called **cyclic group generated by *a*** iff $\exists\ a \in$ G such that

$$G = <a> = \{a^{k} : k \in Z\}$$

Examples 2.23: In (Z_9, $+_9$) find the cyclic subgroup generated by $\bar{2}$**,** $\bar{3}$**,** $\bar{1}$

Solution:

$<\bar{2}> = \{a^{k} : k \in Z\} = \{\ldots, (\bar{2})^{-3}, (\bar{2})^{-2}, (\bar{2})^{-1}, (\bar{2})^{0}, (\bar{2})^{1}, (\bar{2})^{2}, (\bar{2})^{3}, \ldots\}$

$= \{\ldots, \bar{3}, \bar{5}, \bar{7}, \bar{0}, \bar{2}, \bar{4}, \bar{6}, \ldots\} = \{\bar{0}, \bar{1}, \bar{2}, \ldots, \bar{8}\} = Z_9$

$\therefore$ Z_9 is cyclic group generated by $\bar{2}$

$<\bar{3}> = \{\ldots, (\bar{3})^{-3}, (\bar{3})^{-2}, (\bar{3})^{-1}, (\bar{3})^{0}, (\bar{3})^{1}, (\bar{3})^{2}, (\bar{3})^{3}, \ldots\}$

$= \{\ldots, \bar{3}, \bar{6}, \bar{0}, \bar{3}, \bar{6}, \bar{0}, \ldots\} = \{\bar{0}, \bar{3}, \bar{6}\}$ is a cyclic subgroup of Z_9

$<\bar{1}> = \{\ldots, (\bar{1})^{-3}, (\bar{1})^{-2}, (\bar{1})^{-1}, (\bar{1})^{0}, (\bar{1})^{1}, (\bar{1})^{2}, (\bar{1})^{3}, \ldots\}$

$= \{\ldots, \bar{6}, \bar{7}, \bar{8}, \bar{0}, \bar{1}, \bar{2}, \bar{3}, \ldots\} = \{\bar{0}, \bar{1}, \bar{2}, \ldots, \bar{8}\} = Z_9$

$\therefore$ Z_9 is a cyclic group generated by $\bar{1}$

Examples 2.24: In (Z, +) fined a cyclic group generated by 1, 2, – 1.

Solution:

$<1> = \{1^{k} : k \in Z\} = \{\ldots, 1^{-3}, 1^{-2}, 1^{-1}, 1^{0}, 1^{1}, 1^{2}, 1^{3}, \ldots\}$

$= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} = Z$

$<2> = \{2^{k} : k \in Z\} = \{\ldots, 2^{-3}, 2^{-2}, 2^{-1}, 2^{0}, 2^{1}, 2^{2}, 2^{3}, \ldots\}$

$= \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\} \neq Z$

$<-1> = \{(-1)^{k} : k \in Z\}$

$= \{\ldots, (-1)^{-3}, (-1)^{-2}, (-1)^{-1}, (-1)^{0}, (-1)^{1}, (-1)^{2}, (-1)^{3}, \ldots\}$

$= \{\ldots, 2, 1, 0, -1, -2, \ldots\} = Z$

$\therefore$ (Z, +) is cyclic group generated by 1 and – 1.

Examples 2.25: Is $(S_3,\circ)$ cyclic group ?

Solution: $<f_1> = \{f_1\} \neq S_3$

$<f_2> = \{f_2^k : k \in Z\} = \{\ldots, f_2^{-2}, f_2^{-1}, f_2^0, f_2^1, f_2^2, \ldots\}$

$= \{\ldots, f_2, f_3, f_1, f_2, f_3, \ldots\} = \{f_1, f_2, f_3\} \neq S_3$

$<f_3> = \{f_1, f_2, f_3\} \neq S_3$

$<f_4> = \{f_1, f_4\} \neq S_3$

$<f_5> = \{f_1, f_5\} \neq S_3$

$<f_6> = \{f_1, f_6\} \neq S_3$

$\therefore$ $(S_3,\circ)$ is not cyclic group.

Examples 2.26: Is $(G,\cdot)$ cyclic group, such that $G = \{1, -1, i, -i\}$?

Solution:

$<1> = \{1^k : k \in Z\} = \{\ldots, 1^{-3}, 1^{-2}, 1^{-1}, 1^0, 1^1, 1^2, 1^3, \ldots\}$

$= \{\ldots, -1, 1, -1, 1, 1, 1, 1, 1, \ldots\} = \{1, -1\} \neq G$

$<-1> = \{(-1)^k : k \in Z\}$

$= \{\ldots, (-1)^{-3}, (-1)^{-2}, (-1)^{-1}, (-1)^0, (-1)^1, (-1)^2, (-1)^3, \ldots\}$

$= \{\ldots, 1, 1, 1, -1, 1, -1 \ldots\} = \{1, -1\} \neq G$

$<i> = \{i^k : k \in Z\} = \{\ldots, i^{-2}, i^{-1}, i^0, i^1, i^2, i^3, \ldots\}$

$= \{\ldots, -1, i, 1, i, -1, -i, \ldots\} = \{1, -1, i, -i\} = G$

$<-i> = \{(-i)^k : k \in Z\} = \{\ldots, (-i)^{-2}, (-i)^{-1}, (-i)^0, (-i)^1, (-i)^2, (-i)^3, \ldots\}$

$= \{\ldots, -1, i, 1, -i, -1, i, \ldots\} = \{1, -1, i, -i\} = G$

$\therefore$ $(G,\cdot)$ is a cyclic group generated by i and – i.

Examples 2.27: In $(Z_6, +_6)$ find cyclic group generated by $\bar{1}, \bar{2}, \bar{5}$

(Home Work)

Theorem 2.28: Every cyclic group is commutative.

Proof: Let $(G, *)$ be acyclic group

$\therefore \exists\, a \in G$ s.t. $G = <a> = \{a^k : k \in Z\}$. To prove, G is commutative group

Let $x, y \in G$. To prove, $x*y = y*x, \forall\, x, y \in G$

$\because x \in G = <a> \Rightarrow x = a^m \ni m \in Z$ and $y \in G = <a> \Rightarrow y = a^n \ni n \in Z$

$x*y = a^m * a^n = a^{m+n} = a^{n+m} = a^n * a^m = y * x$

$\therefore$ G is commutative group.