Chapter Two : Subgroups and Cyclic Groups

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Definition 2.1:

Let (G, ∗) be a group and H ⊆ G, H is a non-empty subset of G. Then (H, ∗) is a subgroup of (G,∗) if (H,∗) is itself a group.

Definition 2.2:

Let (G, ∗) be a group and H ⊆ G, Then (H, ∗) is subgroup of G if :

(1) ∀ a, b ∈ H ⇒ a ∗ b ∈ H

(2) The identity element of G is an identity element of H. e ∈ G ⇒ e ∈ H

(3) ∀ a ∈ H ⇒ a^{-1}∈ H

Remark 2.3:

Each group (G, ∗) has at least two subgroup ({e}, ∗) and (G, ∗), these subgroups are known **trivial subgroups** or **improper**, any subgroup different from these subgroups known a **proper subgroup**.

Examples 2.4:

(1) $(Z, +)$ is a proper subgroup of (R, +)

(2) H = {1, – 1} ⊆ {1, –1, i , –i}, then (H, .) is a subgroup of ({1,–1, i ,–i}, .)

(3) H = $\{\bar{0},\ \bar{2}\} \subseteq Z_4$

(H, $+_4$) is a proper subgroup of (Z_4, $+_4$).

But $\{\bar{0},\ \bar{3}\}$ is not subgroup of (Z_4, $+_4$).

Since $\bar{3} +_4 \bar{3} = \bar{6}$ (Mod4) $= \bar{2}\ \notin \{\bar{0}, \bar{3}\}$,

it follows that closure is not true in $\{\bar{0}, \bar{3}\}$.

(4) (Q\{0}, ×) is a subgroup of (R\{0}, ×).

Theorem 2.5: Let (G, $*$) be a group and $\phi \neq H \subseteq G$. Then (H, $*$) is a subgroup of (G, $*$) iff $a * b^{-1} \in H, \forall a, b \in H$

Proof:

($\Rightarrow$) let (H, $*$) be a subgroup and a, b $\in$ H, then

a, $b^{-1} \in H \Longrightarrow a * b^{-1} \in H$ (since $*$ closure)

($\Leftarrow$) Let $a * b^{-1} \in H$ To prove, (H, $*$) is subgroup

(1) Since $H \neq \phi \Longrightarrow \exists\, b \in H$ s.t. $b * b^{-1} \in H \Longrightarrow e \in H$.

(2) Since $b \in H$ and $e \in H \Longrightarrow e * b^{-1} \in H \Longrightarrow b^{-1} \in H$

(3) Let $a \in H$ and $b^{-1} \in H$ [by (2)] $\Longrightarrow a * (b^{-1})^{-1} \in H \Longrightarrow a * b \in H$

$\therefore$ By definition (2.2) $\Rightarrow$ (H, $*$) is a subgroup of (G, $*$)

Example 2.6: Let (Z, +) be a group and H ={5a: a $\in$ Z}.

Show that (H, +) is a subgroup of (Z, +)

Solution: By Theorem (2.5) above, let x, y $\in$ H, To prove, $x + y^{-1} \in H$

$x \in H \Longrightarrow x = 5a, a \in Z, \quad y \in H \Longrightarrow y = 5b, b \in Z$

$x + y^{-1} = 5a + (5b)^{-1} = 5a + 5(-b)$

$\quad = 5\underbrace{(a - b)}_{\in Z} \in H$

$\Longrightarrow$ (H, +) is a subgroup of (Z, +)

Theorem 2.7: If (H_i, $*$) is the collection of subgroups of (G, $*$), then ($\cap H_i$, $*$)) is also subgroup of (G, $*$)

Proof:

(1) Since $\exists\, e \in H_i, \forall i \Longrightarrow e \in \cap H_i \Longrightarrow \cap H_i \neq \phi$

(2) Let $x, y \in \cap H_i$ To prove, $x * y^{-1} \in \cap H_i$

Since $x, y \in \cap H_i \Longrightarrow x, y \in H_i\ \forall i$

$\Longrightarrow x * y^{-1} \in H_i, \forall i$ (since H_i subgroups)

$\Longrightarrow x * y^{-1} \in \cap H_i$

$\therefore$ ($\cap H_i$, $*$) is subgroup of (G, $*$)

Theorem 2.8: Let $(H_i, *)$ is the collection of subgroups of $(G, *)$ and let H_k , $H_j \in \{H_i\}$ such that $\exists H_\ell \in \{H_i\}$, $H_k \subseteq H_\ell$ and $H_j \subseteq H_\ell$, then $(\cup H_i, *)$ is also subgroup.

Proof:

(1) Since $\exists\ e \in H_i$ for some $i \Longrightarrow e \in \cup H_i \Longrightarrow \cup H_i \neq \phi$

(2) Let $x, y \in \cup H_i$, then $x, y \in H_k$ or $x, y \in H_j$, so $x, y \in H_\ell$

$\Longrightarrow x * y^{-1} \in H_\ell$, (since H_ℓ subgroup)

$\Longrightarrow x * y^{-1} \in \cup H_i$

$\therefore$ $(\cup H_i, *)$ is subgroup of $(G, *)$

Theorem 2.9: Let $(H_1, *)$ and $(H_2, *)$ are two subgroups of $(G, *)$ then $(H_1 \cup H_2, *)$, is a subgroup of $(G, *)$ iff $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.

Proof: $(\Rightarrow)$ Let $(H_1 \cup H_2, *)$ is a subgroup, To prove, $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$

Suppose that $H_1 \not\subset H_2$ and $H_2 \not\subset H_1$

$\therefore \exists\ a \in H_1$, $a \notin H_2$ and $\exists\ b \in H_2$, $b \notin H_1$

$\therefore a * b \in H_1 \cup H_2 \Longrightarrow a * b^{-1} \in H_1 \cup H_2$

$\Longrightarrow a * b^{-1} \in H_1$ or $a * b^{-1} \in H_2$

$\Longrightarrow a, b \in H_1$ or $a, b \in H_2$ C!!!! (تناقض)

$\therefore$ $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$

$(\Leftarrow)$ Let $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$ To prove, $(H_1 \cup H_2, *)$ is a subgroup

If $H_1 \subseteq H_2 \Longrightarrow H_1 \cup H_2 = H_2$ is a subgroup.

If $H_2 \subseteq H_1 \Longrightarrow H_1 \cup H_2 = H_1$ is a subgroup

$\therefore$ $H_1 \cup H_2$ is a subgroup in two cases.

Remark 2.10: $(H_1 \cup H_2, *)$ need not be a subgroup of $(G, *)$.

For example:

$H_1 = \{r_1, r_3\}$ is a subgroup of G_s, and $H_2 = \{r_1, v\}$ is a subgroup of G_s.

But $H_1 \cup H_2 = \{r_1, r_3, v\}$ is not a subgroup of G_s, since $r_3 \circ v = h \notin H_1 \cup H_2$