Definition 1.9: (**Mathematical System**)

A Mathematical System or (Mathematical Structure) is a non-empty set of elements with one or more binary operations defined on this set.

Example 1.10:

$(R,+),(R,.),(R,-),(R\backslash\{0\},\div),(R,+,.),(N,+),(E,+,\times)$ are Math. System. But $(N,-),(R,\div),(O,+,-)$are not Math. System.

Definition 1.11: **(Semi group)**

A semi group is a pair $(S,\ *)$ in which S is an empty set and $*$ is a binary operation on S with associative law.

(i.e.) $(S,*)$ is semi group $\Leftrightarrow$ **(1)** $S \neq \emptyset$,

(2) $*$ is a binary operation,

(3) $\forall\ a,b,c \in S$, (a ∗b) ∗c = a ∗(b ∗c).

Example 1.12:

(1) $(Z,+),(Z,\times),(N,+),(N,\times),(E,+),(E,\times)$ are semi groups.

(2) $(O,+),(Z,-),(\mathrm{E},-),(R\backslash\{0\},\div)$ are not semi groups.

Definition 1.13: (**The identity element**)

Let $(S,*)$ be a Mathematical System and $e \in S$. Then e is called an identity element if $a * e = e * a = a, \forall\ a \in S.$

Definition 1.14: (**The inverse element**)

Let $(S,*)$ be a Mathematical System and $a,b \in S$. Then b is called an inverse of a if $a * b = b * a = e$ and dented by $b = a^{-1}$.

Definition 1.15: (**The Group**)

The pair $(G,*)$ is a group iff $(G,*)$ is a semi group with identity in which each element of G has an inverse.

Definition 1.16: (**The Group**)

A group $(G,*)$ is a non-empty set G and a binary operation $*$, such that the following axioms are satisfied:

(1) The binary operation $*$ is associative.

$$\text{(i.e.) } (a * b) * c = a * (b * c),\ \forall\ a, b, c \in G$$

(2) There is an element e in G such that

$$a * e = e * a = a, \forall a \in G.$$

This element e is an identity element for $*$ on G.

(3) For each a in G, there is an element b in G such that

$$a * b = b * a = e.$$

The element b is an inverse of a and denoted by a^{-1}.

Remark 1.17:

Every group is a semi group but the converse is not true as in the following example shows.

$(N, +)$ is a semigroup but not group because $\nexists a^{-1} \in N, \forall a \in N$.

Definition 1.18: (**Commutative group**)

A group $(G,*)$ is called a Commutative group iff $a * b = b * a, \forall a, b \in G$.

Example 1.19:

(1) $(Z, +), (E, +), (Q, +), (C, +)$ are commutative groups .

(2) $(Z^+, +)$ is not a group because there is no identity element for $+$ in Z^+.

(3) $(Z^+, \times)$ is not a group because there is an identity element 1 but no inverse of 5.

(4) $(G = \{1, 0, -1, 2\}, +)$ is not group since $+$ is not a binary operation on G, $1+2 = 3 \notin G$.

(5) $(G = \{1, -1\}, \times)$ is comm. Group.

(6) $(R\backslash\{0\}, \times)$, $(Q\backslash\{0\}, \times)$, $(C\backslash\{0\}, \times)$ are comm. Groups.

Example 1.20: Let $G = \{a, b, c, d\}$ be a set. Define operation $*$ on G by the following table. (**Klein 4-group)**

$*$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

Is $(G,*)$ a commutative group?

Solution:

(1) Closure is true.

(2) Asso. ?

(a ∗ b) ∗ c = a ∗ (b ∗ c) ?

b ∗ c = a ∗ d

d = d

b ∗ (a ∗ c) = b ∗ c = d = (b ∗ a) ∗ c

c ∗ (a ∗ b) = c ∗ b = d = (c ∗ a) ∗ b

d ∗ (a ∗ c) = d ∗ c = b = (d ∗ a) ∗ c ….→

⇒ ∗ is asso.

(3) The identity: To prove $\exists e \in G \ \ s.t. a * e = e * a = a, \forall a \in G.$

$a * a = a, b * a = b, c * a = c, d * a = d.$

$\Rightarrow e = a$ is an identity element of G.

(4) The inverse: $a * a = a \Rightarrow a^{-1} = a$

$b * d = a \Rightarrow b^{-1} = d$

$c * c = a \Rightarrow c^{-1} = c$

$a * a = a \Rightarrow a^{-1} = a$

$d * b = a \Rightarrow d^{-1} = b$

(5) Comm. ?

$a * b = b * a$?

$b = b$

$a * c = c * a = c$

$a * d = d * a = d$

$b * c = c * b = d$

$b * d = d * b = a$

$c * d = d * c = b$

$\Rightarrow$ $*$ is a comm.

Therefore $(G,*)$ is a comm. group and called **Klein 4-group**.