نظرية الزمر

Groups Theory

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المصادر العربية :

[1] مقدمة في الجبر المجرد الحديث. تاليف ديفيد بيرتون وترجمه عبد العالي جاسم.

English References

[1] Introduction to modern abstract algebra. By David M. Burton.
[2] A first course in abstract algebra. By J.B. Fraleigh.
[3] Group theory. By M. Suzuki

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Chapter One : Groups Theory
الفصل الاول : نظرية الزمر

Definition 1.1: Binary Operations

Let A be a non empty set. A binary operation on a set A is a function from $A \times A$ into A. (i.e.)

$*: A \times A \to A$ is a binary operation iff

(1) $a * b \in A, \forall a, b \in A$ (Closure)

(2) If $a, b, c, d \in A$ such that $a = c$ and $b = d$, then $a * b = c * d$ (well-define).

Remark 1.2: Some time we used the symbols $*$, $\circ$, $\#$, $\odot$, ... to denote a binary operation.

Example 1. 3:

(1) The operations $\{+, \ \times\}$arebinary operations on R, Z, Q, C.

(2) The operation $"-"$ is not binary operation on N.

(3) The operations $\{+, \ -\}$ are not binary operations on O **(**odd number).

(4) The operation $\div$ is abinary operation on $R\backslash\{0\}, Q\backslash\{0\}, C\backslash\{0\}$.

Example 1.4:

Let $a * b = a + b + 2, \forall a, b \in Z^+$**.** Is $*$ a binary operation on Z^+?

Solution:

(1) Closure : Let $a, b \in Z^+$, then $a * b = \overbrace{a + b}^{\in Z^+} + 2 \in Z^+$.

(2) well-define : Let $a, b, c, d \in A$ such that $a = c$ and $b = d$,

then $a * b = a + b + 2 = c + d + 2 = c * d$

$\Rightarrow *$ is a binary operation on Z^+.

Example 1.5:

Let $a * b = a^b, a, b \in Z$. Is $*$ is a binary operation on Z.

Solution:

(1) Closure : if $a = 3$ and $b = -1$. Then $a * b = 3^{-1} = \frac{1}{3} \notin \mathrm{Z}$

$\Rightarrow \ *$ is not a binary operation on Z.

Exercises (1): which of the following are binary operations?

[1] $a * b = a + b, \forall a, b \in R\backslash\{0\}$.

[2] $a \odot b = \frac{a}{b}, \forall a, b \in Z$.

[3] $a \# b = a + b - 3, \forall a, b \in N$. **(Home Work 1).**

[4] $a \circ b = a + 2b - 5, \forall a, b \in R$.

[5] $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}, \forall \frac{a}{b}, \frac{c}{d} \in Q\backslash\{0\}$.

Definition 1.6: (Commutative)

A binary operation $*$ on a set A is called a commutative if and only if

$$a * b = b * a \quad \forall \quad a, b \in A.$$

Definition 1.7: (Associative)

A binary operation $*$ on a set A is called an associative if

$$(a * b) * c = a * (b * c) \quad \forall \quad a, b, c \in A.$$

Example 1.8: Let R be a set of real numbers and $*$ be a binary operation on R defined as $a * b = a + b - ab$. Is $*$ commutative and associative.

Solution:

Let $a, b \in R$, then

$a * b = a + b - ab = b + a - ba = b * a$

Which implies that $*$ is commutative.

Let $a, b, c \in R$, then

$$\begin{aligned}(a * b) * c &= (a + b - ab) * c \\ &= (a + b - ab) + c - (a + b - ab)c \\ &= a + b + c - ab - ac - bc + abc \ldots\ldots\ldots\ldots..(1)\end{aligned}$$

$$\begin{aligned}a * (b * c) &= a * (b + c - bc) \\ &= a + (b + c - bc) - a(b + c - bc) \\ &= a + b + c - bc - ab - ac + abc \ldots\ldots\ldots\ldots..(2)\end{aligned}$$

$\Rightarrow$ (1) = (2) $\Rightarrow$ $*$ is associative.

Exercises (2): Which of the following binary operations is a comm., asso.?

[1] $a * b = a - b, \quad \forall a, b \in Z$.

[2] $a \odot b = 2ab, \quad \forall a, b \in E$. **(Home Work 2).**

[3] $a \# b = a^3 + b^3, \quad \forall a, b \in R$.