Section (1.2): Derivation of Partial Differential Equation by the Elimination of Arbitrary Constants

For the given relation $F(x, y, z, a, b) = 0$ involving variables x, y, z and arbitrary constants a and b, the relation is differentiated partially with respect to independent variables x and y. Finally arbitrary constants a and b are eliminated from the relations $F(x, y, z, a, b) = 0$, $\frac{\partial F}{\partial x} = 0$ and $\frac{\partial F}{\partial y} = 0$

The equation free from a and b will be the required partial differential equation.

Three situations may arise:

Situation (1):

When the number of arbitrary constants is less than the number of independent variables, then the elimination of arbitrary constants usually gives rise to more than one partial differential equation of order one.

Example: Consider $z = ax + y$ …………(1)

where a is the only arbitrary constant and x, y are two independent variables.

Differentiating (1) partially w.r.t. x, we get

$\frac{\partial z}{\partial x} = a$ …………(2)

Differentiating (1) partially w.r.t. y, we get

$\frac{\partial z}{\partial y} = 1$ …………(3)

Eliminating abetween (1) and (2) yields

$$z = x\left(\frac{\partial z}{\partial x}\right) + y \qquad \ldots\ldots\ldots\ldots(4)$$

Since (3) does not contain arbitrary constant, so (3) is also PDE under consideration thus, we get two PDEs (3) and (4).

Situation (2):

When the number of arbitrary constants is equal to the number of independent variables, then the elimination of arbitrary constants shall give rise to a unique PDE of order one.

Example: Eliminate a and b from

$$az + b = a^2x + y \qquad \ldots\ldots\ldots\ldots(1)$$

Differencing (1) partially w.r.t. x and y, we have

$$a\left(\frac{\partial z}{\partial x}\right) = a^2 \qquad \ldots\ldots\ldots\ldots(2)$$

$$a\left(\frac{\partial z}{\partial y}\right) = 1 \qquad \ldots\ldots\ldots\ldots(3)$$

Eliminating a from (2) and (3), we have

$$\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = 1$$

which is the unique PDE of order one.

Situation (3):

When the number of arbitrary constants is greater than the number of independent variables. Then the elimination of arbitrary constants leads to a partial differential equations of order usually greater than one.

Example: Eliminate $a,$ b and c from

$z = ax + by + cxy$ …………(1)

Differentiating (1) partially w.r.t. x and y, we have

$\frac{\partial z}{\partial x} = a + cy$ …………(2) $\frac{\partial z}{\partial y} = b + cx$ …………(3)

from (2) and (3) $\frac{\partial^2 z}{\partial x^2} = 0,\ \frac{\partial^2 z}{\partial y^2} = 0$ …………(4)

$\frac{\partial^2 z}{\partial x \partial y} = c$ …………(5)

Now, multiply (2) by x and (3) by y, we get

$x\frac{\partial z}{\partial x} = ax + cxy$ ………………(6)

and

$y\frac{\partial z}{\partial y} = by + cxy$ …….. ……(7), by adding (6) to (7), we obtain

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \underbrace{ax + by + cxy} + cxy$$

from (1) and (5)

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z + xy\frac{\partial^2 y}{\partial x \partial y}$ …………(8)

Thus, we get three PDEs given by (4) and (8) which are all of order two.

Examples:

Example1: Find a PDE by eliminating a and b from

$$z = ax + by + a^2 + b^2$$

Sol. Given $z = ax + by + a^2 + b^2$ …………(1)

Differentiating (1) partially with respect to x and y,

we get $\frac{\partial z}{\partial x} = a$ and $\frac{\partial z}{\partial y} = b$

Substituting these values of a and b in (1), we see that the arbitrary constants a and b are eliminated and we obtain

$$z = x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2$$

which is required PDE.

Example2: Eliminate arbitrary constants a and b from

$z = (x - a)^2 + (y - b)^2$ to form the PDE.

Sol. Given $z = (x - a)^2 + (y - b)^2$ …………(1)

Differentiating (1) partially with respect to x and y, to get

$\frac{\partial z}{\partial x} = 2(x - a)$, $\frac{\partial z}{\partial y} = 2(y - b)$

Squaring and adding these equations, we have

$$\left(\frac{\partial z}{\partial x}\right)^2 + (\frac{\partial z}{\partial y})^2 = 4(x - a)^2 + 4(y - b)^2$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + (\frac{\partial z}{\partial y})^2 = 4[(x - a)^2 + (y - b)^2]$$

$\left(\frac{\partial z}{\partial x}\right)^2 + (\frac{\partial z}{\partial y})^2 = 4z$ using (1)

Example 3: Find the PDEs by eliminating arbitrary constants a and b from the following relations:

(a) $z = a(x + y) + b$ (b) $z = ax + by + ab$

(c) $z = ax + a^2y^2 + b$ (d) $z = (x + a)(y + b)$

Sol. (a) Given $z = a(x + y) + b$ …………(1)

Differentiating (1) w.r.t. x and y, we get

$\frac{\partial z}{\partial x} = a \quad , \frac{\partial z}{\partial y} = a$

Eliminating a between these, we get

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$ which is the required PDE.

(b)Try by yourself (c)Try by yourself (d)Try by yourself

… Exercises …

Ex.(1): Eliminate a and b from $z = axe^y + \frac{1}{2}a^2e^{2y} + b$ to form the PDE.

Ex.(2): Eliminate h and k from the equation $(x - h)^2 + (y - k)^2 + z^2 = \alpha^2$to form the PDE.

Ex.(3): Eliminate a and b from the equation $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ to form the PDE.

Ex.(4): Eliminate the arbitrary constants indicated in brackets from the following equations and form corresponding PDEs

(1) $z = ax^3 + by^3$,(a and b)

(2) $4z = \left[ax + \left(\frac{y}{a}\right) + b\right]^2$,(a and b)

(3) $z = ax^2 + bxy + cy^2$, (a,b,c)
