جامعة بغداد

كلية التربية للعلوم الصرفة ابن الهيثم

قسم الرياضيات

المرحلة الثانية

المعادلات التفاضلية الاعتيادية

Ordinary Differential Equations

CHAPTER FIVE

THE DIFFERENTIAL EQUATIONS OF HIGHER ORDER

اساتذة المادة

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Section 5.6: The non-homogenous linear equation with constant coefficients

المعادلة الخطية اللامتجانسة ذات المعاملات الثابتة

The general solution of the non-homogenous linear equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = f(x)$$

Is: $\boldsymbol{y = y_c(x) + y_p(x)}$ ……………(26)

Where

$y_c(x)$ is the complementary solution of the eq. $\boldsymbol{F(D)y = 0}$

&$y_p(x)$ is the particular solution of the eq. $\boldsymbol{F(D)y = f(x)}$

We already found $y_c(x)$ and here we will find $y_p(x)$

الحل العام للمعادلة اللامتجانسة : هو الحل العام للمعادلة المتجانسة والذي نسميه بالدالة المتممة $y_c(x)$ مضافا اليه الحل الخاص للمعادلة اللامتجانسة والذي يسمى بالتكامل الخاص $y_p(x)$

لقد درسنا بالبند السابق طريقة ايجاد الدالة المتممة للمعادلة اللامتجانسة وذلك بوضع $F(D)y = 0$ ويبقى علينا ايجاد $y_p(x)$ الذي هو الحل الخاص للمعادلة اللامتجانسة

وسنتطرق هنا الى بعض الطرق المبسطة لايجاد الحل الخاص

5.6.1:Un determined coefficient method: طريقة المعاملات غير المحددة

The first case : If $f(x)$ was a polynomial متعددة الحدود

The method for finding the particular solution for non- homogenous diff. eq. is a polynomial that degree equal to the degree of $f(x)$ then after that we find the coefficient of the given polynomial

Example1 : Solve the eq. $3y'' - 5y' - 2y = 6x^2 - 7$

Sol.: 1- we find the complementary function ($y_c(x)$)

$$3y'' - 5y' - 2y = 0$$

The char. Eq. is : $3m^2 - 5m - 2 = 0$

$(3m+1)(m-2) = 0 \quad \Longrightarrow \quad m = -\frac{1}{3} \quad , \quad m = 2$

Then the complementary function is

$y_c(x) = c_1 e^{2x} + c_2 e^{-\frac{1}{3}x}$, where $c_1 \& c_2$ are arb. cons.

2-The particular sol. $y_p(x)$

we note that $f(x) = 6x^2 - 7$ its polynomial of the 2- degree so we suppose the solution is a polynomial of 2-degree is

$y = ax^2 + bx + c$

we must find the coefficient a, b, c , but the first we find y, y', y'' then we make it up in the original equation

$y' = 2ax + b \quad ; \quad y'' = 2a$

بالتعويض في المعادلة الاصلية

$3(2a) - 5(2ax + b) - 2(ax^2 + bx + c) = 6x^2 - 7$

نساوي معاملات قوى x بالطريقة التالية

$-2a = 6 \quad \Rightarrow \quad a = -3$

$-10a - 2b = 0 \quad \Rightarrow \quad -10(-3) - 2b = 0 \quad \Rightarrow \quad b = 15$

$6a - 5b - 2c = -7 \quad \Rightarrow \quad 6(-3) - 5(15) - 2c = -7$

$\Rightarrow \quad -18 - 75 - 2c = -7$

$\Rightarrow \quad c = -43$

The particular sol. Is $y_p(x) = -3x^2 + 15x - 43$

and the general sol. Is $y = y_c(x) + y_p(x)$

$\Rightarrow \quad y = c_1 e^{2x} + c_2 e^{-\frac{1}{3}x} - 3x^2 + 15x - 43$ the general sol

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<u>The second case</u> : If the function is $f(x) = be^{ax}$

1- If (a) is not a root of the char. Eq. then we assume that :

- $\boldsymbol{y = Ae^{ax}}$ a ليس جذر للمعادلة

حيث ان A هو المجهول ونجده بالتعويض عن y واشتقاقه في المعادله المعطاة

2- If (a) is one of the roots of the char. Eq. and **<u>not repeated</u>** then we assume:

- $\boldsymbol{y = Axe^{ax}}$ a جذر غير مكرر

3- If (a) is one of the roots of the char. Eq. and **<u>repeated</u>** then we assume:

- $\boldsymbol{y = Ax^n e^{ax}}$ a جذر مكرر

Example2 : Solve the eq. $\mathbf{3y'' - 5y' - 2y = 5e^{3x}}$

Sol.: we find the char. Eq.

$$3y'' - 5y' - 2y = 0 \quad \Longrightarrow \quad 3m^2 - 5m - 2 = 0$$

$$\Longrightarrow \quad (3m+1)(m-2) = 0 \qquad \Longrightarrow m = -\frac{1}{3} \quad , \quad m = 2$$

$$y_c = c_1 e^{2x} + c_2 e^{-\frac{1}{3}x}$$

$\because a = 3$ not root of the char. eq. so we suppose the function as :

$$y = Ae^{3x} \quad \rightarrow \quad y' = 3Ae^{3x} \quad ; \quad y'' = 9Ae^{3x}$$

We sub. y , y' , y'' by the diff. eq.

$$(27 - 15 - 2)Ae^{3x} = 5e^{3x} \qquad \rightarrow \quad A = \frac{5}{10} = \frac{1}{2}$$

So the particular solution is

$y_p(x) = \frac{1}{2} e^{3x}$ and the general sol is

$y = c_1 e^{2x} + c_2 e^{-\frac{1}{3}x} + \frac{1}{2} e^{3x}$ where $c_1 \& c_2$ are arb. cons.

Example3: Find the sol. of $\mathbf{3y'' - 5y' - 2y = 5e^{2x}}$

Sol.: suppose the particular sol. is

$$y = Axe^{2x}$$

Since $a = 2$ is a root of the char. Eq. from the last example where $m = -\frac{1}{3}$,2

a=2 هو احد جذور المعادلةالمميزة و غير مكرر

$$y' = Ae^{2x} + 2Axe^{2x}$$

$$y'' = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x}$$

Substituting y'', y', y in the original equation , we get:

$$3(2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x}) - 5(Ae^{2x} + 2Axe^{2x}) - 2Axe^{2x} = 5e^{2x}$$

$(12-10-2)Axe^{2x}+(12-5)Ae^{2x}=5e^{2x}$

$\Longrightarrow\ \ 0+7Ae^{2x}=5e^{2x}\ \ \Longrightarrow\ \ A=\frac{5}{7}$

$\therefore$ the particular sol. is $y_p(x)=\frac{5}{7}xe^{2x}$

And the general sol. is $y=c_1e^{2x}+c_2e^{-\frac{1}{3}x}+\frac{5}{7}xe^{2x}$ where $c_1\&c_2$ are arb. cons.

Example 4: Solve the eq. $\boldsymbol{y''-4y'+4y=x^2e^{2x}}$

Sol.: the char. Eq. of homogenous eq. $y''-4y'+4y=0$ is:

$m^2-4m+4=0 \quad \Longrightarrow \quad (m-2)(m-2)=0$

وبما ان $a=2$ وان جذور المعادلة المميزة هو 2 ومكرر مرتين اذن نفرض الحل الخاص كالاتي:

$y=Ax^2e^{2x}$

$y'=2Ax^2e^{2x}+2Axe^{2x}$

$y''=4Ax^2e^{2x}+4Axe^{2x}+2Ae^{2x}+4Axe^{2x}$

وبالتعويض عن y,y',y'' بالمعادلة التفاضلية نجد ان

$$4Ax^2e^{2x}+4Axe^{2x}+2Ae^{2x}+4Axe^{2x}-4(2Ax^2e^{2x}+2Axe^{2x})+4(Ax^2e^{2x}) = x^2e^{2x}$$

$4Ax^2+4Ax+2A+4Ax-8Ax^2-8Ax+4Ax^2=x^2$

$\Longrightarrow \quad 2A=x^2 \quad \Longrightarrow \quad A=\dfrac{x^2}{2}$

Then the particular solution is:

$y_p=\frac{1}{2}x^4e^{2x}$

The general sol. is :

$y=c_1e^{2x}+c_2xe^{2x}+\frac{1}{2}x^4e^{2x}$, where $c_1\&c_2$ are arb. cons.

<u>The third case</u> If the function was $\boldsymbol{f(x)=b\sin ax}$ ***or*** $\boldsymbol{f(x)=b\cos ax}$

To find the particular sol. of non-homogenous equation as this case suppose that

1- If (ai) is not a root of the char. Eq. then we assume that :

$y=A\cos ax+B\sin ax$

Where A *and* B are unknown constants.

Substituting y and its derivatives in the original equation to obtain A and B.

ثم نعوض عن y واشتقاقاته بالمعادلة التفاضلية الاصلية للحصول على B , A

2- If (ai) is one of the roots of the char. Eq. and **not repeated** then we assume:

$$y = x(A\cos ax + B\sin ax)$$

3- If (ai) is one of the roots of the char. Eq. and **repeated** then we assume:

$$y = x^n(A\cos ax + B\sin ax)$$

Example 5: Solve the eq. $\boldsymbol{3y'' - 5y' - 2y = 4\sin 2x}$

Sol.:

نلاحظ انه ليس للمعادلة المميزة جذورا عقدية حيث $m = -\frac{1}{3}, 2$

Let $y = A\cos 2x + B\sin 2x$

$\rightarrow \quad y' = -2A\sin 2x + 2B\cos 2x$

$\rightarrow \quad y'' = -4A\cos 2x - 4B\sin 2x$

substituting $y,\ y',\ y''$ in the original eq.

$$3(-4A\cos 2x - 4B\sin 2x) - 5(-2A\sin 2x + 2B\cos 2x) - 2(A\cos 2x + B\sin 2x) = 0\cos 2x + 4\sin 2x$$

نساوي معاملات $\sin 2x$, $\cos 2x$ نحصل على

$-14A - 10B = 0$

$10A - 14B = 4$

$$\Rightarrow A = \frac{5}{37} \quad , \quad B = \frac{-7}{37}$$

$\therefore$the particular sol. is $y = \frac{5}{37}\cos 2x - \frac{7}{37}\sin 2x$

$\therefore$ the general sol. is

$$y = c_1e^{2x} + c_2e^{-\frac{1}{3}x} + \frac{5}{37}\cos 2x - \frac{7}{37}\sin 2x$$

where $c_1\&c_2$ are arb. const.

ملحوظة: يمكن ايجاد الحل الخاص بهذه الطريقة للدوال التي تمثل حاصل ضرب دالتين مثل

$f(x) = x^n sinax \;\; or \;\; f(x) = x^n cosax \;\; \& \;\; f(x) = e^{bx} sinax \;\; or \;\; f(x) = e^{bx} cosax$

ولكن لطول حلولها ووجود طرق اقصر كما سنبين في البند القادم فلم نتطرق لها هنا

5-6-2: The inverse operator method طريقة المؤثر العكسي

The Particular sol. of the linear differential eq. is

$\boldsymbol{F(D)y = f(x)}$

With the constant coefficient is given as the form

$\boldsymbol{y = \frac{1}{F(D)} f(x)}$ ……………………………………………………………… (**27)**

where $F(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$.

And $\frac{1}{F(D)}$ is called the inverse operator

وباستخدام خواص المؤثر التي تم شرحها سابقا

1- $F(D)e^{bx} = F(b)e^{bx}$
2- $F(D)\{e^{bx}y\} = e^{bx}F(D+b)y$
3- $F(D^2)\sin bx = F(-b^2)\sin bx$
4- $F(D^2)\cos bx = F(-b^2)\cos bx$

To find the particular sol. for non-homogenous eq. we take the following cases

لايجاد الحل الخاص للمعادلة التفاضلية الخطية اللامتجانسة سوف نتعامل مع الحالات الاتية

The first case

If $\boldsymbol{f(x) = e^{bx}}$ then there are two cases

1- If $F(b) \neq 0$, then the particular sol. is:

$$\boldsymbol{F(D)y = e^{bx} \quad \Longrightarrow \quad y = \frac{1}{F(D)} . e^{bx} = \frac{1}{F(b)} e^{bx}} \quad \ldots\ldots\ldots\ldots (28)$$

2- If $F(b) = 0$, then the particular sol. will be as a form

$$\boldsymbol{y = \frac{1}{G(b)} . \frac{x^r}{r!} e^{bx}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (29)$$

where $F(D) = (D-b)^r G(D)$, r is a positive integer, $G(b) \neq 0$

اي في الحالة الثانية عندما يكون المؤثر =صفر فلايجوز تطبيق الحالة الاولى عندئذ نحلل المؤثر $F(D)$فنحصل على عامل من نوع $(D-b)^r$ وهو المتسبب في جعل المؤثر= صفر ويبقى من المعادلة $G(b) \neq 0$ وتكون النتيجة حسب القانون اعلاه

Example1: **Solve the eq.** $\boldsymbol{y''' - 2y'' - 5y' + 6y = e^{4x}}$

Sol.: the Eq. is

$(D^3 - 2D^2 - 5D + 6)y = e^{4x}$

1-to find y_c

$(D-1)(D-3)(D+2)y = 0$

the char. Eq. is

$(m-1)(m-3)(m+2) = 0$

$\rightarrow \quad m = 1 \ ,3 \ ,-2$

$\therefore \quad y_c = c_1 e^x + c_2 e^{3x} + c_3 e^{-2x}$

Where $c_1, c_2 \& c_3$ are arbitrary constants

2- the particular sol. y_p is :

$$y = \frac{1}{(D-1)(D-3)(D+2)} . e^{4x}$$

$F(D) = (D-1)(D-3)(D+2)$, b=4

$\rightarrow F(4) = (4-1)(4-3)(4+2) = 18 \neq 0$

Note that $F(b) \neq 0$

$\therefore y_p = \frac{1}{F(b)} e^{bx} \rightarrow y_p = \frac{1}{18} e^{4x}$

The general sol. is

$y = c_1 e^x + c_2 e^{3x} + c_3 e^{-2x} + \frac{1}{18} e^{4x}$, Where $c_1, c_2 \& c_3$ are arbitrary constants

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Example 2: Solve the eq. $\boldsymbol{y''' - 2y'' - 5y' + 6y = e^{3x}}$

Sol.: the comp. fun.y_c is: من المثال السابق

$y_c = c_1 e^x + c_2 e^{3x} + c_3 e^{-2x}$

To find the particular sol. y_p since b=3

$$F(b)=(b-1)(b-3)(b+2)\rightarrow F(3)=(3-1)(3-3)(3+2)=0$$

$$y_p=\frac{1}{(D-1)(D-3)(D+2)}.e^{3x}$$

G(3)=(3-1)(3+2)=10≠ 0 نلاحظ ان $(D-3)$ هو الذي يصفر المقام وان

r=1

The p. Sol. y_p is

$$y_p=\frac{1}{G(b)}\cdot\frac{x^r}{r!}e^{bx}=\frac{1}{10}xe^{3x}$$

<u>Or:</u>

$$y_p=\frac{1}{(D-3)^1}\left\{\frac{1}{(D-1)(D+2)}e^{3x}\right.$$

$$=\frac{1}{(D-3)^1}\left\{\frac{1}{10}e^{3x}\right\}=\frac{1}{10}\frac{x^1}{1i}e^{3x}=\frac{1}{10}xe^{3x}$$

The general sol. is $y=c_1e^x+c_2e^{3x}+c_3e^{-2x}+\frac{1}{10}xe^{3x}$

Where $c_1,c_2\&c_3$ are arbitrary constants

<u>Remark:</u> $\frac{1}{D}f(x)=\int f(x)dx$

Prove: let $\frac{1}{D}f(x)=z \quad\Longrightarrow\quad f(x)=Dz\Longrightarrow f(x)=\frac{dz}{dx}$

$$\therefore\quad z=\int f(x)dx\Longrightarrow\frac{1}{D}f(x)=\int f(x)dx$$

$\frac{1}{D}$ التاثير العكسي للمؤثر D اي ان $\frac{1}{D}$ يمثل التكامل بالنسبة الى x بينما $\frac{1}{D^k}$ يمثل التكامل بالنسبة الى x عدد k من المرات

$$\therefore\quad\frac{1}{D^1}=\int dx=x$$

$$\therefore\quad\frac{1}{D^2}=\iint(dx)^2$$

$$=\frac{1}{2}x^2$$

$\therefore\ \frac{1}{D^3}=\iiint(dx)^3=\frac{1}{6}x^3$

Example3: Solve the eq. $\boldsymbol{y'''+y''-y'-y=e^x}$

Sol.: $(D^3+D^2-D-1)y=0$

$D(D^2-1)+(D^2-1)y=0$

$(D^2-1)(D+1)y=0$

$(D-1)(D+1)(D+1)y=0$

$\rightarrow\ (D-1)(D+1)^2y=0$

Then the char. Eq. is

$(m-1)(m+1)^2=0$

$m_1=m_2=-1, m_3=1$

$y_c=(c_1+c_2x)e^{-x}+c_3e^x$

To find the particular sol.

$F(b)=(b-1)(b+1)^2$, b=1

$F(1)=(1-1)(1+1)^2$=0

=0,∴r=1 =4=G(1)

$$y_p=\frac{1}{(D-1)(D+1)^2}e^x$$

$$=\frac{1}{4}\frac{x}{1!}e^x = \frac{x}{4}e^x$$

Then the general sol. is:

$y=y_C+y_P$

$=(c_1+c_2x)e^{-x}+c_3e^x+\frac{x}{4}e^x$

Where $c_1, c_2 \& c_3$ are arbitrary constants

Example4: Solve the eq. $\boldsymbol{y'''-8y=10e^{3x}}$

Solution: $(D^3-8)y=0 \quad \rightarrow \quad (D-2)(D^2+2D+4)y=0$

The char. Eq. $(m-2)(m^2+2m+4)=0$

$$\rightarrow \quad m = 2, -1 \pm i\sqrt{3}$$

$$y_c = c_1 e^{2x} + e^{-x}(c_2 \cos\sqrt{3}\,x + c_3 \sin\sqrt{3}\,x)$$

The particular sol. is

$F(D) = D^3 - 8$

$F(b) = b^3 - 8 \rightarrow F(3) = 27 - 8 = 19 \neq 0$

$$y_p = 10\frac{1}{F(D)}e^{3x} = 10\frac{1}{F(3)}e^{3x} = \frac{10}{19}e^{3x}$$

The general sol. is:

$$y = c_1 e^{2x} + e^{-x}\left(c_2 \cos\sqrt{3}\,x + c_3 \sin\sqrt{3}\,x\right) + \frac{10}{19}e^{3x}$$

Where $c_1, c_2 \& c_3$ are arbitrary constants

The second case

If $\boldsymbol{f(x) = \sin bx}$ *or* $\boldsymbol{f(x) = \cos bx}$ to find the particular solution in this case we can use one of the three cases:

A) from $e^{ibx} = \cos bx + i \sin bx$ we get:

$$\boldsymbol{cosbx = \frac{e^{ibx} + e^{-ibx}}{2}} \quad \text{and} \quad \boldsymbol{sinbx = \frac{e^{ibx} - e^{-ibx}}{2i}}$$

After that we find the particular solution of e^{ibx} as in the first case when

$F(D) = 0 \ \& F(D) \neq 0$.

اي ان في الحالة الاولى سوف نستخدم قانون اويلر لتحويل الدوال المثلثية(sin bx, cos bx) الى دوال اسية والعمل على الحالة الاولى ويتم اللجوء الى هذه الطريقة في حالة ان تكون $(F(D) = 0)$ اي ان المقام يصبح صفرا بعد التعويض وهذا لايجوز لذلك نلجا الى الطريقة الاولى وتستخدم ايضا عندما تكون $(F(D) \neq 0)$

اي انها طريقة عامة لهذه الحالة

B) If $f(x) = \sin bx$ *or* $\cos bx$ then the particular sol. is:

$$\left.\begin{aligned} &\boldsymbol{y_p = \frac{1}{F(D^2)}\sin bx \quad \Rightarrow \quad y_p = \frac{1}{F(-b^2)}\sin bx \quad , \quad F(-b^2) \neq 0} \\ &\boldsymbol{or \quad y_p = \frac{1}{F(D^2)}\cos bx \quad \Rightarrow \quad y_p = \frac{1}{F(-b^2)}\cos bx \quad , \quad F(-b^2) \neq 0} \end{aligned}\right\} \ldots \boldsymbol{(30)}$$

C) If we have the following formula:

$\frac{1}{D^3+a^3}.\cos bx$ or when F(D) of odd order then the particular Sol. is:

عندما يكون المؤثر ذو رتبة فردية او ظهر فيه رتب فردية فالحل الخاص سيكون باستخدام طريقة العامل المرافق كالاتي:

$$\frac{1}{D^3+a^3}.\cos bx = \frac{1}{a^3-b^2D}.\cos bx$$

$$= \frac{a^3+b^2D}{a^6-b^4D^2}.\cos bx \qquad \text{(نضرب بالعامل المرافق)}$$

$$= \frac{a^3+b^2D}{a^6+b^6}.\cos bx$$

$$= \frac{1}{a^6+b^2}(a^3\cos bx - b^3\sin bx)$$

Example1: **Solve the diff. eq.** $\boldsymbol{(D^2+4)y = \sin 4x}$

Solution: the complementary function is:

$y_c = c_1\cos 2x + c_2\sin 2x$, Where c_1, c_2 are arbitrary constants

The particular sol is

$F(D^2) = D^2 + 4$

$F(-b^2) = -b^2 + 4$, b=4 $\to F(-16) = -16 + 4 = -12$

$y_p = \frac{1}{D^2+4}\sin 4x = \frac{1}{-(4)^2+4}\sin 4x = -\frac{1}{12}\sin 4x$

The general solution is $y = y_c + y_p$

$y = c_1\cos 2x + c_2\sin 2x - \frac{1}{12}\sin 4x$

Where c_1, c_2 are arbitrary constants

Example2: Solve the diff. eq. $\boldsymbol{\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \cos 3x}$

Solution: $(D^2 - 2D + 1)y = 0$

$(D-1)^2y = 0$

the char. Eq. is $(m-1)^2 = 0 \quad \to \quad m = 1,1$

$\therefore \quad y_c(x) = (c_1 + c_2 x)e^x \quad$, Where c_1, c_2 are arbitrary constants

$$y_p = \frac{1}{D^2 - 2D + 1} . \cos 3x$$

$$= \frac{1}{-3^2 - 2D + 1} . \cos 3x$$

$$= -\frac{1}{2} . \frac{1}{(D+4)} . \cos 3x = -\frac{1}{2} \frac{D-4}{(D^2 - 4^2)} \cos 3x$$

$$= -\frac{1}{2}(D-4) \frac{1}{-3^2 - 4^2} \cos 3x = \frac{1}{50}(D \cos 3x - 4 \cos 3x)$$

$$= \frac{1}{50}(-3 \sin 3x - 4 \cos 3x)$$

$$\therefore y = y_c(x) + y_p(x)$$

$$= (c_1 + c_2 x)e^x + \frac{1}{50}(-3 \sin 3x - 4 \cos 3x)$$

Example3: Solve the eq. $\boldsymbol{y'' + 4y = 8\cos 2x}$

Solution: the char. Eq. is $\quad m^2 + 4 = 0 \quad \rightarrow \quad m = \pm 2i$

$\therefore \quad y_c = c_1 \cos 2x + c_2 \sin 2x$

We know that $F(D^2) = D^2 + 4 \rightarrow F(-2^2) = -2^2 + 4 = 0$; b=2

بما ان المقام صفر لايجوز الحل بالطرق المباشرة فنستخدم قانون اويلردالة cos2x

$$\Longrightarrow \quad (D^2 + 4)y = 8 \frac{e^{2ix} + e^{-2ix}}{2}$$

حيث نجد الجزء الحقيقي يمثل الحل الخاص للدالة(e^{2ix}) لان الدالة الاصلية $\cos x$

$$y_p = \underbrace{\frac{4}{(D-2i)(D+2i)} e^{2ix}}_{u} + \underbrace{\frac{4}{(D-2i)(D+2i)} e^{-2ix}}_{w}$$

To find u : b=2i then:

$$u = \frac{4}{4i} \cdot \frac{x}{1!} e^{2ix} = \frac{1}{i} x e^{2ix}$$

To find w; b=-2i then:

$$w = \frac{4}{-4i} \cdot \frac{x}{1!} e^{-2ix} = -\frac{1}{i} x e^{-2ix}$$

$y_p = u + w = \frac{1}{i} x e^{2ix} - \frac{1}{i} x e^{-2ix}$

$= 2\frac{xe^{2ix} - xe^{-2ix}}{2i}$

$= 2x \sin 2x$

$\therefore \quad y = c_1 \cos 2x + c_2 \sin 2x + 2x \sin 2x$

, Where c_1, c_2 are arbitrary constants

Example4: Solve the eq. $(D^2 + 3D - 4)y = \sin 3x$

Solution: 1- the char. Eq. $(m-1)(m+4) = 0 \quad \rightarrow \ y_c = c_1 e^x + c_2 e^{-4x}$

2- To find y_p

$y_p = \frac{1}{D^2+3D-4} . \sin 3x$

$= \frac{1}{-9+3D-4} \sin 3x = \frac{1}{3D-13} \sin 3x = \frac{3D+13}{9D^2-169} \sin 3x$

$= \frac{-1}{250}(3D + 13) \sin 3x$

$= \frac{-1}{250}(9 \cos 3x + 13 \sin 3x)$

The general sol. is $y = c_1 e^x + c_2 e^{-4x} + \frac{-1}{250}(9 \cos 3x + 13 \sin 3x)$

Where c_1, c_2 are arbitrary constants

The third case

If the $f(x)$ is a polynomial of x to find the particular solution of the eq. $F(D)y = x^m$ where m is a positive integer number we write $\frac{1}{F(D)}$ as a power ascending to D

(i.e):

a- $\frac{1}{F(D)}=\frac{1}{1-D}=1+D+D^2+D^3+\cdots+D^m+\cdots$

b- $\frac{1}{F(D)}=\frac{1}{1+D}=1-D+D^2-D^3+\cdots+D^m+\cdots$

c- $\frac{1}{F(D)}=\frac{1}{(1+D)^2}=1-2D+3D^2-4D^3+\cdots+(m+1)D^m+\cdots$

d- $\frac{1}{F(D)}=\frac{1}{(1-D)^2}=1+2D+3D^2+4D^3+\cdots+(m+1)D^m+\cdots$

Since the $D^{m+r}x^m=0$ the particular solution is

$$y=\frac{1}{F(D)}.x^m=(1+D+D^2+D^3+\cdots+D^m)x^m$$
$$=x^m+mx^{n-1}+\cdots+m!$$

Example to explain: $D^2x^1=0$, $D^3x^2=0$

Example1: Solve the diff. eq. $y''+4y=8x^3$

Solution: $D^4+4=0 \quad \rightarrow \quad D=\pm 2i$

the char. Eq. is

$m^4+4=0 \quad \rightarrow \quad m=\pm 2i$

$y_c(x)=c_1\cos 2x+c_2\sin 2x$

The part. Sol. is

$$y_p=\frac{8}{4+D^2}.x^3=\frac{8}{4\left(1+\frac{D^2}{4}\right)}.x^3$$

$$=2\left(1-\left(\frac{D^2}{4}\right)+\left(\frac{D^2}{4}\right)^2+\cdots\right)x^3=2\left(x^3-\frac{6x}{4}\right)=2x^3-3x$$

The general sol. is

$y=c_1\cos 2x+c_2\sin 2x+2x^3-3x$

Where c_1, c_2 are arbitrary constants

Example 2: Solve the diff. eq. $(2D^2-5D-3)y=x^2+2x-1$

Solution: $y_c=c_1e^{-\frac{1}{2}x}+c_2e^{3x}$ الدالة المتممة

$$y_p = \frac{1}{2D^2 - 5D - 3}(x^2 + 2x - 1) = \frac{1}{-3\left(1 + \left(\frac{5}{3}D - \frac{2}{3}D\right)^2\right)}.(x^2 + 2x - 1)$$

$$= -\frac{1}{3}\left(1 - \left(\frac{5}{3}D - \frac{2D^2}{3}\right) + \left(\frac{5}{3}D - \frac{2D^2}{3}\right)^2 + \cdots\right)(x^2 + 2x - 1)$$

$$= -\frac{1}{3}(x^2 + 2x - 1) - \frac{5}{3}(2x + 2) - \frac{2}{3}(2) + \frac{25}{9}(2)$$

$$= -\frac{1}{3}x^2 - 4x + \frac{11}{9}$$

Then the general sol. is :

$$y = c_1 e^{-\frac{1}{2}x} + c_2 e^{3x} - \frac{1}{3}x^2 - 4x + \frac{11}{9}$$

Where c_1, c_2 are arbitrary constants

<u>The fourth case</u>

If the $f(x) = e^{ax}v(x)$ where

$$v(x) = \begin{cases} x^m \\ \sin bx \\ \cos bx \end{cases}$$

The particular solution $\boldsymbol{y_p(x)}$ is

$$\boldsymbol{y_p(x) = \frac{1}{F(D)}.e^{ax}.v(x) = e^{ax}\frac{1}{F(D+a)}.v(x)} \quad \textbf{.................................(31)}$$

Example1: Solve the diff. eq. $\boldsymbol{(D^2 + 2D + 5)y = xe^x}$

Solution: $(D^2 + 2D + 5)y = 0$

The char. Eq. is:

$$m^2 + 2m + 1 = -4$$

$$(m+1)^2 = -4 \quad \rightarrow \quad m + 1 = \pm 2i \rightarrow \quad m = -1 \pm 2i$$

$$\therefore \quad y_c(x) = e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$$

$$y_p(x) = \frac{1}{D^2+2D+5}.x.e^x = e^x \frac{1}{(D+1)^2+2(D+1)+5}x$$

$$= e^x \frac{1}{8\left(1+\frac{D}{2}+\frac{D^2}{8}\right)} . x$$

$$= \frac{1}{8} e^x \left(1 - \frac{1}{2} D + \cdots\right) x$$

$$= \frac{1}{8} e^x \left(x - \frac{1}{2}\right)$$

The general sol. is:

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = e^{-x}(c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{8} e^x \left(x - \frac{1}{2}\right)$$

Where c_1, c_2 are arbitrary constants

Example2: Solve the eq. $\boldsymbol{\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2 e^{3x}}$

Solution:

$$(D^2 - 2D + 1)y = 0$$

The char. Eq. is

$$m^2 - 2m + 1 = 0 \quad \rightarrow \quad (m-1)^2 = 0 \quad \rightarrow \quad m_1 = m_2 = 1$$

$$\therefore \quad y_c(x) = (c_1 x + c_2)e^x$$

لايجاد الحل الخاص نكتب بدلالة المؤثر D

$$(D^2 - 2D + 1)y = x^2 e^{3x}$$

$$\rightarrow \quad y_p = \frac{1}{D^2 - 2D + 1} . x^2 . e^{3x}$$

$$= e^{3x} \frac{1}{(D+3)^2 - 2(D+3) + 1} . x^2$$

$$= e^{3x} \frac{1}{D^2 + 6D + 9 - 2D - 6 + 1} . x^2$$

$$= e^{3x} \frac{1}{D^2 + 4D + 4} . x^2$$

$$= e^{3x} \frac{1}{4\left(1 + D + \frac{D^2}{4}\right)} . x^2$$

$$= \frac{1}{4} e^{3x} \left(1 - \left(\frac{D^2}{4} + D\right) + \left(\frac{D^2}{4} + D\right)^2 + \cdots\right) . x^2$$

$$= \frac{1}{4}e^{3x}\left(x^2 - \frac{1}{2} - 2x + 2\right)$$

$$= \frac{1}{4}e^{3x}\left(x^2 - 2x + \frac{3}{2}\right)$$

$\rightarrow \quad y = (c_1x + c_2)e^x + \frac{1}{4}e^{3x}(x^2 - 2x + \frac{3}{2})$

Where c_1, c_2 are arbitrary constants

Example3: solve the diff. eq. $\boldsymbol{(D^2 + 4D + 4)y = e^x \sin 2x}$

Solution: $y_c = c_1e^{-2x} + c_2xe^{-2x}$

The particular sol. is

$$y_p = e^x \frac{1}{(D+1)^2 + 4(D+1) + 4}.\sin 2x$$

$$= e^x \frac{1}{D^2+2D+1+4D+4+4}.\sin 2x$$

$$= e^x \frac{1}{D^2+6D+9}\sin 2x$$

$$= e^x \frac{1}{-4+6D+9}.\sin 2x$$

$$= e^x \frac{1}{6D+5}\sin 2x$$

$$= e^x \frac{6D-5}{36D^2-25}\sin 2x$$

$$= e^x \frac{-1}{169}(6D-5)\sin 2x$$

$$y_p = -\frac{e^x}{169}(12\cos 2x - 5\sin 2x)$$

The general sol. is:

$$y = y_c + y_p$$

$$y = c_1e^{-2x} + c_2xe^{-2x} - \frac{e^x}{169}(12\cos 2x - 5\sin 2x)$$

Where c_1, c_2 are arbitrary constants

The fifth case الحالة الخامسة

If $\boldsymbol{f(x) = x^m v(x)}$ where $\boldsymbol{v(x) = \cos ax}$ **or** $\boldsymbol{v(x) = \sin ax}$

To find the particular solution for this case we take the following examples.

Example1: Solve the eq. $(D^2+1)y = x^2 \sin 2x$

Solution: To find y_c, we have

$$m^2+1=0 \quad \rightarrow \quad m=\pm i$$

$\therefore \quad y_c(x) = c_1 \cos x + c_2 \sin x$, Where c_1, c_2 are arbitrary constants

$$y_p(x) = \frac{1}{D^2+1}.x^2.\sin 2x = e^{2ix}.\frac{1}{(D+2i)^2+1}x^2$$

$$= e^{2ix}\frac{1}{D^2+4iD-3}.x^2 = e^{2ix}\frac{1}{-3\left(1-\frac{4i}{3}D-\frac{D^2}{3}\right)}.x^2$$

$$= -\frac{1}{3}e^{2ix}\left[1+\frac{4iD}{3}+\frac{D^2}{3}+\left(\frac{4iD}{3}+\frac{D^2}{3}\right)+\cdots\right].x^2$$

$$= -\frac{1}{3}e^{2ix}\left(1+\frac{4iD}{3}+\frac{D^2}{3}-\frac{16D^2}{9}+\cdots\right)x^2$$

$$= -\frac{1}{3}e^{2ix}\left(x^2+\frac{8ix}{3}-\frac{26}{9}\right)$$

$$= -\frac{1}{3}(\cos 2x + i\sin 2x)\left[\left(x^2-\frac{26}{9}\right)+\frac{8ix}{3}\right]$$

$$= -\frac{1}{3}\left[\left(x^2-\frac{26}{9}\right)\sin 2x + \frac{8x}{3}\cos 2x\right]$$

$$y = y_c(x) + y_p(x)$$

Example2: Find the general sol. of $(D+1)y = x\sin x$

Solution: The char. Eq. is: $m+1=0 \quad \rightarrow \quad m=-1$

$\rightarrow \quad y_c = c_1 e^{-x}$ the comp. sol.

The particular sol.

$$\rightarrow \quad y_p = \frac{1}{D+1}x\sin x$$

$$\rightarrow \quad y_p = \frac{1}{D+1}\{xe^{ix}\}$$

$$\rightarrow \quad y_p = e^{ix}\frac{1}{D+i+1}\cdot x$$

$$\rightarrow \quad y_p = e^{ix}\frac{1}{(i+1)(1+\frac{D}{i+1})}x$$

$$\rightarrow \quad y_p = \frac{e^{ix}}{i+1}.\frac{1-i}{1-i}\left(1-\frac{D}{i+1}+\cdots\right)x$$

$$\rightarrow \quad y_p = \frac{e^{ix}(1-i)}{2}\left(x - \frac{1}{i+1}\right)$$

$$= \frac{e^{ix}}{2}\left(x(1-i) - \frac{1-i}{1-i}.\frac{1-i}{1-i}\right.$$

$$= \frac{\cos x + i\sin x}{2}\left(x(1-i) + i\right))$$

$$= \frac{x}{2}(\sin x - \cos x) + \frac{1}{2}\cos x$$

$$y = y_c + y_p$$

<u>Exercises</u>

Solve the following diff. equations.

1- $(D^2 + 5D - 6)y = e^{3x}$

2- $(D^2 + 5D - 6)y = e^{-6x}$

3- $(D^4 + 8D^2)y = \sin 2x$

4- $(D^2 + 3D + 2)^2 y = 0$

5- $(D^2 + 2D + 2)y = \cos 5x$

6- $y''' - 27y = e^{2x}$

7- $y''' - 27y = x^4$

8- $(D^2 + 5D)y = e^x x^2$

9- $(5D^2 + 15)y = e^{2x}\cos x$

10- $(2D + 1)y = x\sin 2x$

11- $(D^4 + 8D^2)(D^2 + 2D + 2)y = \sin 3x$

12- $(D^3 - 64)^3 y = \sinh(-4x)$

13- D $(D^2 - 100)^3\ (D^2 - 3D + 2)^2\ (D^4 + 0.5D^2)y$ =0

Section 2: Reducing the order of a differential equation(with variable coefficients)

The second order differential equation is in the form

$\boldsymbol{F(x, y, y', y'') = 0}$ ....(1)

And it can be reduced to the first order according to its type, we have two types here:

The first type: if the dependent variable y does not appear in the equation

اذا لم يظهر المتغير المعتمد y في المعادلة (بمعنى يظهر خالي من المشتقات)

Then we suppose that

$$y' = p \quad \Longrightarrow \quad y'' = \frac{dp}{dx} = p'$$

Then equation (1) will be

$\boldsymbol{G(x, p, p') = 0}$ **................ (2)**

And this is an equation of first order can be solved as in ch.2 to get p, then we return the variable p and solve it to get an equation in term of x and y represents the general sol.

مما سبق نفهم انه لحل معادلة تفاضلية من الرتبة الثانية ذات معاملات متغيرة باستخدام طريقة تخفيض الرتبة توجد حالتين

أ- عندما لا يظهر المتغير المعتمد y في المعادلة

$$y' = p \quad \Longrightarrow \quad y'' = \frac{dp}{dx} = p'$$

نعوض في المعادلة الاصلية فنحصل على معادلة من الرتبة الاولى تحل كما في الفصل الثاني من حيث كونها قابلة للفصل او خطية ...ثم نقوم بارجاع p بدلالة المشتقة $\frac{dy}{dx}$ ونحل المعادلة الناتجة ايضا كما في الفصل الثاني فنحصل على الحل العام

Example (1): solve the following eq.

$$\boldsymbol{x^2 y'' - (y')^2 - 2xy' = 0} \qquad \textbf{..........(3)}$$

Solution:

Note that eq. (34) does not contain the variable y

Let $y' = p \quad \Longrightarrow \quad y'' = \frac{dp}{dx} = p'$

Sub. in (3)

$$x^2\frac{dp}{dx} - p^2 - 2xp = 0$$

$$x^2\frac{dp}{dx} - 2xp = p^2 \qquad \div\ x^2$$

$\frac{dp}{dx} - \frac{2}{x}p = \frac{1}{x^2}p^2$ برنولي ………………………………….. (4)

$p^{-2}\frac{dp}{dx} - \frac{2}{x}p^{-1} = \frac{1}{x^2}$ ……………………………………. (5)

Let $z = p^{-1} \quad \Longrightarrow \quad \frac{dz}{dx} = -p^{-2}\frac{dp}{dx}$

Sub. In (5)

$$\left[-\frac{dz}{dx} - \frac{2}{x}z = \frac{1}{x^2}\right] * -1$$

$\frac{dz}{dx} + \frac{2}{x}z = \frac{-1}{x^2}$ linear

$$I.F = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$x^2 z = \int -\frac{1}{x^2}.x^2 dx + c_1$$

$$x^2 z = -\int dx + c_1$$

$x^2 z = -x + c_1$ ………………………..…………..(6)

$z = \frac{-x+c_1}{x^2}$

Replacing $z = p^{-1}$

$$\frac{1}{p} = \frac{-x+c_1}{x^2} \quad \Longrightarrow \quad p = \frac{x^2}{c_1 - x}$$

$$dy = \left(-x - c_1 + \frac{c_1^2}{c_1 - x}\right)dx$$

Integrating both sides

$y = \frac{-x^2}{2} - c_1 x - c_1^2 \ln|c_1 - x| + c_2$ c_1& c_2 are constant

The second type: If the independent variable x does not appear in the equation

اذا لم يظهر المتغير المستقل x في المعادلة

Then we suppose that

$$y' = p \quad \Longrightarrow \quad y'' = \frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dx}.\frac{dy}{dy} = \frac{dp}{dy}.\frac{dy}{dx} = p\frac{dp}{dy}$$

Then equation (1) will be

$$\boldsymbol{E\left(y, p, p\frac{dp}{dy}\right) = 0} \qquad \textbf{................ (7)}$$

And this equation is of the first order with dependent variable p and independent variable y solve it to get an equation with p & y then return the variable $p = \frac{dy}{dx}$ and solve it to get the general solution.

Example (2): Solve the following eq.

$$\boldsymbol{yy'' + 2y' - (y')^2 = 0} \qquad \textbf{...........(8)}$$

Solution: نلاحظ ان x غير ظاهرة بالمعادلة

$$y' = p \qquad \Longrightarrow \qquad y'' = p\frac{dp}{dy}$$

Sub. in eq. (8), we get:

$$yp\frac{dp}{dy} + 2p - p^2 = 0$$

$$p\left(y\frac{dp}{dy} + 2 - p\right) = 0$$

$p = 0 \quad \Longrightarrow \quad \frac{dy}{dx} = 0 \quad \Longrightarrow \quad y = b$ Singular sol.

Or

$$y\frac{dp}{dy} + 2 - p = 0 \qquad \div y$$

$\frac{dp}{dy} - \frac{1}{y}p = \frac{-2}{y}$ linear

$$I = e^{\int -\frac{1}{y}dy} = e^{-\ln y} = y^{-1}$$

$$y^{-1}p = \int y^{-1}.\frac{-2}{y}dy + c_1$$

$$[y^{-1}p = 2y^{-1} + c_1] * y$$

$p = 2 + c_1 y$ $\quad p$ بارجاع

$$\frac{dy}{dx} = 2 + c_1 y$$

$\frac{dy}{2+c_1 y} = dx \qquad \frac{1}{c_1}\ln|2 + c_1 y| = x + c_2$ where c_1 & c_2 are constants

And this the general sol.

Exercises:

Solve the following equations:

1) $2y'' - (y')^2 + 1 = 0$
2) $y'' + k^2 y = 0$
3) $xy'' = y'$
4) $y'' - k^2 y = 0$
5) $yy'' + (y')^2 = 0$
6) $yy'' + (y')^3 = 0$