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قسم الرياضيات

المرحلة الثانية

المعادلات التفاضلية الاعتيادية

ORDINARY DIFFERENTIAL EQUATIONS

Chapter One

بعض الاساسيات المهمة للمعادلات التفاضلية الاعتيادية

SOME IMPORTANT BASICS OF ORDINARY DIFFERENTIAL EQUATIONS

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Chapter one

SOME IMPORTANT BASICS OF ORDINARY DIFFERENTIAL EQUATIONS

1.1: Introduction: A differential equation is a mathematical equation that relates some function with its derivatives.

The derivatives represent their rates of change, and the equation defines a relationship between two variables.

The differential equations play an important role in many fields such as engineering, physics, economics and biology. Now, Let x be a number in the domain of the function f then we can express the first derivative of the function f for x as follows:

$$If\ \ y = f(x)\ then\ \frac{dy}{dx} = \frac{df(x)}{dx}\ or\ y' = f'(x)\ ,$$

Where the symbols $\frac{d(\cdot)}{dx}$ and $(\cdot)'$ represent the first derivative of the function.

1.2: Definitions

1.2.1: Differential equation

A differential equation is an equation involving derivatives or differentials.

For example:-

$$1 - \left(\frac{dy}{dx}\right)^4 + y = x$$

$$2 - x^2\left(\frac{d^2y}{dx^2}\right)^3 + x\frac{dy}{dx} + y = 0$$

$$3-\frac{d^3y}{dx^3}-\left(\frac{d^2y}{dx^2}\right)^2+\frac{dy}{dx}=x^2+1$$

$$4-y'''+2(y'')^2+y'=cosx$$

$$5-\frac{\partial z}{\partial x}+\frac{\partial^2 z}{\partial y^2}=x^2+y$$

$$6-x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3u$$

1.2.2: Ordinary Differential Equation

Ordinary differential equation is a differential equation involving only ordinary derivatives (i.e.) It has derivatives of one or more dependent variables w.r.t. single independent variable. Such as equations 1,2,3 and4

1.2.3: Partial Differential Equation

A Partial differential equation is a differential equation involving partial derivatives (i.e.) It has derivatives of one or more dependent variable w.r.t. more than one independent variable.

For example the equations 5 and 6 are p.d.es

1.2.4: Order of a Differential Equation

The order of the highest order derivative in a differential equation is called the order of a diff. eq.

For example :-

(i) Equations (1) and (6) are of order one

(ii) Equations (2) and (5) are of order two

(iii) Equations (3) and (4) are of order three

1.2.5: Degree of Differential Equation

The degree of differential equation that is algebraic in its derivatives is the algebraic degree of the highest derivative shown in the equation (i.e.) when the equation is free from radicals and fractions in the dependent variable and its derivatives.

For example :-

(i) Equations (3),(4),(5) and (6) are of first degree

(ii) Equation (2) is of the third degree

(iii) Equation (1) is of the fourth degree

Other examples:- Find the order and degree of the following differential equations:

$$1 - \sqrt[3]{\left(\frac{d^2y}{dx^2}\right)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)}$$

$$2 - \sin(y') = y' + x + 3$$

Solution (1): This equation can be written as

$$\left(\frac{d^2y}{dx^2}\right)^{2/3} = \left(1+\frac{dy}{dx}\right)^{1/2}$$

$$\left[\left(\frac{d^2y}{dx^2}\right)^{2/3}\right]^6 = \left[\left(1+\frac{dy}{dx}\right)^{1/2}\right]^6$$

$$\left(\frac{d^2y}{dx^2}\right)^{4} = \left(1+\frac{dy}{dx}\right)^{3}$$

Therefore, this equation is of second order and fourth degree.

Solution (2): It hasn't degree since it is not algebraic in its derivatives.

1.2.6: Linear Differential Equation

The differential of any order shall be linear if the dependent variable and all derivatives are of the first degree and are not multiplied by each other and its general formula is

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = f(x) \qquad \dots(1)$$

Where $a_0, a_1, \dots, a_n$ and $f(x)$ are functions for x on the interval $a \le x \le b$

An equation that is not linear is said to be nonlinear

For example:-

$1 - 3y^{(3)} + 2y' = 5\sin x$ $\qquad$ $Linear$

$2-x\dfrac{d^2y}{dx^2}-y^2=0$ $\qquad non-Linear$

$3-y^{-1}\dfrac{d^2y}{dx^2}+8y=e^x$ $\qquad non-Linear$

$4-x^2y''+2xy'+y=0$ $\qquad Linear$

$5-y^{(5)}+yy'+2x=0$ $\qquad non-Linear$

$6-y''+5xy'+\dfrac{1}{y}=\sqrt{x+1}$ $\qquad non-Linear$

1.2.7: Homogeneous Linear Differential Equation

Equation (1) is said to be homogeneous if $f(x)=0$

(i.e.) $a_0y^{(n)}+a_1y^{(n-1)}+\cdots+a_{n-1}y'+a_ny=0$ ... (2)

Therefore, the equations (2) and (4) are homogeneous and (1), (3), (5) and (6) are non-homogeneous.

Note: If $a_0, a_1, \dots, a_n$ in equation (1) are constant then the equation is said to be linear differential equation with constant coefficients.

Exercises:

Find the order, degree, linear and homogeneity of the following differential equations:

$1-y''+3y'-2y=0$

$2-(y''')^3+(y'')^2+xy=x$

$3-(y')^4+y^2=0$

$4-\sqrt[3]{(y''')^2}=\sqrt{1+(y')^2}$

$5-\frac{d^2y}{dx^2}-3\frac{dy}{dx}+4y=e^x$

1.3: Solution of the Differential Equation

The Solution of the differential equation is a relation between the variables of the equation and satisfies the following:

(i) Its empty of derivatives

(ii) Satisfies the differential equation

(iii) Defined on a certain interval

Example (1): Is $y(x)=A\sin 2x+B\cos 2x$ a solution of the diff. eq. $y''+4y=0$ ... (3)

Sol. First, we must derive the function that given twice

$$y=A\sin 2x+B\cos 2x \qquad \ldots(4)$$

$$y'=2A\cos 2x-2B\sin 2x \qquad \ldots(5)$$

$$y''=-4A\sin 2x-4B\cos 2x \qquad \ldots(6)$$

Substituting (4), (6) in (3), we get

$$-4A\sin 2x-4B\cos 2x+4(A\sin 2x+B\cos 2x)$$

$$=-4A\sin 2x-4B\cos 2x+4A\sin 2x+4B\cos 2x=0$$

Thus, the given function satisfies eq. (3)

$\therefore\ y(x) = A\sin 2x + B\cos 2x$ is a solution of eq. (3)

Example (2): Prove that the function

$$y(x) = x\ln x - x \qquad \dots (7)$$

is a solution of $\quad xy' = x + y \qquad \dots (8)$

Sol. Deriving (7) w.r.t. x we get

$$y'(x) = x\frac{1}{x} + \ln x - 1$$

$$y'(x) = \ln x \qquad \dots (9)$$

Substituting (7),(9) in (8), we get

$$x\ln x = x + x\ln x - x$$

$$x\ln x = x\ln x$$

Hence, the equation (7) is a solution of the diff. eq. (8).

1.3.1: General solution of the differential equation

The general solution of the differential equation is the solution that is free of derivatives and contains a number of arbitrary constants and their number is equal to the order of the equation.

Example (3): Find the general solution of the equation $y''' = 0$

Sol. Integrating both sides three times

$$\int y'''\,dx = 0\cdot dx \qquad \dots (10)$$

$$y'' = c_1 \qquad \dots (11)$$

$$y' = c_1 x + c_2 \qquad \dots (12)$$

$$y = \frac{c_1}{2} x^2 + c_2 x + c_3 \qquad \dots (13)$$

Where c_1, c_2 and c_3 are arbitrary constants.

Note that, the number of the constants is equal to the order of the equation.

1.3.2: The Particular Solution

It's the solution that results after substituting the values of the arbitrary constants in the general solution.

Example (4): write the particular solution of the equation $y''' = 0$ $when$ $c_1 = 2, c_2 = 2, c_3 = 0.$

Sol. : The solution of $y''' = 0$ is $y(x) = \frac{c_1}{2} x^2 + c_2 x + c_3$ (from Ex(3))

Sub. c_1, c_2 and c_3 in it

$$y(x) = \frac{2}{2} x^2 + 2x + 0$$

$$y(x) = x^2 + 2x$$

Remark: A general solution is a set of solutions that represent curves and are not intersected with each other called integral curves while only one of them passes through a given point of existence of these curves and at this point one real value is determined for the arbitrary constant.

Example (5): Find the general solution and the particular solution of the equation $y' = x$ $\quad\quad\quad ...(14)$

that passes through the point $(1,2)$ and sketch the integral curves.

Sol.: Integrating (14) w.r.t. x we get

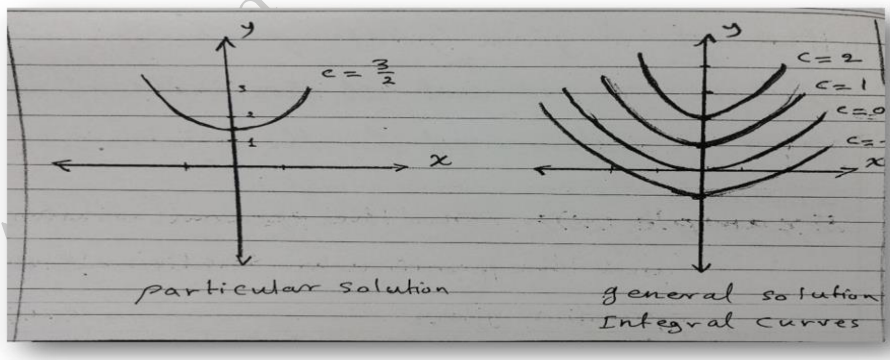
$$y = \frac{x^2}{2} + c \quad\quad\quad ...(15)$$

This is the general solution

To find the particular solution, substituting the point (1,2) in (15)

$$2 = \frac{1}{2} + c \;\rightarrow c = \frac{3}{2}, \text{ then } y = \frac{x^2}{2} + \frac{3}{2} \quad\quad ...(16)$$

And this is the particular solution



1.4: Singular Solution of the Differential Equation