جامعة بغداد

كلية التربية للعلوم الصرفة ابن الهيثم

قسم الرياضيات

المرحلة الثانية

المعادلات التفاضلية الاعتيادية

Ordinary Differential Equations

CHAPTER FOUR

Solution of The Differential Equations of The First Order and Higher Degree

اساتذة المادة

ا.م. اسماء عبد عصواد   أ.م.د. مي محمد هلال

طباعة

أ.م. اسماء عبد عصواد

Solution of The Differential Equations of The First Order and Higher Degree

The general form of the differential equation of the first order and the degree n is:

$$\boldsymbol{p^n + a_1(x,y)p^{n-1} + \cdots + a_{n-1}(x,y)p + a_n(x,y) = 0} \quad \dots\dots\dots \quad \text{(I)}$$

Where n=2,3,4,... and $p = \frac{dy}{dx}$

The differential equations of this type are divided into three cases:

4.1: Equation solvable for $\boldsymbol{p}$

4.2: Equation solvable for $\boldsymbol{y}$

4.3: Equation solvable for $\boldsymbol{x}$

Here we will discuss each of these three types with examples:

4.1: Equation solvable for p:

In this type we can analyze the left side of equation (I), which is considered to be polynomial for p in the form of n linear factors, so we can write equation (I) as the form:

$$\boldsymbol{(p - F_1)(p - F_2)\dots(p - F_n) = 0} \quad \dots\dots\dots\dots \quad \dots\dots\dots\dots\dots. \quad \text{(II)}$$

Where $F_1, F_2, \dots, F_n$ are functions of x and y,

Then equivalent each factor of equation (II) by zero to obtain n of differential equations of order 1 and degree1

في هذه الحالة نقوم بتحليل المعادلة الى حاصل ضرب عدد من العوامل جميعها من الرتبة الاولى والدرجة الاولى باستخدام طرق التحليل المعروفة ثم ناخذ كل عامل على حدة ونرجع p الى المشتقة $\frac{dy}{dx}$ ونكامل

نكرر العملية نفسها لبقية الحدود ثم نضرب الحدود الناتجة فنحصل على الحل العام.

Ex1: Solve $\boldsymbol{x^2p^2 + xyp - 6y^2 = 0}$

Sol: $(xp + 3y)(xp - 2y) = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad (1)$

$$xp + 3y = 0 \Rightarrow xp = -3y \Rightarrow p = \frac{-3y}{x} \Rightarrow \frac{dy}{dx} = \frac{-3y}{x} \Rightarrow \frac{dy}{y} = \frac{-3dx}{x}$$

Integrating both sides, we get:

$ln|y| = -3ln|x| + lnc \Rightarrow$

$ln|y| = ln|x|^{-3} + lnc \Rightarrow y = cx^{-3} \ \Rightarrow \ y - cx^{-3} = 0 \quad \ldots\ldots\ldots\ldots \ (2)$

Now take $(xp - 2y) = 0$ then

$$xp = 2y \Rightarrow x\frac{dy}{dx} = 2y \Rightarrow \frac{dy}{y} = \frac{2dx}{x}$$

Integrating both sides, we get:

$y = cx^2 \ \Rightarrow y - cx^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$

From (2) and (3), we get:

$(\,y - cx^{-3})(y - cx^2) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \quad (4)$

And this is the general solution.

Ex2: Solve the following equation $\boldsymbol{p^2 + py = x^2 + xy}$

Sol: rearrange the equation

$p^2 + py - x^2 - xy = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots. \quad (1)$

$\Rightarrow \ p^2 - x^2 + py - xy = 0$

$\Rightarrow (p - x)(p + x) + y(p - x) = 0$

$\Rightarrow (p - x)(p + x + y) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots. \quad (2)$

Take the first factor,

$$(p - x) = 0 \Rightarrow p = x \Rightarrow \frac{dy}{dx} = x \Rightarrow dy = xdx$$

Integrating both sides to get:

$$y = \frac{x^2}{2} + c \Rightarrow y - \frac{x^2}{2} - c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots. \quad (3)$$

Take the second factor, $dv = e^x dx \Rightarrow v = e^x$

$$p + x + y = 0 \Rightarrow \frac{dy}{dx} + y = -x \ \ and\ this\ is\ a\ linear\ eq.$$

$$I = e^{\int dx} = e^x$$

Substitute in :

$$e^x y = \int -xe^x dx + c \qquad\qquad u = x \Rightarrow du = dx$$

$$dv = e^x dx \Rightarrow v = e^x$$

$$e^x y = -xe^x + e^x + c$$

$$e^x y + xe^x - e^x - c = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

From (3)&(4), we get:

$$\left(y - \frac{x^2}{2} - c\right)(e^x y + xe^x - e^x - c) = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

So eq. (5) is the general sol.

4.2: Equation solvable for *y*

This type of equations can be written as:

$$y = F(x, p) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (III)$$

Differentiating for x we get :

$$\frac{dy}{dx} = \frac{dF}{dx} + \frac{dF}{dp}\cdot\frac{dp}{dx} = F(x, p, \frac{dp}{dx})$$

$$\Rightarrow p = F\left(x, p, \frac{dp}{dx}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (IV)$$

And this equation is of the first order and the first degree to solve it, we analyze the equation into a several factors, one of which contains $\frac{dp}{dx}$ and we get from it the general solution $\emptyset(x, p, c) = 0$

the rest contains p and we get from it the singular solution.

Ex3: Solve $\boldsymbol{2yp - 3x = xp^2}$

Sol: rearrange the equation

$$2y = 3\frac{x}{p} + xp \qquad \text{.......................} \qquad (1)$$

Differentiating both sides:

$$2\frac{dy}{dx} = 3\frac{p - x\frac{dp}{dx}}{p^2} + x\frac{dp}{dx} + p \qquad \text{.......................} \qquad (2)$$

$$2p = \frac{3}{p} - \frac{3x}{p^2} \cdot \frac{dp}{dx} + x\frac{dp}{dx} + p \Rightarrow$$

$$\left[2p - p = \frac{3}{p} - \frac{3x}{p^2} \cdot \frac{dp}{dx} + x\frac{dp}{dx}\ \right] \quad \times p^2$$

$$p^3 - 3p - xp^2\frac{dp}{dx} + 3x\frac{dp}{dx} = 0$$

$$p(p^2 - 3) - x\frac{dp}{dx}(p^2 - 3) = 0$$

$$(p^2 - 3)(p - x\frac{dp}{dx}) = 0 \qquad \text{.......................} \qquad (3)$$

Either $(p - x\frac{dp}{dx}) = 0 \Rightarrow pdx = xdp \Rightarrow \frac{dx}{x} = \frac{dp}{p} \Rightarrow lnp = lnx + lnc$

$$\Rightarrow p = cx \quad \text{..........} \ (4)$$

Substituting (4)in the original equation, we get:

$$2cxy - 3x = c^2x^3 \Rightarrow$$

$$\boldsymbol{y = \frac{cx^2}{2} + \frac{3}{2c}} \qquad \ldots \ldots \ldots (5)$$

Equation (5) is the general solution

Or $p^2 - 3 = 0 \Rightarrow \ p^2 = 3 \Rightarrow p = \pm\sqrt{3}$ .... (6)

Sub. In the original equation, we get:

$$\pm 2\sqrt{3}y - 3x = 3x \Rightarrow$$

$$\boldsymbol{y = \pm\sqrt{3}x} \qquad \ldots \ldots \ldots \ldots \ldots \qquad (7)$$

We note that equation (7) does not contain arbitrary constants, so it does not represent a general solution , but rather a singular solution.

نلاحظ ان المعادلة (7) لا تحوي ثوابت اختيارية وعليه هي لا تمثل حلا عاما وانما تمثل حلا منفردا

Ex4: Solve $16x^2 + 2p^2y - p^3x = 0$ .... **(1)**

Sol: Dividing on p^2we get:

$$2y = px - 16\frac{x^2}{p^2}$$

Deriving for x we get:

$$2\frac{dy}{dx} = p + x\frac{dp}{dx} - 32\frac{x}{p^2} + 32\frac{x^2}{p^3}\frac{dp}{dx}$$

$$\left[\, 2p = p + x\frac{dp}{dx} - 32\frac{x}{p^2} + 32\frac{x^2}{p^3}\frac{dp}{dx} \qquad \right] \times p^3$$

$$p^4 + 32xp - xp^3\frac{dp}{dx} - 32x^2\frac{dp}{dx} = 0$$

$$p(p^3 + 32x) - x(p^3 + 32x)\frac{dp}{dx} = 0$$

$$(p^3 + 32x)\left(p - x\frac{dp}{dx}\right) = 0$$

Either $\left(p - x\frac{dp}{dx}\right) = 0 \Rightarrow p = x\frac{dp}{dx} \Rightarrow \frac{dp}{p} = \frac{dx}{x} \Rightarrow$

$p = cx$ ……….. (2)

Substituting eq. (2) in (1) we get:

$16x^2 + 2c^2x^2y - c^3x^4 = 0$ …………… (3)

Equation (3) is the general solution,

or $p^3 + 32x = 0 \Rightarrow p^3 = -32x \Rightarrow p = \sqrt[3]{-32x}$ ……………(4)

Substituting eq. (4) in (1) we get:

$$16x^2 + 2(32x)^{\frac{2}{3}}y + 32x^2 = 0$$

$$y = \frac{-48x^2}{2(32x)^{\frac{2}{3}}}$$

$$\boldsymbol{y = \frac{-3\,x^{\frac{4}{3}}}{\sqrt[3]{2}}}$$ ……………. (5)

And this is the singular solution.

In this type (equation solvable for y) we have two special cases,

4.2.1. Clairaut's equation

4.2.2. Lagrange's equation

And we will discuss them in the following:

4.2.1. Clairaut's equation: معادلة كليروت

It is a differential equation of the form:

$$\boldsymbol{y = px + f(p)} \qquad \dots\dots\dots\dots\dots\dots\dots \quad (V)$$

we can solve it by differentiating it for x.

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx} \Rightarrow$$

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx} \Rightarrow$$

$$x\frac{dp}{dx} + f'(p)\frac{dp}{dx} = 0 \Rightarrow$$

$$(x + f'(p))\frac{dp}{dx} = 0$$

Either $\frac{dp}{dx} = 0 \Rightarrow p = c \;\; substituting\ in\ (V), we\ get:$

$$\boldsymbol{y = cx + f(c)} \qquad \dots\dots\dots\dots\dots (VI)$$

And this is the general solution.

Or $\left(x + f'(p)\right) = 0 \Rightarrow f'(p) = -x$

Taking $(f')^{-1}$for both sides to get:

$$p = (f')^{-1}(-x)$$

Sub. In (V), we get:

$$\boldsymbol{y = x(f')^{-1}(-x) \; + f(f')^{-1}(-x)} \qquad \dots\dots\dots\dots\dots\dots (VII)$$

And this is the singular solution.

Ex1: Solve $\boldsymbol{y = px + \cos p}$

Sol: $y = px + \cos p$ ....(1)

Deriving eq. (1):

$$\frac{dy}{dx} = p + x\frac{dp}{dx} - sinp \cdot \frac{dp}{dx} \qquad \dots\dots\dots\dots\dots\dots\dots (2)$$

$$p = p + x\frac{dp}{dx} - sinp \cdot \frac{dp}{dx} \quad \Rightarrow$$

$$x\frac{dp}{dx} - sinp \cdot \frac{dp}{dx} = 0 \Rightarrow (x - sinp) \cdot \frac{dp}{dx} = 0$$

Either $(x - sinp) = 0 \Rightarrow \sin p = x \Rightarrow p = sin^{-1}x$

Sub. in (1), we get:

$$\boxed{y = xsin^{-1}x + \cos(sin^{-1}x) \qquad \dots\dots\dots\dots\dots (2)}$$

And this is the singular sol.

Or $\frac{dp}{dx} = 0 \Rightarrow p = c$

Sub. in (1), we get:

$$\boxed{y = cx + \cos c \qquad \dots\dots\dots\dots\dots (3)}$$

And this is the general sol.

Ex 2: Find the general solution of $y = px + \sqrt{4 + p^2}$

Sol: $y = px + \sqrt{4 + p^2} \qquad \dots\dots\dots\dots \quad (1)$

Deriving for x, we get:

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{1}{2\sqrt{4+p^2}}(2p\frac{dp}{dx}) \Rightarrow$$

$$p = p + x\frac{dp}{dx} + \frac{1}{\sqrt{4+p^2}}(p\frac{dp}{dx}) \Rightarrow$$

$$x\frac{dp}{dx} + \frac{1}{\sqrt{4+p^2}}(p\frac{dp}{dx}) = 0 \Rightarrow$$

$$(x + \frac{1}{\sqrt{4+p^2}}p)\frac{dp}{dx} = 0$$

To get the general sol., we take $\frac{dp}{dx} = 0 \Rightarrow p = c$

Sub. in (1), we get:

$$y = cx + \sqrt{4 + c^2} \qquad \dots\dots\dots\dots\dots \quad (2)$$

Eq. (2) represent the general sol.

4.2.2. Lagrange's equation معادلة لاكرانج

It is a differential equation of the form:

$$\boldsymbol{y = xf(p) + g(p) \qquad ; f(p) \neq p \;\&\; p = \frac{dy}{dx}} \qquad \dots\dots \quad \boldsymbol{(VIII)}$$

To find the general solution , we differentiate equation (VIII):

$$\frac{dy}{dx} = f(p) + xf'(p)\frac{dp}{dx} + g'(p)\frac{dp}{dx} \Rightarrow$$

$$p = f(p) + xf'(p)\frac{dp}{dx} + g'(p)\frac{dp}{dx} \Rightarrow$$

$$p = f(p) + (xf'(p) + g'(p))\frac{dp}{dx} \Rightarrow$$

$$\left(xf'(p) + g'(p)\right)\frac{dp}{dx} = p - f(p) \Rightarrow$$

$$\frac{dx}{dp} - \frac{f'(p)}{p-f(p)}x = \frac{g'(p)}{p-f(p)}$$

Where $f'(p) = \frac{d(f(p))}{dp}$, $g'(p) = \frac{d(g(p))}{dp}$,

This equation is a linear differential equation of order 1 with two variables x , p. and I=$e^{\int -\frac{f'(p)}{p-f(p)}dp}$, Q=$\frac{g'(p)}{p-f(p)}$

Ex1: find the general solution of the differential equation

$$\boldsymbol{y = 2px + p^3} \qquad \dots\dots\dots \quad \mathbf{(1)}$$

Sol: Equation (1) in Lagrange form where $f(p) = 2p$ & $g(p) = p^3$, deriving both sides:

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + 3p^2\frac{dp}{dx} \Rightarrow$$

$$p = 2p + (2x + 3p^2)\frac{dp}{dx} \Rightarrow$$

$$-p = (2x + 3p^2)\frac{dp}{dx} \Rightarrow$$

$$\frac{dx}{dp} = \frac{(2x+3p^2)}{-p} \Rightarrow \frac{dx}{dp} + \frac{2x}{p} = -3p \qquad (and\ this\ is\ a\ linear\ eq.)$$

$$I = e^{\int \frac{2}{p} dp} = p^2 \text{ , } Q(p) = -3p$$

Hence, we get:

$$p^2 x = \int (-3p)\, p^2 dp + c$$

$$p^2 x = \frac{-3}{4} p^4 + c \qquad \dots\dots\dots\dots\dots\dots\dots \qquad (2)$$

نرتب المعادلة (2) ثم نستخدم اكمال المربع لايجاد قيمة p

$$p^4 + \frac{4}{3} p^2 x = \frac{4}{3} c$$

$$p^4 + \frac{4}{3} p^2 x + \frac{4}{9} x^2 = \frac{4}{3} c + \frac{4}{9} x^2$$

$$(p^2 + \frac{2}{3} x)^2 = \frac{4}{3} c + \frac{4}{9} x^2$$

نجذر الطرفين

$$p^2 = \pm\sqrt{\frac{4}{3} c + \frac{4}{9} x^2} - \frac{2}{3} x \Rightarrow$$

$$p = \pm(\pm\sqrt{\frac{4}{3} c + \frac{4}{9} x^2} - \frac{2}{3} x)^{\frac{1}{2}} \qquad \dots\dots\dots\dots \quad (3)$$

Sub. (3) in (1), we get:

$$y = \pm 2x(\pm\sqrt{\frac{4}{3} c + \frac{4}{9} x^2} - \frac{2}{3} x)^{\frac{1}{2}} \pm (\pm\sqrt{\frac{4}{3} c + \frac{4}{9} x^2} - \frac{2}{3} x)^{\frac{3}{2}} \dots\dots\dots(4)$$

And this is the general solution.

4.3: Equation solvable for *x*

This equation can be written as

$\boldsymbol{x = f(y,p)}$ ……………… $\boldsymbol{(IX)}$

Deriving for y, we get:

$$\frac{dx}{dy} = \frac{df}{dy} + \frac{df}{dp}\frac{dp}{dy} = F\left(y, p, \frac{dp}{dy}\right) \Rightarrow$$

$$\frac{1}{p} = F\left(y, p, \frac{dp}{dy}\right) \qquad \text{……………….. (X)}$$

Eq. (X) is a diff. eq. of order 1 and degree 1 whose solution is :

$$\emptyset(y, p, c) = 0 \qquad \text{……………… } (XI)$$

From eqs. (IX) &(XI) we get the general solution.

من المعادلتين (IX) &(XI) نحصل على الحل العام عن طريق تعويض احداهما بالاخرى وحذف المتغير p

Ex1: Solve $\boldsymbol{p^3 - 2xyp + 4y^2 = 0}$ ………………$\boldsymbol{(1)}$

Sol: Rearrange equation(1) as:

$$2x = \frac{p^2}{y} + 4\frac{y}{p} \qquad \text{………….(2)}$$

Deriving for y, we get:

$$2\frac{dx}{dy} = \frac{2yp\frac{dp}{dy} - p^2}{y^2} + 4\frac{p - y\frac{dp}{dy}}{p^2} \Rightarrow$$

$$\left[\frac{2}{p} = \frac{2p}{y}\cdot\frac{dp}{dy} - \frac{p^2}{y^2} + \frac{4}{p} - \frac{4y}{p^2}\cdot\frac{dp}{dy}\ \right] \qquad \times p^2y^2 \quad \Rightarrow$$

$$2py^2 = 2p^3y\frac{dp}{dy} - p^4 + 4py^2 - 4y^3\frac{dp}{dy}$$

$$2py^2 - p^4 + 2y(p^3 - 2y^2)\frac{dp}{dy} = 0 \quad \Rightarrow$$

$$p(2y^2 - p^3) - 2y(2y^2 - p^3)\frac{dp}{dy} = 0 \Rightarrow$$

$$(p - 2y\frac{dp}{dy})(2y^2 - p^3) = 0$$

Taking the first term $\left(p - 2y\frac{dp}{dy}\right) = 0 \Rightarrow \frac{dy}{y} = 2\frac{dp}{p}$

Integrating both sides, we get:

$$ln|y| + lnc = 2ln|p| \Rightarrow lnp^2 = ln|cy| \Rightarrow$$

$$p^2 = cy \quad \Rightarrow p = \pm\sqrt{cy} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Sub (3) in (2):

$$2x = \frac{cy}{y} + 4\frac{y}{\sqrt{cy}} \Rightarrow$$

$$2x - c = 4\frac{y}{\sqrt{cy}} \Rightarrow (2x-c)^2 = \frac{16y}{c} \Rightarrow$$

$$y = \frac{c(2x-c)^2}{16} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Eq. (4)is the general sol.

Now taking the second term $(2y^2 - p^3) = 0$, we get:

$$p^3 = 2y^2 \Rightarrow p = \sqrt[3]{2y^2}$$

Sub in eq. (2)

$$2x = \frac{(2y^2)^{2/3}}{y} + 4\frac{y}{\sqrt[3]{2y^2}} \qquad \ldots\ldots\ldots\ldots\ldots (5)$$

And this is the singular solution.

Ex 2: Solve the following diff. eq.

$$\boldsymbol{p^2 x = 2yp - 3} \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

Sol: Dividing on p^2

$$x = \frac{2y}{p} - \frac{3}{p^2} \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

Deriving for y

$$\frac{dx}{dy} = \frac{2}{p} - 2yp^{-2}\frac{dp}{dy} + 6p^{-3}\frac{dp}{dy} \Rightarrow$$

$$\left[\frac{1}{p} = \frac{2}{p} + (-2yp^{-2} + 6p^{-3})\frac{dp}{dy} \qquad \right] \quad \times p^3$$

$$p^2 - 2p^2 = (6 - 2yp)\frac{dp}{dy} \quad \Rightarrow$$

$$-p^2 = (6-2yp)\frac{dp}{dy} \qquad \Rightarrow$$

$$-p^2dy = 6dp - 2ypdp \ \Rightarrow$$

$$[\ 2ypdp - p^2dy = 6dp\ \ ] \qquad \div -p^4$$

$$\frac{p^2dy-2ypdp}{p^4} = \frac{-6}{p^4}dp \qquad \Rightarrow$$

$$d\left(\frac{y}{p^2}\right) = \frac{-6}{p^4}dp$$

Integrating both sides:

$$\frac{y}{p^2} = 2p^{-3} + c$$

$$y = 2p^{-1} + cp^2 \qquad \text{........................(3)}$$

Sub. In eq. (1)

$$p^2x = 2(2p^{-1}+cp^2)p - 3 \Rightarrow$$

$$p^2x = 2cp^3 + 1 \qquad \text{........................(4)}$$

هنا ايجاد p صعب جدا بهذا الشكل لذا سنستفيد من معادلة (3)

From eq. (3):

$$cp^3 = py - 2$$

Sub. In (4):

$$p^2x = 2py - 3 \ \Rightarrow \ p^2x - 2py + 3 = 0$$ بالدستور

$$p = \frac{y \pm \sqrt{y^2-3x}}{x}$$

Sub. In (3), we get:

$$y = \frac{2x}{y \pm \sqrt{y^2-3x}} + c\left(\frac{y \pm \sqrt{y^2-3x}}{x}\right)^2 \qquad \text{......................(5)}$$

Eq. (5) represent the general sol.

■**Question**: Can you solve example (2) using the second case? Is the result you will get equal to the output you have?

Exercises:

Solve the following equations:

1. $y = 3px + 6y^2p^2$

2. $xp^2 + (y - 1 - x^2)p - x(y - 1) = 0$

3. $y^2(1 + p^2) = 1$

4. $y = px + \sqrt{4 + p^2}$

5. $y^2p^2 - 3xyp + 2x^2 = 0$

6. $p^2 - p - 6 = 0$

7. $p^2 - 4xp - 12x^2 = 0$

8. $p^2 + xp + yp + xy = 0$

9. $py - 2p^4 + 2 = 0$

10. $y = p\sin p + \cos p$

11. $x - 2p - lnp = 0$

12. Find the general solution of:

a. $(y - px)^2 = \sin(y - px) + p^2$

b. $e^{y-px} = (y - px)^2 - p^3$

Hint: derive for x

13. $y = px + p^2$

14. $px = y + p^3$

15. $4y - 4px\ ln|x| = p^2x^2$

16. $yp^2 - 2xp + y = 0$

17. $y = 2xp + p\ ln|p|$