جامعة بغداد

كلية التربية للعلوم الصرفة ابن الهيثم

قسم الرياضيات

المرحلة الثانية

المعادلات التفاضلية الاعتيادية

Ordinary Differential Equations

CHAPTER FIVE

THE DIFFERENTIAL EQUATIONS OF HIGHER ORDER

اساتذة المادة

ا.م. اسماء عبد عصواد   أ.م.د. مي محمد هلال   ا.م.سماهر مرز ياسين

طباعة

ا.م. اسماء عبد عصواد

Section 5.1: Linear Differential equations of order n

Definition (5.1.1): The general form of linear differential equation (LDE) of order n is:

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = f(x) \quad \ldots\ldots\ldots\ (1)$$

Where $a_0(x) \neq 0$

■If all the coefficients $a_0, a_1, \ldots, a_n$ are constants then eq.(1) is called ***linear equation with constant coefficients.***

■but if at least one of the coefficients is a function of *x* then eq.(1) is called ***linear equation with variable coefficients.***

■In eq.(1); If *f(x)*=0 then eq.(1) becomes ***homogeneous equation*** and has the form

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \quad \ldots\ \ldots\ldots\ (2)$$

المعادلة (2) معادلة تفاضلية خطية متجانسة لان *f(x)*=0

وعليه المعادلة (1) تسمى معادلة تفاضلية غير متجانسة عندما *f(x)*≠0

Example: The following equations are LDE with constant coefficients

المعادلات الاتية هي معادلات تفاضلية خطية ذات معاملات ثابتة

1) $y'' + y' + y = x^3$
2) $y'' + y = \cos x$
3) $y'' - 3y' + y = 4x^3 + 4$
4) $\frac{d^4y}{dx^4} + \frac{d^3y}{dx^3} + \frac{dy}{dx} = 0$ نلاحظ ان المعادلة 4 متجانسة

Theorem5.1.1: If the functions $y_1, y_2, \ldots, y_n$ are solutions of the homo. eq. (2) and $c_1, c_2, \ldots, c_n$ are constants, then

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n \quad \ldots\ldots.(3)$$

Is a solution of eq. (2) also.

Definition (5.1.2): The functions $y_1, y_2, \ldots, y_n$ are called (***linear dependent***) on the set I if we found numbers $(c_1, c_2, \ldots c_n)$ are <u>not all equal to zero</u> where

$$c_1y_1 + c_2y_2 + \cdots + c_ny_n = 0 \quad \ldots\ldots\ldots.(4)$$

And they are called (***linear independent***) on the set I if we found numbers ($c_1, c_2, \dots c_n$) that all equal to zero (i.e.)

$c_1 = c_2 = \cdots = c_n = 0$ where :

$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$

linear dependent معتمدة خطيا

linear independent مستقلة خطيا

Section 5.2: The Wronskian Determinant محدد رونسكي

The Wronskian determinant of differentiable functions $y_1, y_2, \dots, y_n$ on the interval I is:

$$W = W(y_1, y_2, \dots y_n) = \begin{vmatrix} y_1 & , & y_2 & , \dots , & y_n \\ y'_1 & , & y'_2 & , \dots , & y'_n \\ y''_1 & , & y''_2 & , \dots , & y''_n \\ & & . & & \\ & & . & & \\ & & . & & \\ y_1^{(n-1)} & , & y_2^{(n-1)} & , \dots , & y_n^{(n-1)} \end{vmatrix} \quad \dots\dots\dots\dots(5)$$

Theorem 5.2.1: If the functions $y_1, y_2, \dots, y_n$ are solutions of the homogeneous equation (2) then if:

■ $W = W(y_1, y_2, \dots y_n) = 0$ then $y_1, y_2, \dots, y_n$ are ***linearly dependent solutions***

And if

■ $W = W(y_1, y_2, \dots y_n) \neq 0$ then $y_1, y_2, \dots, y_n$ are ***linearly independent solutions***

Ex1: Prove that the functions $e^x, 2e^x, e^{-x}$ are linearly dependent on the interval (-∞, ∞).

Sol: $W(e^x, 2e^x, e^{-x}) = \begin{vmatrix} e^x & 2e^x & e^{-x} \\ e^x & 2e^x & -e^{-x} \\ e^x & 2e^x & e^{-x} \end{vmatrix}$

$$= e^x \begin{vmatrix} 2e^x & -e^{-x} \\ 2e^x & e^{-x} \end{vmatrix} - 2e^x \begin{vmatrix} e^x & -e^{-x} \\ e^x & e^{-x} \end{vmatrix} + e^{-x} \begin{vmatrix} e^x & 2e^x \\ e^x & 2e^x \end{vmatrix}$$

$$= e^x(2+2) - 2e^x(1+1) + e^{-x}(0)$$

$$= 4e^x - 4e^x = 0$$

Then $e^x, 2e^x, e^{-x}$ are linearly dependent.

Ex2: Prove that $\mathbf{\sin x}$, $\mathbf{\cos x}$ are two linearly independent solutions of the diff. eq. $y'' + y = 0 \ \ in \ \ -\infty < x < \infty$.

Sol:

Let $y_1 = \sin x$, $y_2 = \cos x$ then

$y'_1 = \cos x$, $y'_2 = -sinx$ and

$y''_1 = -\sin x$, $y''_2 = -\cos x$

So

$$y'' + y = 0$$

1- $y''_1 + y_1 = -cosx + cosx = 0$
2- $y''_2 + y_2 = -sinx + sinx = 0$

كلاهما حل للمعادلة التفاضلية الان نتاكد هل ان الحلول مرتبطة ام مستقلة خطيا؟

$$w(cosx, \sin x) = \begin{vmatrix} cosx & sinx \\ -sinx & cosx \end{vmatrix} = \cos^2 x + sin^2 x = 1$$

So the two solutions are linearly independent.

Ex3: Show whether the functions xe^x, x^2e^x ; $x \neq 0$ are linearly dependent or linearly independent.

Sol:

$$\begin{vmatrix} xe^x & x^2e^x \\ xe^x + e^x & x^2e^x + 2xe^x \end{vmatrix} = xe^x(x^2e^x + 2xe^x) - x^2e^x(xe^x + e^x)$$

$$= x^3e^{2x} + 2x^2e^{2x} - x^3e^{2x} - x^2e^{2x}$$

$$= x^2e^{2x} \neq 0$$

the functions xe^x, x^2e^x are linearly independent

<u>H.W1</u>: Show whether the functions $\boldsymbol{e^x cosx}$, $\boldsymbol{e^x sinx}$ are linearly dependent or linearly independent.

<u>H.W2</u>: prove that $\boldsymbol{y_1 = e^{ax} cosbx, y_2 = e^{ax} sinbx}$ are sol. of diff. eq.

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0$$

Then show that the solutions are linearly independent

H.W3: Show whether the functions $\boldsymbol{secx}, \boldsymbol{tanx}$ are linearly dependent or linearly independent.

Section 5.3: The Differential Operator (D) المؤثر التفاضليD

Let D denote differentiation with respect to x that is

$Dy = \frac{dy}{dx}, D^2y = \frac{d^2y}{dx}$ and so on , that is, for positive integer k

$D^k y = \frac{d^k y}{dx^k}$

لاحظ ان هذه الدوال التفاضلية لاتعني شيئا الا اذا اثرت على دالة ما للمتغير المستقل

For example, $D(cos4x) = -4sin4x$

$$D(5x^3 - 6x^2) = 15x^2 - 12x$$

The expression: $\mathrm{F}(D) = a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n$ ………(6)

Is called a differential operator of order n.

It may be defined as that operator which, when applied to any function y, yields the result:

$$F(D)y = (a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y$$

$$= a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny \qquad \text{………………….(7)}$$

Where $a_0, a_1, \dots, a_{n-1}, a_n$ are (constant or variable) coefficients

Properties of the operator D: خواص المؤثر D

1. **Commutative Property** **الخاصية الابدالية**

If the operators of the linear differential equation have constant coefficients, then the operators are commutative. But if some of them are variables then the operators are not commutative.

اذا كانت المؤثرات للمعادلات التفاضلية الخطية ذات معاملات ثابتة فان علاقة الضرب تكون ابدالية عليها، اما اذا كانت بعض المعاملات دوالا فأن العلاقة على المؤثرات تكون غير ابدالية.

Ex1: **Let y be any function,**$\boldsymbol{A = D + 2 \ \ and \ \ B = 3D - 1 \,, prove\ that}$

$\boldsymbol{ABy = BAy}$.

Sol: $By = (3D-1)y = 3\frac{dy}{dx} - y$

Then $A(By) = (D+2)\left(3\frac{dy}{dx} - y\right) = \left(\frac{d}{dx} + 2\right)\left(3\frac{dy}{dx} - y\right)$

$$= 3\frac{d^2y}{dx^2} - \frac{dy}{dx} + 6\frac{dy}{dx}$$

$$= 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y$$

$$= (3D^2 + 5D - 2)y$$

Now to find BAy

$Ay = (D+2)y = \frac{dy}{dx} + 2y$

$\rightarrow B(Ay) = (3D-1)\left(\frac{dy}{dx} + 2y\right)$

$$= \left(3\frac{d}{dx} - 1\right)\left(\frac{dy}{dx} + 2y\right)$$

$$= \frac{3d^2y}{dx^2} + 6\frac{dy}{dx} - \frac{dy}{dx} - 2y$$

$$= 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y$$

$$= (3D^2 + 5D - 2)y$$

Hence $BAy = ABy$ (the operators are commutative).

Ex .2: **Let $\boldsymbol{G = xD + 2}$ *and* $\boldsymbol{H = D - 1}$, Is *G·Hy*=*H·Gy* , show this.**

Sol.: $G(Hy) = (xD+2)(D-1)y = (xD+2)\left(\frac{dy}{dx} - y\right)$

$$= x\frac{d^2y}{dx^2} - x\frac{dy}{dx} + \frac{dy}{dx} - 2y$$

$$= x\frac{d^2y}{dx^2} + (-x)\frac{dy}{dx} - 2y$$

So $GHy = [xD^2 + (2-x)D - 2]y$

On other hand

$H(Gy) = (D-1)(xD+2)y = (D-1)(x\frac{dy}{dx} + 2y)$

$$= x\frac{d^2y}{dx^2} + \frac{dy}{dx} + 2\frac{dy}{dx} - x\frac{dy}{dx} - 2y$$

$$= x\frac{d^2y}{dx^2} + (3-x)\frac{dy}{dx} - 2y$$

Then $HGy = [xD^2 + (3-x)D - 2]y$

$\therefore HGy \neq GHy$ (the operators are not commutative).

2) $D^m f(x) \cdot D^n g(x) = D^n g(x) \cdot D^m f(x)$

3) $D^m . D^n = D^n . D^m = D^{m+n}$

$(i.e)\ \ D^m[D^n f(x)] = D^n[D^m f(x)] = D^{m+n} f(x)$

4) $D[c_1 f(x) + C_2 g(x)] = C_1 Df(x) + C_2 Dg(x)$

Where $f(x), g(x)$differential function and C_1, C_2 are constant.

Theorems about the operator D:

Theorem 5.3.1: if b is a number then $F(D)e^{bx} = F(b)e^{bx}$ ……(8)

Proof:

$$De^{bx} = be^{bx}, D^2e^{bx} = b^2e^{bx} \dots\dots D^n e^{bx} = b^n e^{bx}$$

$$\because F(D)e^{bx} = (D^n + a_1D^{n-1} + \dots + a_{n-1}D + a_n)e^{bx}$$

$$= (b^n + a_1 b^{n-1} + \dots + a_{n-1}b + a_n)e^{bx}$$

$$= F(b)e^{bx}$$

Example1: **find $(D^2 + 3D + 2)e^{3x}$**

Sol. : $b = 3$,

$$F(D) = D^2 + 3D + 2$$

$$F(3) = 3^2 + 3.3 + 2 = 20$$

$$(D^2 + 3D + 2)e^{3x} = F(3)e^{3x} = 20e^{3x}$$

Theorem 5.3.2: If y is a differentiable function and b is a number then

$$F(D).\{e^{bx}y\} = e^{bx}F(D+b)y \qquad \dots\dots\dots (9)$$

Proof:

$$D\{e^{bx}y\} = e^{bx}Dy + be^{bx}y = e^{bx}(D+b)y$$

Also $D^2\{e^{bx}y\} = D\{e^{bx}(D+b)y\} = e^{bx}\{(D+b)(D+b)y\}$

$$= e^{bx}(D+b)^2 y$$

Continuing in the same way, we get:

$D^n\{e^{bx}y\} = e^{bx}(D+b)^n y$

$\therefore F(D)\{e^{bx}y\} = e^{bx}F(D+b)y$

Example2: Find $(\boldsymbol{D^2-4D+1})\{\boldsymbol{e^{2x}y}\}$

Sol.: $\because b = 2$,

$F(D) = D^2 - 4D + 1$

$\Longrightarrow F(D+b) = F(D+2)$

$= (D+2)^2 - 4(D+2) + 1$

$= D^2 + 4D + 4 - 4D - 8 + 1 = D^2 - 3$

$\therefore (D^2 - 4D + 1)\{e^{2x}y\} = e^{2x}(D^2 - 3)y$

$\boxed{\text{Theorem 5.3.3}}$: if it is $\boldsymbol{b}$ number then

a- $\boldsymbol{F(D^2)\sin bx = F(-b^2)\sin bx}$ ………….(10)

b- $\boldsymbol{F(D^2)\cos bx = F(-b^2)\cos bx}$ ………….(11)

Proof:

$\because F(D^2)\sin bx = (D^{2n} + a_1 D^{2(n-1)} + \cdots + a_{n-1}D^2 + a_n)\sin bx$ …... (12)

We will start with derivatives sequentially:

$D(\sin bx) = b\cos bx$

$D^2(sinbx) = -b^2 \sin bx$ …….. (13)

$D^4(\sin bx) = (D^2)^2 \sin bx = (-b^2)^2 \sin bx$ ……... (14)

$D^6(\sin bx) = (D^2)^3 \sin bx = (-b^2)^3 \sin bx$ ……… (15)

And so on,

substituting equations (13),(14), (15) in (12) , we get:

$$F(D^2)\sin bx = [(-b^2)^n + a_1(-b^2)^{n-1} + \cdots + a_{n-1}(-b^2) + a_n]\sin bx$$
$$= F(-b^2)\sin bx$$

And by the same away we prove, $F(D^2)\cos bx = F(-b^2)\cos bx$

Remark:

1) $D(\sin bx) = b\cos bx$
2) $D^3(sin\,bx) = D^2.D\,sin\,bx = (-b^2).D\,sin\,bx = -b^3\cos bx$
3) $D^5(sin\,bx) = D^4.D\,sin\,bx = (b^4).D\,sin\,bx = b^5\cos bx$
   And so on .

Example3: Find

a- $(\boldsymbol{D^4 + 3D^2 - 1})\ \mathbf{sin\,2}\boldsymbol{x}$
b- $(\boldsymbol{D^4 - 2D^2})\ \mathbf{cos\,2}\boldsymbol{x}$

Sol.: a

$$b = 2,\ f(D^2) = D^4 + 3D^2 - 1$$

$$\Longrightarrow f(-2^2) = \{(-2^2)^2 + (-2^2) - 1\} = 16 - 12 - 1 = 3$$

$$\therefore (D^4 + 3\mathrm{D}^2 - 1)\sin 2x = 3\sin 2x$$

Sol.: b

$(D^4 - 2D^2)\cos 2x = \{(-2^2)^2 - 2(-2^2)\}\cos 2x$

$= \{16 + 8\}\cos 2x = 24\cos 2x$

Example4: Prove $(\boldsymbol{D+1})(\boldsymbol{D^2+2})\ \mathbf{sin\,2}\boldsymbol{x} = (\boldsymbol{D^2+2})(\boldsymbol{D+1})\ \mathbf{sin\,2}\boldsymbol{x}$

$\boldsymbol{Sol}$: $(D+1)[D^2\sin 2x + 2\sin 2x]$

$= (D+1)[-4\sin 2x + 2\sin 2x]$

$= -4D\sin 2x + 2D\sin 2x - 4\sin 2x + 2\sin 2x$

$= -8\sin 2x + 4cos2x - 4sin2x + 2sin2x$

$= -4cos2x - 2sin2x$

الاتجاه الثاني واجب بيتي

H.W

1- Is (D + 1)(D + 2x)y = (D + 2x)(D + 1)y
2- Is $(D+x)(D+2x)e^x = (D+2x)(D+x)e^x$
3- Find $(D^2+1)^2 e^{2x}$
4- Find $(D^4 + 2D^2 + 1)\cos 3x$
5- Find $(D^3 + 2D^2)(\sin 2x + e^{6x})$

Section 5.4: Solution the linear differential equation by reduce order to the first order

حل المعادلة التفاضلية الخطية بتخفيض رتبتها الى الرتبة الاولى

قبل اعطاء الصيغة العامة لهذه الطريقة سوف نعطي بعض الامثلة التوضيحية

Example 1: Solve the eq. $\boldsymbol{y'' - 2y' - 3y = 5e^{-2x}}$

سوف نتبع الخطوات الاتية

Sol: $(D^2 - 2D - 3)y = 5e^{-2x}$ — 1- نكتب المعادلة بدلالة المؤثر D ونحللها

$(D + 1)(D - 3)y = 5e^{-2x}$ — 2- نفرض فرضية $(D - 3)y = u$

$Let\ u = (D - 3)y$ — 3- نعوض هذي الفرضية في المعادلة السابقة

$\rightarrow\ (D + 1)u = 5e^{-2x}$

$\frac{du}{dx} + u = 5e^{-2x}$ (linear eq.)

$I = e^{\int dx} = e^x$ integral factor

$u.e^x = \int e^x.5e^{-2x}dx + c$

$[u.e^x = -5e^{-x} + c].e^{-x}$

$\longrightarrow u = -5e^{-2x} + ce^{-x}$

$\longrightarrow (D - 3).y = u\ \longrightarrow (D - 3)y = -5e^{-2x} + c_1e^{-x}$

$I = e^{-3x}$

$I.y = \int e^{-3x}(-5e^{-2x} + ce^{-x})dx + c_1$

$\longrightarrow e^{-3x}.y = \left(e^{-5x} - \frac{c_1}{4}e^{-4x} + c_2\right)e^{3x}$

$y = e^{-2x} - \frac{c_1}{4}e^{-x} + c_2e^{3x}$ The general sol. where $c_1\&c_2$ are arbitrary constants.

Example2: Solve $\boldsymbol{y''' + 2y'' + 5y' - 6y = 0}$

Sol:

$(D^3 + 2D - 5D - 6)y = 0$

$D = -1$ is the root of equation

اي اذا نعوض(1-) سوف تصفر المعادلة بحيث يحقق المعادلة

$D=-1 \Longrightarrow (D+1)=0$

$\Longrightarrow (D+1)(D^2+D-6).y=0$

$(D+1)(D+3)(D-2).y=0$ ………………………………. (*)

Let $u=(D+3)(D-2)y$ ……………………………….. (**)

sub. in (*)

$\Longrightarrow (\mathrm{D}+1)\mathrm{u}=0$

$\rightarrow \dfrac{du}{dx}+u=0$

$\rightarrow \int \dfrac{du}{u}+\int dx=\int 0\,dx \quad \rightarrow \ln u+x=c$

$\rightarrow u=e^{c-x}$

$\rightarrow u=e^{c}.e^{-x} \quad , \quad e^{c}=A$

$\rightarrow u=Ae^{-x}$ ………………………….. (***)

Sub.(**) in (***), we get:

$(D+3)(D-2)y=Ae^{-x}$

Let $u_1=(D-2)y$

$(D+3)u_1=Ae^{-x}$

$\longrightarrow \frac{du_1}{dx}+3u_1=Ae^{-x}$ linear

I=e^{3x}

$\rightarrow e^{3x}.u_1=\int e^{3x}.Ae^{-x}dx+c_1$

$e^{3x}.u_1=A\int e^{2x}dx+c_1 \rightarrow (e^{3x}u_1=\dfrac{A}{2}e^{2x}+c_1)e^{-3x}$

$\rightarrow u_1=\dfrac{A}{2}e^{-x}+c_1e^{-3x} \quad ; \quad let\ \dfrac{A}{2}=c_2$

$\rightarrow u_1=c_2e^{-x}+c_1e^{-3x}$

نعوض عن قيمة u_1 في الفرضية

$\rightarrow (D-2)y=c_1e^{-3x}+c_2e^{-x} \rightarrow \dfrac{dy}{dx}-2y=c_1e^{-3x}+c_2e^{-x}$

$\rightarrow y = \frac{-c_1}{5} e^{-3x} + \frac{-c_2}{3} e^{-x} + c_3 e^{2x}$

$\rightarrow y = Ke^{-3x} + Be^{-x} + c_3 e^{2x}$ the general sol. where K, B, c_3 are arb. const.

Example3: Solve $(D-2)^3 y = 2x$

Sol.: $(D-2)(D-2)(D-2)y = 2x$

Let $u_1 = (D-2)(D-2)y$

$\rightarrow (D-2)u_1 = 2x$

$\rightarrow \frac{du_1}{dx} - 2u_1 = 2x$; $I = e^{-2x}$ Integration factor

$e^{-2x}u_1 = \int 2xe^{-2x}dx + c_1$

$\Longrightarrow e^{-2x}u_1 = \left[-xe^{-2x} - \frac{1}{2}e^{-2x} + c_1\right] * e^{2x}$

$\longrightarrow u_1 = -x - \frac{1}{2} + c_1 e^{2x}$

$(D-2)(D-2)y = -x - \frac{1}{2} + c_1 e^{2x}$

Let $u_2 = (D-2)y$

$\Longrightarrow (D-2)u_2 = -x - \frac{1}{2} + c_1 e^{2x}$

$I = e^{-2x}$

$\Longrightarrow e^{-2x}u_2 = \int -xe^{-2x}dx + \int -\frac{1}{2}e^{-2x} + \int c_1 dx$

$\Longrightarrow e^{-2x}u_1 = \frac{x}{2}e^{-2x} + \frac{1}{4}e^{-2x} + \frac{1}{4}e^{-2x}c_1 x + c_2$

$\Longrightarrow u_2 = \frac{x}{2} + \frac{1}{4} + \frac{1}{4} + c_1 xe^{2x} + c_2 e^{2x}$

$\Longrightarrow u_2 = \frac{x}{2} + \frac{1}{2} + c_1 xe^{2x} + c_2 e^{2x}$

$\Longrightarrow (D-2)y = \frac{x}{2} + \frac{1}{2} + c_1 xe^{2x} + c_2 e^{2x}$

وبنفس الاسلوب نحصل على

$y = -\frac{x}{4} - \frac{3}{8} + (c_1 \frac{x^2}{2} + c_2 x + c_3)e^{2x}$ الحل العام

The general rule of (5-4)

We can find a general solution for linear diff. eq. with fixed coefficients

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = f(x)$$

Where we first write the differential equation with the effect of D

$$(D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n) y = f(x)$$

ثم نكتب الجزء الايسر بدلالة عوامل التاثير

$(D - m_1)(D - m_2) \dots (D - m_n) y = f(x)$ ……………………………….(16)

Let $u_1 = (D - m_2)(D - m_3) \dots (D - m_n) y$ …………………………….....(17)

نعوض قيمة u_1 في المعادلة السابقة

$\Longrightarrow (D - m_1) u_1 = f(x)$ linear equation

The general solution of the above equation is

$$u_1 . e^{-m_1 x} = \int e^{-m_1 x} . f(x) dx + c$$

$$u_1 = e^{m_1 x} \int e^{-m_1 x} f(x) dx + c e^{m_1 x}$$

نعوض في الفرضية لنحصل على معادلة خطية من الرتبة (n-1)

$$(D - m_2)(D - m_3) \dots (D - m_n) . y = e^{m_1 x} \int e^{-m_1 x} f(x) dx + c\, e^{m_1 x}$$

$(D - m_2) u_2 = e^{m_1 x} \int e^{-m_1 x} f(x) dx$ $e^{-m_1 x}$ عامل التكامل

$$u_2 . e^{-m_2 x} = \int e^{-m_2 x} . e^{m_1 x} \int e^{-m_1 x} . f(x) (dx)^2$$

$$u_2 = e^{m_2 x} \int e^{(m_1 - m_2) x} . \int e^{-m_1 x} . f(x) (dx)^2$$

ونستمر بهذا الاسلوب حيث سيكون الحل العام للمعادلة التفاضلية الخطية ذات المعاملات الثابتة

$$\boxed{\boldsymbol{y = e^{m_n x} \int e^{(m_{n-1} - m_n) x} . \int e^{(m_{n-2} - m_{n-1}) x} \int e^{(m_{n-3} - m_{n-2}) x} \dots \int e^{-m_1 x} f(x) (dx)^n} \; \textbf{…(18)}}$$

تحسب التكاملات بصورة متتالية

We will re-solve the previous examples using the general rule

Example 4: Solve the eq. $\boldsymbol{y'' - 2y' - 3y = 5e^{-2x}}$ **(the same ex.1)**

Sol.:$m_1 = -1, m_2 = 3, f(x) = 5e^{-2x}$ and

$$y = e^{m_2 x}\int e^{(m_1-m_2)x}.\int e^{-m_1 x} f(x)(dx)^2$$

$$y = e^{3x}\int e^{-4x}.\int e^{x}(5e^{-2x})dx \cdot dx$$

$$y = e^{3x}\int e^{-4x}.\int 5e^{-x}dx \cdot dx$$

$$y = e^{3x}\int e^{-4x}.\ (-5e^{-x}+c_1)dx$$

$$y = e^{3x}\int(-5e^{-5x}+c_1e^{-4x})\ dx$$

$$y = e^{3x}(e^{-5x}+\frac{c_1}{-4}e^{-4x}+c_2)$$

$y = e^{-2x}+\frac{c_1}{-4}e^{-x}+c_2e^{3x}$ the general sol.

Example5: Solve $(D-2)^3 y = 2x$ (the same Ex.3)

Sol.: $m_1 = m_2 = m_3 = 2$

$$y = e^{m_3 x}\int e^{(m_2-m_3)x}.\int e^{(m_1-m_2)x}\cdot\int e^{-m_1 x} f(x)(dx)^3$$

$$y = e^{2x}\int\int\int e^{-2x}(2x)\,dx\cdot\ dx\cdot\ dx$$

$$y = e^{2x}\int\int(-xe^{-2x}-\frac{1}{2}e^{-2x}+c_1)dx\cdot\ dx$$

$$y = e^{2x}\int(\frac{1}{2}xe^{-2x}+\frac{1}{4}e^{-2x}+\frac{1}{4}e^{-2x}+c_1x+c_2)dx$$

$$y = e^{2x}\int(\frac{1}{2}xe^{-2x}+\frac{1}{2}e^{-2x}+c_1x+c_2)dx$$

$$y = e^{2x}(\frac{-1}{4}xe^{-2x}-\frac{1}{8}e^{-2x}-\frac{1}{4}e^{-2x}+\frac{c_1}{2}x^2+c_2x+c_3)$$

$y = \frac{-1}{4}x-\frac{3}{8}+(\frac{c_1}{2}x^2+c_2x+c_3)e^{2x}$ the general solution

Exercises : Solve the following equations

1-$(D^2-1)y = x$

2- $y''-6y'+8y = e^x$

3-$(D^2-4D+3)y = 2x+1$

$4-(D^3-8)y = x$

Section 5.5: Solution of homogenous linear diff. eq. with higher order and constant coefficient (the characteristic equation)

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \qquad \ldots\ldots\ldots\ldots\ldots(19)$$

Where $a_1, a_2, \ldots, a_n$ are constants

Or $(D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n)y = 0$

We can write this eq. by diff. operator as a form

معادلة بدلالة المؤثر

$$F(D)y = (D - m_1)(D - m_2) \ldots (D - m_2)y - 0 \qquad \ldots\ldots\ldots\ldots\ldots(20)$$

And the following equation is called the (Auxiliary equation)

$$F(m) = (m - m_1)(m - m_2) \ldots (m - m_n) = 0$$

As for the equation

$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_{n-1} m + a_n = 0 \qquad \ldots\ldots\ldots(21)$$

Is called (characteristic equation) المعادلة المميزة

لحل هذا النوع من المعادلات نبدا بدراسة حل المعادلة المميزة اولا

There are three cases of the roots of characteristic eq.

<u>The first case 1: Different Real Roots</u>

If the roots of the characteristic eq.(21) are different and real , let $m_1 and\ m_2$ are two roots of an equation of order 2 and $m_1 \neq m_2$;

$m_1, m_2 \in R$

يكون جذرا المعادلة المميزة مختلفين وحقيقيين

we note that the solution which called(the complementary function y_c) is:

$$\boldsymbol{y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

In general, for n roots $(m_1, m_2, \ldots, m_n)$ the complementary function y_c is:

$$\boxed{\boldsymbol{y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}} \qquad \ldots\ldots\ldots\ldots(22)}$$

Where$c_1, c_2, \ldots, c_n$ are an arbitrary constants.

<u>**Example**1</u>: Find the general sol(**the complementary solution)**of the eq.

$$\boldsymbol{y'' + 3y' - 4y = 0}$$

Sol.:

$(D^2+3D-4)y=0$

the Ch. Eq. is نكتب بدلالة المعادلة المساعدة

$\Rightarrow m^2+3m-4=0$

$(m+4)(m-1)=0 \Rightarrow \quad m=-4 \quad , \quad m=1$ different roots

$\therefore$ the general sol. is

$y=y_c=c_1e^{-4x}+c_2e^x$ الحل العام(الحل المتمم)

Where c_1, c_2 are two arbitrary constants

Example 2 : Solve $\boldsymbol{y'''+2y''+5y'-6y=0}$

Sol.:

$(D^3+2D^2-5D-6)y=0$

The Ch. Eq. is

$m^3+2m^2-5m-6=0$

$(m+1)(m+3)(m-2)=0$

$\downarrow \quad \downarrow \quad \downarrow$

$m_1=-1, m_2=-3, m_3=2$ (three different real roots)

The general sol. is :

$y=c_1e^{-x}+c_2e^{-3x}+c_3e^{2x}$ where c_1, c_2 $and c_3$ are an arbitrary constants.

<u>The second case 2: Equal Real Roots</u>

If the roots of the characteristic eq.(21) are equal and real suppose that , if m_1, m_2 are the roots of eq. of order 2 and $m_1=m_2$; $m_1, m_2 \in R$ then

the complementary function y_c is:

$y=y_c=c_1e^{m_1x}+c_2xe^{m_1x}$

Or $y=y_c=(c_1+c_2x)\,e^{m_1x}$

Where c_1, c_2 are an arbitrary constant

In general, for n roots $(m_1, m_2, \dots, m_n)$ the complementary function y_c is:

$$\boldsymbol{y=y_c=(c_1+c_2x+\cdots+c_nx^{n-1})\,e^{m_1x}} \qquad \dots\dots(23)$$

Where $c_1, c_2, \dots, c_n$ are an arbitrary constants.

Example3: Find the general solution of the eq.

$$\boldsymbol{y'' - 4y' + 4y = 0}$$

Sol.: $(D^2 - 4D + 4)y = 0$

The Ch. Eq. is:

$m^2 - 4m + 4 = 0 \qquad \rightarrow \qquad (m-2)^2 = 0$

$\Longrightarrow m = 2,2$ جذران حقيقيان ومكرران

The general sol. is:

$y = (c_1 + c_2 x)e^{2x}$

Where c_1, c_2 are an arbitrary constant

Example4: Find the general solution of the eq

$$\boldsymbol{(D^3 - 5D^2 + 7D - 3)y = 0}$$

Sol.: The Ch. Eq. is:

$$m^3 - 5m^2 + 7m - 3 = 0$$

$(m-1)(m-3)(m-3) = 0$

$(m-1)^2(m-3) = 0$

$\downarrow \qquad\qquad \downarrow$

$m_1 = m_2 = 1, m_3 = 3$

جذران حقيقيان مكرران وجذر مختلف

The general solution is:

$y = (c_1 + c_2 x)e^{x} + c_3 e^{3x}$

Where c_1, c_2, c_3 are an arbitrary constants.

<u>The third case : Complex roots</u>

If the roots of the characteristic eq. is complex numbers, suppose m_1 and m_2 are complex numbers then

If was $m_1 = a + ib \quad , \quad m_2 = a - ib$

The solution of the diff. eq. of the first case is

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$$\Rightarrow y = c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x} = c_1 e^{ax}.e^{ibx} + c_2 e^{ax}.e^{-ibx}$$
$$= e^{ax}(c_1 e^{ibx} + c_2 e^{-ibx})$$

$e^{ibx} = cos\, bx + i \sin bx$ (Euler's low)

$e^{-ibx} = cos\, bx + i \sin bx$

$$\Rightarrow y = e^{ax}[c_1(\cos bx + i \sin bx) + c_2(\cos bx + i \sin bx)]$$
$$= e^{ax}[c_1 \cos bx + c_1 i \sin bx + c_2 \cos bx - c_2 i \sin bx]$$
$$= e^{ax}[(c_1 + c_2)\cos bx + i(c_1 - c_2)\sin bx]$$
$$= e^{ax}[A \cos bx + B \sin bx]$$

Where $A = c_1 + c_2$, $B = i(c_1 - c_2)$, then

$$\boldsymbol{y = y_c = e^{ax}[A\cos bx + B \sin bx]} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(24)}$$

Example5 : Find the general sol of the eq.

$$\boldsymbol{y'' + 2y' + 5y = 0}$$

Sol.:

$(D^2 + 2D + 5)y = 0$ نكتب بدلالة المعادلة المساعدة

$m^2 + 2m + 5 = 0$

$\therefore m = \frac{-2\pm\sqrt{4-20}}{2} = -1 + 2i$

$\therefore$ the general sol. $y = e^{-x}[c_1 \cos 2x + c_2 \sin 2x]$

Example 6: Find the general sol of the eq.

$$\boldsymbol{y'' + 9y = 0}$$

Sol.: $(D^2 + 9)y = 0$

$m^2 + 9 = 0 \quad \Longrightarrow \quad m^2 = -9 \quad \Longrightarrow \quad m = \pm 3i$

The general sol.

$y = c_1 \cos 3x + c_2 \sin 3x$

➡ If the roots of the char. Eq. are complex numbers and equal, then the sol. is:

$$\boldsymbol{y = e^{ax}\left[\left(c_1 + c_2x + c_3x^2 + \cdots + c_kx^{k-1}\right)\cos bx + \left(c_{k+1} + c_{k+2}x + c_{k+3}x^2 + \cdots + c_nx^{k-1}\right)\sin bx\right.}$$ **; n=2k**

…………………..(25)

For example:

If n=6→$m_1 = m_2 = m_3 = a + ib$ *and* $m_4 = m_5 = m_6 = a - ib$

Then the complementary solution is:

$$y = e^{ax}[(c_1 + c_2x + c_3x^2)\cos bx + (c_4 + c_5x + c_6x^2)\sin bx$$

Example7: **Find the general sol of the eq.** $\boldsymbol{(D-2)^3(D+3)^2(D-4)y = 0}$ **.**

Sol.: $(m-2)^3(m+3)^2(m-4) = 0$

$$\Longrightarrow \quad m = 2,2,2,-3,-3,4$$

The general sol. is

$$y = (c_1 + c_2x + c_3x^2)e^{2x} + (c_4 + c_5x)e^{-3x} + c_6e^{4x}$$

Example 8 : Solve the equation $\boldsymbol{(D^4 + 8D^2 + 16)y = 0}$

Sol.: $m^4 + 8m^2 + 16 = 0$ نكتب بدلالة المعادلة المميزة

$$(m^2+4)^2 = 0 \qquad \Longrightarrow \qquad m = \pm 2i \;,\; \pm\; 2i$$

$(i.e.)$ $m_1 = m_2 = 2i \,, m_3 = m_4 = -2i$ (complex roots)

$$y = (c_1 + c_2x)cos2x + (c_3 + c_4x)sin2x$$

Where $c_1, c_2\,, c_3 \& c_4$ are arbitrary constant.

Example 9: Solve the equation $\boldsymbol{(D^3 - 3D^2 + 9D + 13)y = 0}$

Sol.: $m^3 - 3m^2 + 9m + 13 = 0$ بالتخمين

$m = -1$,

$$(-1)^3 - 3(-1)^2 + 9(-1) + 13 = 0$$

$$-1 - 3 - 9 + 13 = 0$$

تحل بالدستور

$$(m+1)(m^2 - 4m + 13) = 0$$

$$\begin{array}{r|l} & m^2 - 4m + 13 \\ \hline m+1 & m^3 - 3m^2 + 9m + 13 \\ & \pm m^3 \pm m^2 \\ \hline & -4m^2 + 9m \\ & \pm 4m^2 \pm 4m \\ \hline & 13m + 13 \\ & \pm 13m \pm 13 \\ \hline & 0 + 0 \end{array}$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{+4 \pm \sqrt{16 - 4.1.13}}{2.1}$$

$$= \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i$$

$m_1 = -1 \quad , \quad m_2 = 2 + 3i \quad , \quad m_3 = 2 - 3i$

Then the general sol. is

$$y = c_1 e^{-x} + c_2 e^{2x} cos3x + c_3 e^{2x} sin3x$$

Where c_1, c_2 &c_3 are arbitrary constant.

Example 10: **Solve** $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0$

Sol.: The char. Eq. is

$$m^3 - 4m^2 + m + 6 = 0$$

$$m = -1 \quad \Longrightarrow \quad (-1)^3 - 4(-1)^2 + (-1) + 6 = 0$$

$$\Longrightarrow (m+1)(m^2 - 5m + 6) = 0$$

$$(m+1)(m-3)(m-2) = 0$$

$\Longrightarrow m_1 = -1, m_2 = 3, m_3 = 2$ جذور حقيقية مختلفة

$$\begin{array}{r|l} & m^2 - 5m + 6 \\ \hline m+1 & m^3 - 4m^2 + m + 6 \\ & \pm m^3 \pm m^2 \\ \hline & -5m^2 + m \\ & \pm 5m^2 \pm 5m \\ \hline & 6m + 6 \\ & \pm 6m \pm 6 \\ \hline & 0 + 0 \end{array}$$

The general sol. is:

$$y = c_1 e^{-x} + c_2 e^{3x} + c_3 e^{2x}$$

Where c_1, c_2 &c_3 are arbitrary constant.