

جامعة بغداد

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قسم الرياضيات

المرحلة الثانية

المعادلات التفاضلية الاعتيادية

Ordinary Differential Equations

CHAPTER THREE

Applications of first order and first degree differential equations

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CHAPTER THREE

Applications of first order and first degree differential equations

The emergence and development of differential equations came from physical application and most of the laws of nature were derived and proven using differential equations, so in this chapter we will review some applications of the equation of first order and first degree in different sciences.

ان ظهور المعادلات التفاضلية وتطويرها جاء من التطبيقات الفيزيائية كما ان معظم قوانين الطبيعة اشتقت وبرهنت باستخدام المعادلات التفاضلية, لذا في هذا الفصل سوف نستعرض بعض التطبيقات للمعادلات التفاضلية من الرتبة الاولى والدرجة الاولى في علوم مختلفة

3.1: Geometrical applications تطبيقات هندسية

Some relationships related to curve such as slope, tangent parts and perpendicular are expressed using derivative. The relationship based on one of these quantities forms a differential equation and by solving it we get the equation of integration curves for that differential equation. In this section we will deal with such relationships.

We know that the tangent slope (m_t) to a given curve in any point (x, y) on a function, let it be $f(x, y)$ then

$$\frac{dy}{dx} = f(x, y)$$

By solving this differential equation, we get the required equation.

يعبر عن بعض العلاقات التي تتعلق بمنحنى ما, مثل الميل واجزاء المماس والعمود باستخدام المشتقات فالعلاقة التي تقوم على احدى هذه الكميات تؤلف معادلة تفاضلية وبحلها نحصل على معادلة منحنيات التكامل لتلك المعادلة التفاضلية, في هذه الفقرة سوف نتعامل مع مثل هذه العلاقات.

من المعلوم لدينا ان ميل المماس m_t لمنحنى ما لأي نقطة (x, y) واقعة عليه هو دالة ولتكن $f(x, y)$ وعليه :

$$\frac{dy}{dx} = f(x, y)$$

وبحل هذه المعادلة نحصل على المعادلة المطلوبة

Example 1:

Find the equation of the curve that passes through the point (3,4) and its slope at any point (x, y) on it is equal to $(-\frac{y-1}{x-1})$

Solution:

$$\frac{dy}{dx} = -\frac{y-1}{x-1} \dots\dots\dots (1) \quad \text{Where } m_t = \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{y-1} + \frac{dx}{x-1} = 0 \dots\dots\dots (2) \quad \text{Integrating both sides}$$

$$\Rightarrow \ln|y-1| + \ln|x-1| = \ln c \dots\dots\dots (3)$$

$$\Rightarrow (y-1)(x-1) = c \dots\dots\dots (4)$$

Sub. (3,4) in eq. (4), we get

$$(4-1)(3-1) = c \quad \rightarrow \quad c = 6 \quad \text{Sub. In (4)}$$

$$(y-1)(x-1) = 6 \dots\dots\dots (5)$$

Eq. (5) is the required equation.

Example 2:

Find the function that equal to its derivative and satisfied with the point (1,3)

Solution:

Let the function equal y , then

$$y = \frac{dy}{dx} \dots\dots\dots (1)$$

$$\frac{dy}{y} = dx \quad \text{Integrating both sides, we get}$$

$$\ln y = x + c \dots\dots\dots (2)$$

$$y = e^{x+c}$$

$$y = ke^x \quad \text{Where } k = e^c \dots\dots\dots (3)$$

Sub. The point (1,3) in eq. (3), we get

$$3 = ke^1 \quad \rightarrow \quad k = \frac{3}{e} \quad \text{Sub. In eq. (3)}$$

$$y = \frac{3}{e} e^x \quad \rightarrow \quad y = 3e^{x-1} \quad \dots\dots\dots (4)$$

Eq. (4) is the required eq.

3.1.1: Orthogonal Trajectories **المسارات المتعامدة**

When all the curves of one family of curves $F(x, y, c_1) = 0$ intersect orthogonally all the curves another family $(x, y, c_2) = 0$, then the families are said to be orthogonal trajectories i.e. $m_t = \frac{-1}{m_\perp}$

EX.: (a) - The family $y = -\frac{1}{2}x + c_1$ and $y = 2x + c_2$ are orthogonal trajectories.

(b) - The families $y = c_1x^3$ and $x^2 + 3y^2 = c_2$ are orthogonal trajectories.

(c) - The families $y = c_1x$ through the origin of the family $x^2 + y^2 = c_2$ of concentric circles are orthogonal trajectories.

(Prove the above cases)

Remark: • There is a solution depended on the slopes on two orthogonal trajectories at the point of intersection.

• If $m_1 = f(x, y)$ is the slope of $F(x, y, c) = 0$ at (x, y) , then the slope of orthogonal trajectories is $m_2 = -\frac{1}{f(x, y)}$ at the same point.

Example 3:

Find the orthogonal trajectories of concentric circles (دوائر متحدة المراكز) at the origin $x^2 + y^2 = c^2$

Solution:

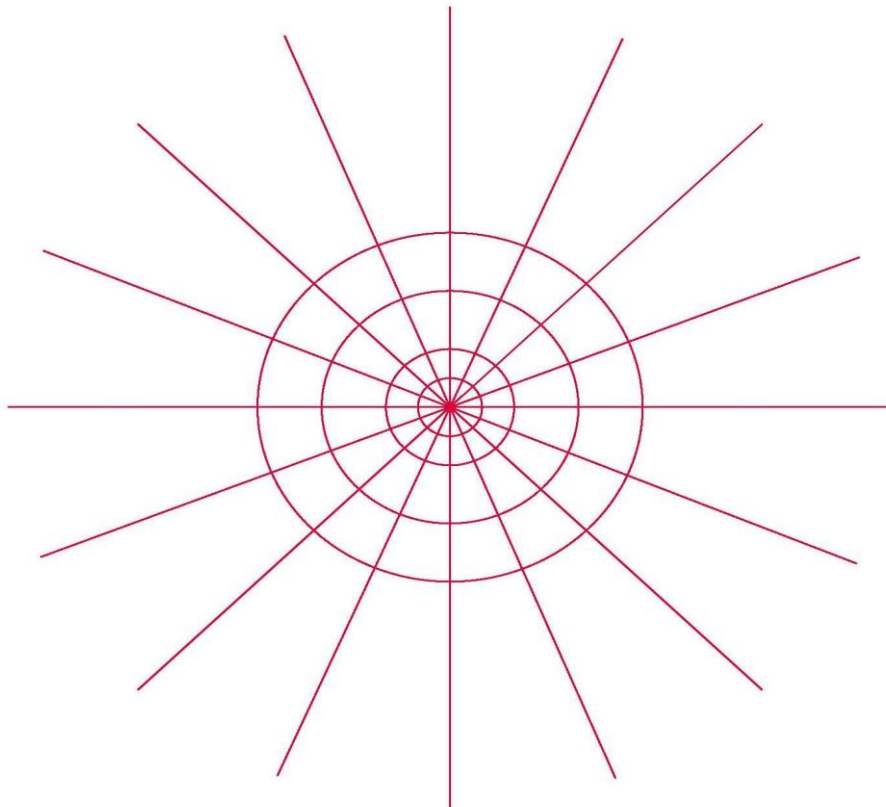
$$2x + 2y \frac{dy}{dx} = 0 \Rightarrow x + y \frac{dy}{dx} = 0$$

$$y \frac{dy}{dx} = -x \Rightarrow m_1 = \frac{dy}{dx} = -\frac{x}{y}$$

$$(m_1 = -\frac{1}{m_2}) \text{ لايجاد المسارات المتعامدة}$$

$$\Rightarrow m_2 = \frac{y}{x} \Rightarrow \frac{dy}{dx} = \frac{y}{x} \Rightarrow \ln y = \ln x + \ln c$$

$$\Rightarrow y = cx \text{ تمثل المسارات المتعامدة}$$



Example 4:

Find the orthogonal trajectories of $y^2 = 4ax$.

Solution:

$$\frac{y^2}{x} = 4a \quad \rightarrow \quad \frac{2yy'x - y^2}{x^2} = 0$$

$$2yy'x - y^2 = 0 \quad \Rightarrow \quad y' = \frac{y}{2x}$$

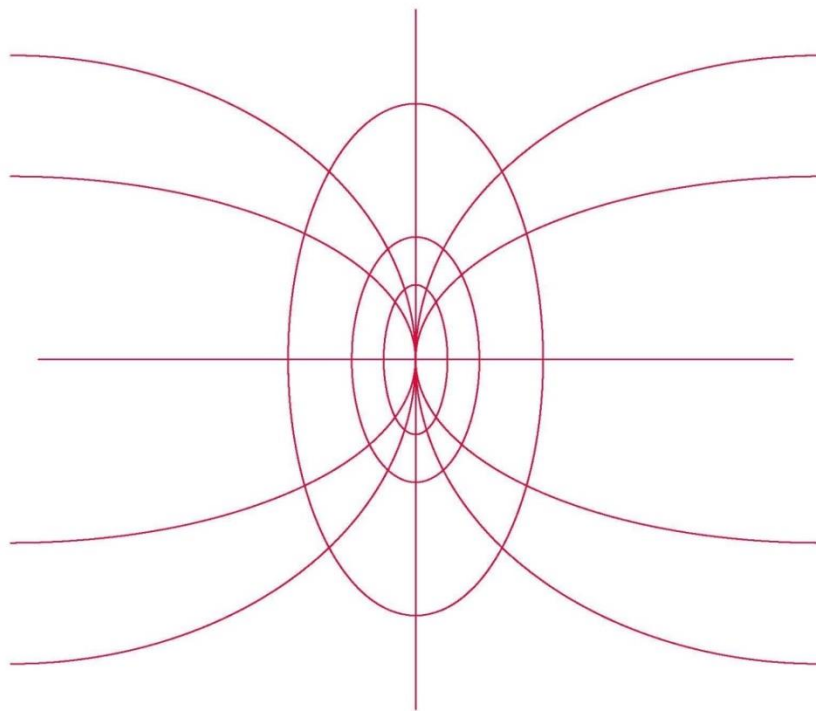
$$\frac{dy}{dx} = \frac{y}{2x} = m_1$$

$$\Rightarrow m_2 = -\frac{1}{y/2x} \quad \Rightarrow \quad m_2 = \frac{-2x}{y} \quad \text{المتعامدة}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x}{y} \quad \Rightarrow \quad ydy = -2xdx$$

$$-\frac{1}{2}y^2 = -x^2 + c_1 \quad \Rightarrow \quad y^2 + 2x^2 = c_2 \quad \text{قطوع ناقص}$$

Where $c_2 = -2c_1$



3.1.2: The polar coordinate الاحداثيات القطبية

To find the orthogonal trajectories to the curves $F(r, \theta, c) = 0$

- I) Form its diff. eq. in the form $f\left(r, \theta, \frac{dr}{d\theta}\right) = 0$ by eliminating c
- II) Replacing in this diff. eq. $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$

Remark: $\tan \varphi = \frac{r}{r'}$ & $r' = \frac{dr}{d\theta} \Rightarrow \tan \varphi = r \frac{d\theta}{dr}$

Example 5:

Find the orthogonal trajectory of $r = a \cos \theta$, where a is a constant

Solution:

$$r = a \cos \theta \quad \text{----- (1)}$$

Derive both sides of eq. (1) w.r.t. θ

$$\frac{dr}{d\theta} = -a \sin \theta \quad \text{----- (2)}$$

Put $(-r^2 \frac{d\theta}{dr})$ instead of $(\frac{dr}{d\theta})$

$$\begin{aligned} \Rightarrow -r^2 \frac{d\theta}{dr} &= -a \sin \theta && \div r \\ r \frac{d\theta}{dr} &= \frac{a \sin \theta}{r}, && \because r = a \cos \theta \end{aligned}$$

$$\Rightarrow r \frac{d\theta}{dr} = \tan \theta$$

$$\Rightarrow \frac{dr}{r} = \frac{d\theta}{\tan \theta} \quad \text{----- (3)}$$

Integrating both sides, we get:

$$\Rightarrow \ln|r| = \int \frac{\cos \theta}{\sin \theta} d\theta + \ln|c|$$

$$\ln|r| = \ln|\sin \theta| + \ln|c|$$

$$\Rightarrow r = c \sin \theta \quad \text{----- (4)}$$

Eq.(4) is the orthogonal trajectory of $r = a \cos \theta$

Example 6:

Find the orthogonal trajectory of $r^2 = a^2 \cos \theta$.

Solution: $r^2 = a^2 \cos \theta$ ----- (1)

Derive eq. (1) w.r.t. θ

$$2r \frac{dr}{d\theta} = -a^2 \sin \theta \quad \text{----- (2)}$$

$$(\div 2r)$$

$$\frac{dr}{d\theta} = -\frac{1}{2} \cdot \frac{a^2 \sin \theta}{r} \quad \rightarrow \quad \frac{dr}{d\theta} = -\frac{1}{2} r \cdot \frac{a^2 \sin \theta}{r^2}$$

$$\frac{dr}{d\theta} = -\frac{1}{2} r \cdot \tan \theta \quad , \quad \text{since} \quad r^2 = a^2 \cos \theta$$

$$-2 \frac{dr}{d\theta} = r \tan \theta$$

بوضع $-r^2 \frac{dr}{d\theta}$ بدل من $\frac{dr}{d\theta}$ فان المعادلة التفاضلية للمسارات المتعامدة تكون

$$-2r^2 \frac{d\theta}{dr} = r \tan \theta$$

$$\Rightarrow \quad r = c \sin^2(\theta)$$

وهذه تمثل مجموعة المنحنيات التي تتقاطع على التعامد مع مجموعة المنحنيات المعطاة

Exercises:

Find the orthogonal trajectory of

1- $y = x + e^x + c$

2- $x^2 - y^2 = c$

3- $xy = a$

4- $r = a(1 - \cos \theta)$

5- $r = a(2 + \sin \theta)$

3.2: Growth and Decay of population**نمو واضمحلال السكان**

The initial-value problem: $\frac{dx}{dt} = kx$, $x(t_0) = x_0$

Where k is constant of proportionality occurs in many physical theories involving either growth or decay:

For example: in biology it is often observed that the rate at which certain bacteria grow is proportional to the number of bacteria present at any time t :

Then if $p(t)$ is the size of bacteria at t , then we can write:

$$\frac{dp(t)}{dt} = kp(t) \text{ and } p(0) = p_0 \quad \dots\dots\dots (1)$$

Where k a positive constant and the sol. , then

$$\frac{dp(t)}{p(t)} = k dt \rightarrow \ln p(t) = kt + c \rightarrow p(t) = e^{kt+c}$$

$$p(t) = p_0 e^{kt} \quad \dots\dots\dots (2)$$

Where $p_0 = e^c$

Example (1):

If the rate of growth is proportional to the number of bacteria present if this number at some times is $(10)^6$ bacteria and after one hour be $\frac{3}{2}(10)^6$ bacteria. Find the number of bacteria after four hours.

Solution: Let $p(t)$ is the number of bacteria, then :

$$\frac{dp(t)}{dt} = kp(t) \rightarrow \frac{dp(t)}{p(t)} = k dt \rightarrow \ln p(t) = kt + c$$

$$\rightarrow p(t) = p_0 e^{kt} \quad , \quad p_0 = (10)^6 \quad , \quad p(1) = \frac{3}{2}(10)^6$$

$$\frac{3}{2}(10)^6 = (10)^6 e^{1.k} \quad \Rightarrow \quad e^k = \frac{3}{2} \quad \Rightarrow \quad k = \ln\left(\frac{3}{2}\right)$$

$$p(t) = (10)^6 e^{t \ln 3/2} \quad \Rightarrow \quad p(t) = (10)^6 (3/2)^t$$

$$p(4) = (10)^6 \cdot \left(\frac{3}{2}\right)^4 = 5.0625 \times 10^6 \quad \text{Bacteria}$$

Example (2): Cane sugar dissolved in water, under certain conditions, turns into dextrose at a rate proportional to the unconverted quantity at each moment. If the amount of cane sugar dissolved at the beginning ($t=0$) was 75 grams, and 8 grams were transformed after 30 minutes. How much cane sugar turn into after 90 minutes?

يتحول سكر القصب المذاب في الماء تحت شروط معينة الى سكر عنب بمعدل يتناسب مع الكمية غير المحولة في كل لحظة. فاذا كانت كمية قصب السكر المذابة في البداية عندما ($t = 0$) هي 75 غراما وتحول منها 8 غرامات بعد 30 دقيقة. كم يتحول من سكر القصب بعد مرور 90 دقيقة؟

Solution:

Let the amount of cane sugar equal p , and the dextrose equal D ,

When $p_0 = 75$, $p(30) = 75 - 8 = 67$, we get:

$$\frac{dp(t)}{dt} = -kp(t) \rightarrow \frac{dp(t)}{p(t)} = -kdt$$

$$\rightarrow \ln p(t) = -kt + c$$

$$\rightarrow p(t) = p_0 e^{-kt}$$

$$\rightarrow p(30) = p_0 e^{-30k}$$

$$\rightarrow 67 = 75e^{-30k}$$

$$\rightarrow k = \frac{\ln(\frac{67}{75})}{-30}$$

$$p(t) = 75e^{t \ln \frac{67}{75} / 30}$$

$$\text{When } t=90 \rightarrow p(90) = 75e^{90 \ln \frac{67}{75} / 30}$$

$$p(90) = 75\left(\frac{67}{75}\right)^3 = 53.468 \approx 53.5 \text{ gr}$$

$$D = 75 - p(90) = 75 - 53.5 = 21.5 \text{ gr} \quad \text{السكر المتحول}$$

Exercises:

- 1- If the population of a city doubles in 50 years, in how many years will the population become three times the original number, if you know that the rate of population increase is proportional to the number of population.
- 2- A herd of buffaloes has 1000 buffaloes in it, and the population is growing exponentially. At time $t=4$ it has 2000 buffaloes. Write a formula for the number of buffaloes at arbitrary time t .
- 3- A herd of elephants is growing exponentially. At time $t=2$ it has 1000 elephants in it, and at time $t=4$ it has 2000 elephants. Write a formula for the number of elephants at arbitrary time t .
- 4- A colony of bacteria is growing exponentially. At time $t=0$ it has 10 bacteria in it, and at time $t=4$ it has 2000. At what time will it have 100,000 bacteria?
- 5- A colony of ants is growing exponentially. At time $t=0$ it has 1000 ants in it, and at time $t=3$ it has 3000. At what time will it have 10,000 ants?

3.2: Cooling problem مسائل التبريد

Physical information: Experiment show that the time rate of change dT/dt of the temperature T of a body is proportional to the difference between T and T_m (the temperature of the surrounding medium). This is called Newton's law of cooling.

وفقا لقانون نيوتن للحرارة (تبريد/تسخين), درجة حرارة الجسم تتغير كنسبة بين التغير في درجة حرارة الجسم نفسه ودرجة حرارة الوسط المحيط بالجسم المسماة (درجة الحرارة المحيطة).

لتكن $T(t)$ تمثل درجة حرارة الجسم عند زمن t , T_m درجة الحرارة المحيطة, $\frac{dT}{dt}$ التغير في درجة حرارة الجسم بالنسبة لزمن t , وعليه فالترجمة الرياضية لقانون نيوتن للتبريد والتسخين سيكون:

$$\frac{dT}{dt} = k(T - T_m) \quad \text{حيث } k \text{ يمثل ثابت التناسب}$$

$$\text{or } T' = k(T - T_m)$$

لحل هذه المعادلة التفاضلية من الرتبة الاولى نحصل على

$$\frac{dT}{T - T_m} = k dt \quad \Rightarrow \quad \ln|T - T_m| = kt + c_1$$

$$\therefore T - T_m = ce^{kt} \quad \Rightarrow \quad T = T_m + ce^{kt}$$

Example (1):

Suppose that you turn off the heat in your home at night 2 hours before you go to bed, call this time $t = 0$, if the temp. T at $t = 0$ is 66°F and at time you go to bed ($t = 2$) has dropped to 63°F , what temp. can you expect in the morning, say 8 hours later ($t = 10$)? (of course, this process of cooling off will depend on the outside temp. T_m which we assume to be constant at 32°F).

نفرض انك اطفأت التدفئة في غرفتك قبل الخلود للنوم بساعتين اي عند زمن $t = 0$ وكانت درجة الحرارة 66°F عند $t = 0$ هي (اي قبل النوم بساعتين) ثم بعد ساعتين ($t = 2$) اصبحت درجة الحرارة للغرفة 63°F ما هي درجة الحرارة المتوقعة عند الصباح اي بعد 8 ساعات من النوم ($t = 10$) علما ان درجة الحرارة في الخارج (خارج الغرفة) 32°F ($T_m = 32$)

Solution:

$$T(0) = 66, \quad T(2) = 63, \quad T_m = 32$$

$$\therefore T = T_m + ce^{kt} \Rightarrow T = 32 + ce^{kt}$$

$$66 = 32 + ce^{kt}, \quad \because t = 0 \Rightarrow 66 = 32 + c$$

$$\Rightarrow c = 34$$

$$63 = 32 + 34e^{2k} \quad (\text{when } t = 2)$$

$$e^{2k} = \frac{63 - 32}{34} \Rightarrow e^{2k} = 0.9118$$

$$2k = \ln(0.9118) \Rightarrow 2k = -0.0923 \Rightarrow k = -0.04615$$

$$T(10) = 32 + 34e^{-0.04615 \times 10} = 32 + 34e^{-0.4615}$$

$$= 32 + 34(0.630) = 32 + 21.42 \approx 53^\circ\text{F}$$

Example (2):

Suppose we know that the temperature of an object is 95°C , that the ambient temperature is 20°C , and that exactly 20 minutes after the object began to cool its temperature was 70°C . Write a formula for the temperature at arbitrary time t .

على فرض اننا نعلم درجة حرارة جسم ما وهي 95°C , وان درجة حرارة المحيط هي 20°C . بعد 20 دقيقة بالضبط من ابتداء الجسم بالبرودة اصبحت درجة حرارته 70°C . اكتب صيغة تمثل درجة حرارة الجسم لأي وقت t .

Solution:

$$T(0) = 95^\circ\text{C}, T(20) = 70^\circ\text{C}, T_m = 20$$

$$\therefore T(t) = T_m + ce^{kt} \text{ -----(1)}$$

$$\rightarrow T(t) = 20 + ce^{kt}$$

When $t=0$, we get:

$$T(0) = 20 + c \rightarrow 95 - 20 = c \rightarrow c = 75$$

Sub. in eq. (1)

$$T(t) = 20 + 75e^{kt} \text{ -----(2)}$$

When $t=20$, we get:

$$T(20) = 20 + 75e^{20k}$$

$$70 = 20 + 75e^{20k} \rightarrow 50 = 75e^{20k} \rightarrow \frac{2}{3} = e^{20k}$$

$$\rightarrow k = \frac{1}{20} \ln\left(\frac{2}{3}\right) \approx -0.02027$$

$$T(t) = 20 + 75e^{\frac{t}{20} \ln\left(\frac{2}{3}\right)} \text{ -----(3)}$$

Eq.(3) is the required equation

Exercises:

1. Suppose we know that the temperature of an object is 95°C , that the ambient temperature is 20°C , and that exactly 20 minutes after the object began to cool its temperature was 70°C .
 - a) Determine the object's temperature 45 minutes after cooling began.
 - b) Determine the number of minutes, after cooling began, until the object reaches a temperature of 35°C .
2. If the rate of decrease in the temperature of a hot object is proportional to the increase in the temperature of the object over the ambient temperature. (i.e.)

$$\frac{dT}{dt} = -\lambda(T - T_m)$$

Where T is the object temperature , T_m is the ambient temperature and λ is the constant of proportionality. Prove that the temperature of the object after t time is :

$$T - T_m = (T_1 - T_m)e^{-\lambda t}$$

Where T_1 is the initial temperature of the object.