جامعة بغداد
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قسم الرياضيات

المرحلة الثانية

المعادلات التفاضلية الاعتيادية

Ordinary Differential Equations

Chapter 2

أساتذة المادة

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CHAPTER TWO

The Ordinary Differential Equation of the first order and first degree

An ordinary differential equation of first order and first degree is written in one of the following forms:

المعادلة التفاضلية الاعتيادية من الرتبة الأولى والدرجة الأولى تكتب بأحدى الاشكال التالية:

$$M(x,y)dx + N(x,y)dy = 0$$

$$\boldsymbol{or} \quad M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

$$\boldsymbol{or} \quad \frac{dy}{dx} = f(x,y)$$

Such that f, M, N does not contain derivative.

Although this kind of differential equations seems simple, there is no general way of solving, but several methods depending on the type of the equation. Therefore, the equations that can be solved directly divided into several types, the most important ones are:

(رغم ان هذا النوع من المعادلات التفاضلية تبدو بسيطة الا انه لا توجد طريقة عامة للحل وانما عدة طرق حسب نوع المعادلة، وعلى ذلك تنقسم المعادلات التي يمكن إيجاد حلها بطريقة مباشرة الى عدة أنواع أهمها:)

2.1) Separable equation.

2.2) Homogenous equation.

2.3) Differential equation with linear coefficients.

2.4) Exact differential equation

2.5) integral factors.

2.6) Bernoulli's equation

2.7) Ricatt's Equation

2.8) The diff. eq. of the form $\boldsymbol{f'(y)\frac{dy}{dx}+P(x)f(y)=Q(x)}$

2.9) Equation that is solved using a suitable substitution.

Now let's start:

2.1 Separable of Variables:

Definition1: An equation in the form

$$\frac{dy}{dx}=\frac{g(x)}{h(y)} \quad \text{or} \quad h(y)d\,y=g(x)\,dx$$

is said to be separable or to have separable variable. And if both g and h are differentiable, then:

$$\int h(y)\,dy = \int g(x)\,dx + c$$

<u>Ex 1 :</u> Solve : a) $\boldsymbol{\frac{dy}{dx} = 1 + e^{5x}}$ **b)** $\boldsymbol{\frac{dy}{dx} = sin(x)}$

<u>Solution:</u>

a) $dy = (1 + e^{5x})dx \Rightarrow y = x + \frac{1}{5}e^{5x} + c$

b) $dy = \sin x\, dx \Rightarrow y = -\cos x + c$

<u>Ex 2 :</u> Solve : $\boldsymbol{(1 + x^2)\, y\, dy + (y + 3)\, dx = 0}$

<u>Solution:</u> $[(1 + x^2)\, y\, dy + (y + 3)\, dx = 0] * \frac{1}{(1+x^2)(y+3)}$

$$\Rightarrow \left(\frac{y}{y+3}\right) dy + \frac{dx}{1+x^2} = 0$$

$$\Rightarrow \int \left(1 - \frac{3}{y+3}\right) dy + \tan^{-1} x = c$$

$$\Rightarrow \int dy - \int \frac{3}{y+3} dy + \tan^{-1} x = c$$

$$\Rightarrow \mathrm{y} - 3 \ln\ |\mathrm{y} + 3|\ + \tan^{-1} x = c,\ c \in \mathbb{R} \qquad (c\ is\ an\ arb.cons.)$$

Remark: $\frac{y}{y+3}$ can be solved by two methods

يمكن تبسيط المقدار $\frac{y}{y+3}$ بطريقتين: اما *<u>القسمة الطويلة</u>* او *<u>بالتجزئة</u>*.

1) القسمة الطويلة:

$$\begin{array}{r|l} & \;1 \\ \hline y+3 & \;y \end{array}$$

2) التجزئة:

$$\frac{y}{y+3} = \frac{y+3-3}{y+3} = \frac{y+3}{y+3} - \frac{3}{y+3} = 1 - \frac{3}{y+3}$$

<u>Ex 3 :</u> Solve $\boldsymbol{y' = e^{x-y}}$

<u>Solution:</u> $\frac{dy}{dx} = e^x . e^{-y} \Rightarrow dy = e^x . e^{-y}\, dx$

$$\Rightarrow e^y\, dy = e^x\, dx \Rightarrow \int e^y dy = \int e^x\, dx$$

$\Rightarrow\ e^y = e^x + c$, (The general solution),(c is an arb. cons.)

<u>Ex 4 :</u> Solve $\boldsymbol{x\,y\,dy + (2yx^2 + 4x^2 - y - 2)dx = 0}$

<u>Solution:</u> بواسطة التحليل نحصل على:

$$[x\,y\,dy + (y+2)(2x^2-1)dx = 0] * \frac{1}{x(y+2)}$$

$$\Rightarrow \frac{y}{y+2}\,dy + \frac{2x^2-1}{x}\,dx = 0$$

$$\Rightarrow \int (1 - \frac{2}{y+2})\,dy + \int (2x - \frac{1}{x})\,dx = \int 0$$

$\Rightarrow y - 2\ln|y+2| + x^2 - \ln|x| = c$, (The general solution of O.D.E.)

<u>Ex 5 :</u> Solve $\boldsymbol{sin^2x\;cos\,y\,dx + sin\,y\;sec\,x\,dy = 0}$

<u>Solution:</u> $\frac{\sin^2 x}{\sec x}\,dx + \frac{\sin y}{\cos y}\,dy = 0$

$$\Rightarrow \frac{\sin^2 x}{\frac{1}{\cos x}}\,dx + \frac{-\sin y}{-\cos y}\,dy = 0$$

$$\Rightarrow \int \sin^2 x \ \cos x \ dx + \int \frac{-\sin y}{-\cos y} \, dy = \int 0$$

Then the general solution is:

$-\ \frac{\sin^3 x}{3} - \ln\ |\cos y| = c$, (Where c is an arb. cons.)

<u>Ex 6 :</u> Solve $\frac{dy}{dx} + e^x y = e^x y^2$

<u>Solution:</u> $\frac{dy}{dx} = e^x\, y^2 - e^x\, y \Rightarrow \frac{dy}{dx} = e^x\,(y^2 - y\,)$

$$\Rightarrow \frac{dy}{y(y-1\,)} = e^x\, dx$$

$\int \frac{1}{y(y-1\,)}\, dy$ (We use the fraction law) (قانون التجزئة)

$$\frac{A}{y} + \frac{B}{y-1} = \frac{A\,(y-1) + B\,y}{y(y-1)}$$

$$\Rightarrow Ay - A + B\,y\ = 1$$

$$\Rightarrow (A+B)y\ = 0 \qquad \ldots(1)$$

$$\Rightarrow -A\ = 1 \qquad \ldots(2)$$

$$\Rightarrow A + B\ = 0$$

$A = -B$

$From$ (2): $A = -1$

$From$ (1): $A + B = 0 \longrightarrow -1 + B = 0 \rightarrow -1 = -B \longrightarrow B = 1$

Now, $\int \frac{1}{(y^2 - y)} dy = \int (\frac{-1}{y} + \frac{1}{y-1}) dy$

$= \int \frac{-1}{y} dy + \int \frac{1}{y-1} dy = -ln|y| + ln|y - 1|$

Then the general solution is:

$-\ln|y| + \ln|y - 1| = e^x + c,$ (Where c is an arb. con.)

2.2 Homogenous differential equation

Definition2: The function $f(x, y)$ is homogeneous function of degree n if

$$f(t\,x, t\,y) = t^n f(x, y) \qquad \dots\dots \qquad \text{(I)}$$

where t is a constant.

Definition3: The differential equation

$$M(x,y)dx + N(x,y)dy = 0 \qquad \ldots. \qquad \text{(II)}$$

is called homogenous if $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree.

To solve the homogeneous differential equation:

$$M(x,y)dx + N(x,y)dy = 0$$

Can be written in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \qquad \ldots. \qquad \text{(III)}$$

Let $v = \frac{y}{x}$, then equation (III) becomes

$$\frac{dy}{dx} = f(v) \qquad \ldots. \qquad \text{(IV)}$$

Since $v = \frac{y}{x}$, then $y = vx$, and so

$$dy = vdx + xdv$$

We substitute the above expression.

The new equation is separable, we solve it and in the last we put $v = \frac{y}{x}$ to get the general solution.

<u>Ex 1 :</u> Solve $\boldsymbol{x\, y^2\, dy - (x^3 + y^3)dx = 0}$

<u>Solution:</u> $M(x,y) = (x^3 + y^3)$

$$N(x,y) = xy^2$$

$$M(t\,x, t\,y) = (tx)^3 + (ty)^3 = t^3x^3 + t^3y^3$$

$= t^3(x^3 + y^3) = t^3M(x,y),$ الدالة متجانسة من الدرجة الثالثة

$$N(t\,x, t\,y) = t\,x\,(t\,y)^2 = tx\,(t^2\,y^2)$$

$= t^3(x\,y^2) = t^3\,N(x,y),$ الدالة متجانسة من الدرجة الثالثة

So, M and N are both homogeneous, and have the same degree, so the diff. eq. is homogeneous.

Let $y = vx \longrightarrow dy = v\,dx + x\,dv$

بالتعويض في المعادلة التفاضلية نحصل على:

$$xv^2x^2(vdx + xdv) - (x^3 + v^3x^3)dx = 0$$

$$\Longrightarrow x^3v^3dx + x^4v^2dv - x^3dx - v^3x^3dx = 0$$

$$\Longrightarrow [x^4v^2dv - x^3dx = 0] * \frac{1}{x^4}$$

$$\Longrightarrow \int v^2dv - \int \frac{1}{x}dx = \int 0$$

$$\Longrightarrow \frac{v^3}{3} - \ln|x| = c$$

When $v = \frac{y}{x} \Longrightarrow \frac{y^3}{3x^3} - \ln|x| = c$, (The general solution)

<u>Ex 2 :</u> Solve $\boldsymbol{xy\,dx + (x^2 - 2y^2)dy = 0}$

<u>Solution:</u> The equation is homogeneous (prove it), so

Let $v = \frac{y}{x} \longrightarrow y = v\,x \longrightarrow dy = v\,dx + x\,dv$

$$x(vx)\,dx + (x^2 - 2v^2x^2)(v\,dx + x\,dv) = 0$$

$$\Longrightarrow vx^2\,dx + x^2v\,dx + x^3dv - 2v^3x^2dx - 2v^2x^3dv = 0$$

$$\Longrightarrow 2vx^2\,dx - 2v^3x^2dx + x^3(1 - 2v^2)dv = 0$$

$$\Longrightarrow 2x^2(v - v^3)dx + x^3(1 - 2v^2)dv = 0$$

$$\Longrightarrow [2x^2(v-v^3)dx + x^3(1-2v^2)dv = 0\,] * \frac{1}{x^3(v-v^3)}$$

$$\Longrightarrow \frac{2x^2}{x^3}dx + \frac{1-2v^2}{v-v^3}\,dv = 0$$

$$\Longrightarrow \int \frac{2}{x}dx + \int \frac{1-2v^2}{v-v^3}\,dv = 0$$

$$\int \frac{1-2v^2}{v-v^3}\,dv\ = ?$$

$$\frac{1-2v^2}{v-v^3} = \frac{1-2v^2}{v(1-v^2)}$$

$$= \frac{1-2v^2}{v(1-v)(1+v)}$$

$$= \frac{A}{v} + \frac{B}{1-v} + \frac{C}{1+v} \qquad \dots (*)$$

$$= \frac{A(1-v^2) + Bv(1+v) + cv(1-v)}{v(1-v)(1+v)}$$

$$= \frac{A - Av^2 + Bv + Bv^2 + Cv - Cv^2}{v(1-v^2)}$$

$$-A + B - C = -2 \qquad \dots\ (1)$$

$$B + C\ =\ 0 \qquad \dots\ (2)$$

$A = 1$ ... (3)

By substituting (3) in (1), we get:

$-1 + B - C = -2 \longrightarrow B - C = -1$ .... (4)

Eq. (2) + Eq. (4):

$$B - C = -1$$

$$B + C = 0$$

$$2B = -1 \longrightarrow B = \frac{-1}{2}$$

From Eq. (2) $\longrightarrow C = \frac{1}{2}$

So, $A = 1,\ B = \frac{-1}{2}, and\ C = \frac{1}{2}$

Substituting A,B,C in eq.(*), we get:

بتعويض قيم A, B, C في المعادلة (*)، نحصل على:

$$\int \frac{1 - 2v^2}{v - v^3}\, dv = \left(\frac{1}{v} + \frac{\frac{-1}{2}}{1 - v} + \frac{\frac{1}{2}}{1 + v}\right) dv$$

$$= \ln v - \frac{1}{2} \ln|1 - v| + \frac{1}{2} \ln|1 + v| = c$$

$$\therefore 2\ln|x| + \ln|v| - \frac{1}{2}\ln|1-v| + \frac{1}{2}\ln|1+v| = c$$

$$\Rightarrow 2\ln|x| + \ln\left|\frac{y}{x}\right| - \frac{1}{2}\ln\left|1-\frac{y}{x}\right| + \frac{1}{2}\ln\left|1+\frac{y}{x}\right| = c,$$ (The general sol.)

Ex 3 : Prove the following diff. eq. is homogeneous, then find the general solution:

$$\boldsymbol{xdy - ydx = \sqrt{x^2 + y^2}\,dx}$$

Solution: We must prove the degree of

$$M(x,y)dx = N(x,y)dy$$

$$x\,dy = \left(y + \sqrt{x^2+y^2}\right)dx \qquad (1)$$

$$\because M(tx,ty) = (ty + \sqrt{(tx)^2 + (ty)^2}\,)$$

$$= ty + \sqrt{t^2(x^2+y^2)}$$

$$= ty + t\sqrt{x^2+y^2}$$

$$= t(y + \sqrt{x^2+y^2}\,)$$

$$= tM(x,y),$$

$\Longrightarrow \; M(x,y)$, متجانسة من الدرجة الاولى

$\because N(tx,ty) = tx$

$\Longrightarrow \; N(x,y)$, متجانسة من الدرجة الاولى

Then the diff. eq. is homogeneous. Now, we find the general solution:

Let $y = vx \longrightarrow dy = vdx + x\,dv$, by substituting in the diff. eq.(1), we get:

$$x\,(vdx + x\,dv) - vx\,dx = \sqrt{x^2 + v^2x^2}dx$$

$$xvdx + x^2dv - vxdx = x\,\sqrt{1+v^2}dx$$

$$[x^2dv = x\,\sqrt{1+v^2}dx] * \frac{1}{x^2\sqrt{1+v^2}}$$

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{1}{x}\,dx$$

$$\int \frac{dv}{\sqrt{1+v^2}} = ?$$

Let $v = \tan\theta \;\longrightarrow dv = \sec^2\theta\,d\theta$

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{\sec^2\theta\,d\theta}{\sqrt{1+\tan^2\theta}} = \int \frac{\sec^2\theta}{sec\,\theta}\,d\theta = \int \sec\,\theta\,d\theta$$

$$= \ln|sec\theta + \tan\theta| = \ln\left|v + \sqrt{1+v^2}\right| = l\,n\left|\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right|$$

Then the general solution is:

$$\therefore \quad l\,n\left|\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right| = \ln x + c \quad \text{(c is an arb. cons.)}$$

2.3 Differential Equation with Linear Coefficients (Equation that reduce to homogeneous equation)

These equations can be expressed as:

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \qquad \dots \quad (V)$$

Two lines:

$a_1x + b_1y + c_1 = 0$, and $a_2x + b_2y + c_2 = 0 \qquad \dots \quad (VI)$

a) Intersect if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, (المستقيمان متقاطعان)

or: Intersect if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$

b) Parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, (المستقيمان متوازيان)

or: Parallel if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$

Case (1): If the two lines that in (VI) intersect , i.e. $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, we seek a translation of axes using the form:

$x = x_1 + h \longrightarrow dx = dx_1$

$y = y_1 + k \longrightarrow dy = dy_1$

where (h, k) is the point of intersection

Then the substitution $x = x_1 + h, y = y_1 + k \;\; (dx = dx_1, dy = dy_1)$

transform equation (V) into the homogeneous equation.

Remark: First we find the intersection point.

Ex 1 : Solve the following O.D.E. :

$$\boldsymbol{(2x - 3y + 4)dx + (3x - 2y + 1)dy = 0}$$

Solution: $\left.\begin{matrix} 2x - 3y + 4 = 0 \\ 3x - 2y + 1 = 0 \end{matrix}\right]$ $\quad \frac{a_1}{a_2} = \frac{2}{3}, \frac{b_1}{b_2} = \frac{-3}{-2}$

$\frac{2}{3} \neq \frac{-3}{-2} \longrightarrow$ The two lines are intersection.

<u>or</u>: $\begin{vmatrix} 2 & -3 \\ 3 & -2 \end{vmatrix} = 5 \neq 0 \longrightarrow$ The two lines intersect.

Now, we must find the intersection point:

$$2x - 3y + 4 = 0 * (-3) \qquad (1)$$
$$3x - 2y + 1 = 0 * (2) \qquad (2)$$

$\longrightarrow 5y - 10 = 0 \longrightarrow y = 2$, [by substituting in Eq.(1)]

$\longrightarrow 2x - 3(2) + 4 = 0 \longrightarrow x = 1$

$\therefore (h, k) = (1,2)$ [The intersection point]

Let $y = y_1 + 2 \longrightarrow dy = dy_1$

$x = x_1 + 1 \longrightarrow dx = dx_1$

By substituting in the D.E., we get:

$$(2(x_1 + 1) - 3(y_1 + 2) + 4)dx_1 + (3(x_1 + 1) - 2(y_1 + 2) + 1)dy_1 = 0$$

$$(2x_1 + 2 - 3y_1 - 6 + 4)dx_1 + (3x_1 + 3 - 2y_1 - 4 + 1)dy_1 = 0$$

$$(2x_1 - 3y_1)dx_1 + (3x_1 - 2y_1)dy_1 = 0$$

The above equation is homogeneous

Let $y_1 = vx_1 \longrightarrow dy_1 = v\,dx_1 + x_1\,dv$

$$(2x_1 - 3vx_1)dx_1 + (3x_1 - 2vx_1)(vdx_1 + x_1dv) = 0$$

$$(2x_1 - 3vx_1)dx_1 + 3x_1vdx_1 + 3x_1^2dv - 2v^2x_1dx_1 - 2vx_1^2dv = 0$$

$$2x_1dx_1 - 3vx_1dx_1 + 3x_1vdx_1 + 3x_1^2dv - 2v^2x_1dx_1 - 2vx_1^2dv = 0$$

$$[2x_1(1 - v^2)dx_1 + x_1^2(3 - 2v)dv = 0] * \frac{1}{x_1^2(1 - v^2)}$$

$$\frac{2}{x_1}dx_1 + \frac{3 - 2v}{1 - v^2}dv = 0$$

$$2\int \frac{dx_1}{x_1} + \int \frac{3 - 2v}{(1 + v)(1 - v)}dv = \int 0$$

$$\because \frac{3 - 2v}{(1 + v)(1 - v)} = \frac{A}{(1 + v)} + \frac{B}{(1 - v)} = \frac{A - Av + B + Bv}{(1 + v)(1 - v)}$$

$$= \frac{(A + B) + (B - A)v}{(1 + v)(1 - v)}$$

$\therefore \quad A + B = 3$

$\quad B - A = -2$

$\longrightarrow B = \frac{1}{2}$, and $A = \frac{5}{2}$

$$2\int \frac{dx_1}{x_1} + \int \left(\frac{\frac{5}{2}}{(1+v)} + \frac{\frac{1}{2}}{(1-v)} \right) dv = \int 0$$

$$4\int \frac{dx_1}{x_1} + 5\int \frac{dv}{(1+v)} + \int \frac{dv}{(1-v)} = \int 0$$

$$4\ln|x_1| + 5\ln|1+v| - \ln|1-v| = c_1$$

$$\ln x_1^4 + \ln(1+v)^5 - \ln(1-v) = c_1$$

$$\ln x_1^4 + \ln \frac{(1+v)^5}{(1-v)} = c_1$$

$\ln(x_1^4 . \frac{(1+v)^5}{(1-v)}) = c_1$ (c_1 is an arb. cons.)

$x_1^4 . \frac{(1+v)^5}{1-v} = e^{c_1},$ let $e^{c_1} = c$

When $v = \frac{y_1}{x_1} \longrightarrow x_1^4 . \frac{\left(1+\frac{y_1}{x_1}\right)^5}{1-\frac{y_1}{x_1}} = c$

$$\longrightarrow x_1^4 . \frac{\frac{(x_1+y_1)^5}{x_1^5}}{\frac{x_1-y_1}{x_1}} = c$$

$$\longrightarrow (x_1+y_1)^5 = C(x_1-y_1)$$

$\longrightarrow (x-1+y-2)^5 = C(x-1-(y-2))$

$\longrightarrow (x+y-3)^5 = C(x-y+1),$ (The general solution)

<u>Case (2):</u> If the two lines that in (VI) are parallel (i.e. $\frac{a_1}{a_2} = \frac{b_1}{b_2}$)then the solution will be using the hypothesis $\boldsymbol{z = ax + by}$ as shown in the following example

<u>Ex 2 :</u> Solve the following O.D.E. $\boldsymbol{(x-y+2)dx = (x-y-3)dy}$

نلاحظ انها معادلة تفاضلية ذات معاملات خطية.

<u>Solution:</u>

$\because \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} = -1+1 = 0$

اذن المستقيمان متوازيان

$(x-y+2)dx-(x-y-3)dy = 0$

Let $z = x-y \longrightarrow dz = dx-dy \longrightarrow dy = dx-dz$

$\longrightarrow (z+2)dx-(z-3)dy = 0$

$zdx+2dx-(z-3)(dx-dz) = 0$

$$zdx + 2dx - zdx + zdz + 3dx - 3dz = 0$$

$$\int 5dx + \int (z-3)dz = \int 0$$

$$5x + \frac{z^2}{2} - 3z = c$$

$5x + \frac{(x-y)^2}{2} - 3(x-y) = c,$ (The general solution)

<u>ملاحظة:</u> يمكن حل المثال السابق(Ex 2) بأسلوب اخر.

$$(\boldsymbol{x - y + 2})\boldsymbol{dx} = (\boldsymbol{x - y - 3})\boldsymbol{dy}$$

$\longrightarrow \frac{dy}{dx} = \frac{x-y+2}{(x-y-3)}$ … (1)

Let $z = x - y$. To solve for $\frac{dy}{dx}$, we differentiate $z = x - y$ with respect to (w.r.t.) x to obtain $\frac{dz}{dx} = 1 - \frac{dy}{dx}$, and so $\frac{dy}{dx} = 1 - \frac{dz}{dx}$, substitute into Eq. (1) yields:

$$1 - \frac{dz}{dx} = \frac{z+2}{z-3} \Rightarrow \frac{dz}{dx} = 1 - \frac{z+2}{z-3} = \frac{-5}{z-3}$$

$$\Rightarrow (z-3)dz = -5dx \Rightarrow \frac{z^2}{2} - 3z = -5x + c$$

$\Rightarrow \frac{(x-y)^2}{2} - 3(x-y) = -5x + c,$ (The general solution)