# "Limits of Functions"

## غايات الدوال

**Definition:** Let $f: X \to Y$ be a function, then

$$\lim_{x \to a} f(x) = L$$

is read ( the limit of $f(x)$ as $x$ approaches to $a$).
It means $f(x)$ gets closer and closer to $L$ as $x$ gets closer and closer to $a$.

**Properties of Limits:** خصائص الغايات

1. $\lim_{x \to a} c = c$

2. $\lim_{x \to a} x = a$

3. $\lim_{x \to a} (c f(x)) = c \lim_{x \to a} f(x) \quad , \; c \in \mathbb{R}$

4. $\lim_{x \to a} ( f(x) \mp g(x)) = \lim_{x \to a} f(x) \mp \lim_{x \to a} g(x)$

5. $\lim_{x \to a} \left( \frac{f(x)}{g(x)} \right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \; , \; \lim_{x \to a} g(x) \neq 0$

6. $\lim_{x \to a} ( f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

7. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ , $n$ is any positive integer number and $\lim_{x \to a} f(x) \geqslant 0$.

**Remarks:**

1. $\lim_{x \to a} f(x) = \frac{0}{0}$ imposible (undefined)

2. $\lim_{x \to a} f(x) = \frac{k}{0}$ imposible , $k \in \mathbb{R}$. (undefined)



**Examples:** Compute the limits for the following functions.

1. $\lim_{x \to 3} \sqrt[3]{2(x+1)} = \sqrt[3]{2(3+1)} = \sqrt[3]{8} = 2$

2. $\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = \frac{-8+8-1}{5+6} = \frac{-1}{11}$

3. $\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} = \frac{\sqrt{9} - 3}{9-9} = \frac{0}{0}$ كمية غير معرفة

$$\Rightarrow \lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} = \lim_{x \to 9} \frac{\cancel{\sqrt{x} - 3}}{\cancel{(\sqrt{x} - 3)}(\sqrt{x} + 1)}$$

$$= \lim_{x \to 9} \frac{1}{\sqrt{x} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6}.$$

4. $\lim_{x \to 0} \frac{(4+x)^2 - 16}{x} = \frac{0}{0}$ كمية غير معرفة

$$\Rightarrow \lim_{x \to 0} \frac{(4+x)^2 - 16}{x} = \lim_{x \to 0} \frac{\cancel{16} + 8x + x^2 - \cancel{16}}{x}$$

$$= \lim_{x \to 0} \frac{\cancel{x}(8+x)}{\cancel{x}}$$

$$= \lim_{x \to 0} (8+x) = 8+0 = 8.$$

**Hw:**

Find the limits for the following functions.

1. $\lim_{x \to 0} \frac{\sqrt{5x+4}}{x^2 - 2}$

2. $\lim_{x \to -3} \frac{x^2 + 4x + 3}{x + 3}$

## "Limits at Infinity"

## الغايات اللانهائية

**Remarks:**

1. $\lim\limits_{x \to \infty} f(x) = \frac{0}{1} = 0$

2. $\lim\limits_{x \to \infty} f(x) = \frac{0}{k} = 0 \ , k \in \mathbb{R}$

3. $\lim\limits_{x \to \infty} f(x) = \frac{1}{\infty} = 0$

4. $\lim\limits_{x \to \infty} f(x) = \frac{k}{\infty} = 0 \ , k \in \mathbb{R}$

5. $\lim\limits_{x \to \infty} f(x) = \frac{0}{\infty}$ impossible (undefined)

6. $\lim\limits_{x \to \infty} f(x) = \frac{\infty}{\infty}$ impossible (undefined)

**Examples:** Find the limits for the following functions.

1. $\lim\limits_{x \to \infty} \left(-4 + \frac{1}{x}\right) = -4 + \frac{1}{\infty} = -4$

2. $\lim\limits_{x \to \infty} \frac{3}{x^2} = \frac{3}{\infty} = 0$

3. $\lim\limits_{x \to \infty} \frac{x}{7x+4} = \frac{\infty}{7(\infty)+4} = \frac{\infty}{\infty}$ (كمية غير معينة)

$\Rightarrow \lim\limits_{x \to \infty} \frac{x}{7x+4} = \lim\limits_{x \to \infty} \frac{\frac{x}{x}}{\frac{7x}{x} + \frac{4}{x}} = \frac{1}{7 + \frac{4}{\infty}} = \frac{1}{7}$.

4. $\lim\limits_{x \to \infty} \frac{5x+2}{2x^2+1} = \frac{\infty}{\infty}$ كذلك

$\rightarrow \lim\limits_{x \to \infty} \frac{5x+2}{2x^2+1} = \lim\limits_{x \to \infty} \frac{\frac{5x}{x^2} + \frac{2}{x^2}}{\frac{2x^2}{x^2} + \frac{1}{x^2}} = \lim\limits_{x \to \infty} \frac{\frac{5}{x} + \frac{2}{x^2}}{2 + \frac{1}{x^2}}$

$= \frac{\frac{5}{\infty} + \frac{2}{(\infty)^2}}{2 + \frac{1}{(\infty)^2}} = \frac{0+0}{2+0} = \frac{0}{2} = 0$

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5. $\lim\limits_{x \to \infty} \frac{\sqrt{x^3 - 3x^2}}{\sqrt[4]{x^6 + 2x}}$

نلاحظ أن أعلى قوة للمتغير x في المقام هي $x^{\frac{6}{4}} = x^{\frac{3}{2}}$
(الجذر الرابع للمقدار $x^6$) بقسمة كل من البسط والمقام
على $x^{\frac{3}{2}} = \sqrt{x^3}$ نحصل على:

$\lim\limits_{x \to \infty} \frac{\sqrt{\frac{x^3}{x^3} - \frac{3x^2}{x^3}}}{\sqrt[4]{\frac{x^6}{x^6} + \frac{2x}{x^6}}} = \lim\limits_{x \to \infty} \frac{\sqrt{1 - \frac{3}{x}}}{\sqrt[4]{1 + \frac{2}{x^5}}}$

$= \frac{\sqrt{1 - \frac{3}{\infty}}}{\sqrt[4]{1 + \frac{2}{\infty}}} = \frac{\sqrt{1-0}}{\sqrt[4]{1+0}} = 1$

**Hw:**

Compute the limits for the following functions.

① $\lim\limits_{x \to \infty} 2 - \frac{1}{\sqrt{x^2+1}}$

② $\lim\limits_{x \to \infty} \frac{3x^2+1}{5x^2+7x-39}$

③ $\lim\limits_{x \to \infty} \frac{3x+7}{\sqrt{4x^2+5}}$.

## "One-side Limits"

الغايات من جهة واحدة

① Right-hand side Limit: الغاية من جهة اليمين

Let $f$ be a function, then $\lim\limits_{\substack{x \to a^+ \\ (x>a)}} f(x) = L_1$

is called right-hand limit.

② Left-hand side Limit: الغاية من جهة اليسار

Let $f$ be a function, then $\lim\limits_{\substack{x \to a^- \\ (x<a)}} f(x) = L_2$

is called left-hand limit.

Remark:

If $\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^-} f(x)$, then we say that

$f(x)$ has a limit as $x \to a$ ($\lim\limits_{x \to a} f(x) = L$ exists).

Example:

Let $f(x) = \begin{cases} 2x+5 & , x > 3 \\ x^2+2 & , x \leq 3 \end{cases}$ find the limit.

Sol:

$\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+} (2x+5) = 2(3)+5 = 11$

$\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-} (x^2+2) = (3)^2+2 = 11$

$\because \lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^-} f(x) \Rightarrow \lim\limits_{x \to 3} f(x) = 11$.





## "Continuous Functions"

الدوال المستمرة

Definition: A function $f$ is said to be continuous at $a$ if the following conditions are satisfied:

1. $f(a)$ defined

2. $\lim\limits_{x \to a} f(x)$ exists

3. $f(a) = \lim\limits_{x \to a} f(x)$.

Examples:

① Is the function $f(x) = \begin{cases} 2x+3 & , x \leq 4 \\ 7 + \frac{16}{x} & , x > 4 \end{cases}$

continuous? Explain.

Sol: at $(a=4)$

1. $f(4) = 2(4)+3 = 8+3 = 11$

2. $\lim\limits_{x \to 4^+} f(x) = \lim\limits_{x \to 4^+} (7 + \frac{16}{x}) = 7 + \frac{16}{4} = 7+4 = 11$

$\lim\limits_{x \to 4^-} f(x) = \lim\limits_{x \to 4^-} (2x+3) = 2(4)+3 = 8+3 = 11$

$\because \lim\limits_{x \to 4^+} f(x) = \lim\limits_{x \to 4^-} f(x) \Rightarrow \lim\limits_{x \to 4} f(x) = 11$ exists

3. $f(4) = \lim\limits_{x \to 4} f(x)$

$\therefore f(x)$ is continuous at $a=4$.