Sequence and Series of Functions

Sequence of Real-Valued Functions

Definition:

Let $F(S)=\{f: S\to\mathbb{R} : S\subseteq\mathbb{R}\}$
define $f_n: S\to\mathbb{R}$, $\forall n\in\mathbb{N}$
$\langle f_n\rangle = f_1, f_2, \dots$ is called sequence of real-valued functions ($f_n\in F(S)$).

Definition: (Point wise convergent of sequence of real-valued Functions)

Let $\langle f_n\rangle$ be a sequence of real-valued functions defined on S. $\langle f_n\rangle$ is pointwise convergent to f if $\forall x\in S, \forall \varepsilon>0, \exists n_0(x,\varepsilon)>0$ s.t. $|f_n(x)-f(x)|<\varepsilon, \forall n> n_0(x,\varepsilon)$.

Definition: (Uniform convergent of sequence of real-valued Functions)

Let $\langle f_n\rangle$ be a sequence of real-valued functions defined on S. $\langle f_n\rangle$ is uniform convergent to f if $\forall \varepsilon>0, \exists n_0(\varepsilon)>0$ s.t. $|f_n(x)-f(x)|<\varepsilon, \forall n>n_0(\varepsilon), \forall x\in S$.

Remarks:

1. Every uniform convergent is point wise convergent.
2. The converse of (1) may not be true (Ex. 1 or 2 or 3).
3. If f_n pointwise convergent and
   (i) f_n continuous on S, $\forall n$
   (ii) either $f_{n+1}(x)\leq f_n(x)$ or $f_{n+1}(x)\geq f_n(x)$
   (iii) S is compact
   then f_n is uniform convergent.

Examples:

(1) Let $f_n:\mathbb{R}\to\mathbb{R}$ defined as $f_n(x)=\frac{x}{n}$, $\forall x\in\mathbb{R}, n\in\mathbb{N}$.
Is f_n pointwise convergent to $f=0$ on $\mathbb{R}$?
Is f_n uniform convergent to $f=0$ on $\mathbb{R}$?
If $f_n:(0,a]\to\mathbb{R}$, $f_n(x)=\frac{x}{n}$, $\forall x\in(0,a], n\in\mathbb{N}$, is f_n uniform convergent on $(0,a]$?

proof: claim $f_n \xrightarrow[\mathbb{R}]{P} f=0$

$\forall x\in\mathbb{R}$, given $\varepsilon>0$, $|f_n(x)-f(x)|=\left|\frac{x}{n}-0\right|=\left|\frac{x}{n}\right|$
By A.P. $\forall\varepsilon>0, \exists n_0(x,\varepsilon)$ s.t. $|x|<n_0\varepsilon \Rightarrow \frac{|x|}{n_0}<\varepsilon$
$\therefore |f_n(x)-f(x)|=\frac{|x|}{n}<\frac{|x|}{n_0}<\varepsilon, \forall n>n_0(x,\varepsilon)$.

$\therefore f_n\xrightarrow[\mathbb{R}]{P} 0$ ($\langle f_n\rangle$ convergent pointwise to zero on $\mathbb{R}$).

But $\langle f_n\rangle$ dose not uniform convergent to $f=0$ on $\mathbb{R}$ since if $\langle f_n\rangle$ convergent uniformly to $f=0$, so $\exists n_0(\varepsilon)>0$ s.t. $|x|<n_0\varepsilon$ i.e. $x\in(-n_0\varepsilon, n_0\varepsilon)$ C!

If f_n defined on $(0,a]$ then $f_n\xrightarrow[(0,a]]{U} f=0$
($\langle f_n\rangle$ uniform convergent on $(0,a]$)
given $\varepsilon>0$, $|f_n(x)-f(x)|=\left|\frac{x}{n}-0\right|=\frac{|x|}{n}<\frac{a}{n}$
by A.P. $\forall\varepsilon>0, \exists n_0(\varepsilon)$ s.t. $\frac{a}{n_0}<\varepsilon$

$\therefore |f_n(x)-f(x)|=\frac{|x|}{n}<\frac{a}{n}<\frac{a}{n_0}<\varepsilon, \forall n, n>n_0(\varepsilon)$, $\forall x\in(0,a]$.

(2) Let $f_n : [0,1] \to \mathbb{R}$ defined as $f_n(x) = x^n, \forall x \in [0,1], \forall n$.

Is $f_n \xrightarrow[{[0,1]}]{P} f$? Is $f_n \xrightarrow[{[0,1]}]{U} f$? Is $f_n \xrightarrow[{[0,a]}]{U} f$, $0<a<1$?

<u>Proof.</u> since $\langle x^n \rangle$ is decreasing and bounded below by zero
so $\langle x^n \rangle$ is convergent

If $x=0 \Rightarrow f_n(x) = x^n = 0 \longrightarrow 0 = f$

If $x=1 \Rightarrow f_n(x) = x^n = 1 \longrightarrow 1 = f$

i.e. $f = \begin{cases} 0 , & 0 \le x < 1 \\ 1 , & x = 1 \end{cases}$

$\therefore \forall x \in [0,1], \forall \varepsilon > 0, \exists n_0(x,\varepsilon) > 0$ s.t.

$|f_n(x) - f(x)| < \varepsilon , \forall n > n_0(x,\varepsilon)$

$\therefore \langle f_n \rangle$ pointwise convergent to f on $[0,1]$ $\left(f_n \xrightarrow[{[0,1]}]{P} f\right)$.

But $\langle f_n \rangle$ dose not uniform convergent to $f=0$ on $[0,1]$
since, given $\varepsilon > 0$, $\varepsilon = \frac{1}{4} > 0$ and $x = (\frac{1}{2})^{1/n}$

$|f_n(x) - f(x)| = |(\frac{1}{2})^{n/n} - 0| = \frac{1}{2} > \frac{1}{4}$.

If $f_n : [0,a] \to \mathbb{R}$, $0<a<1$, $f_n = x^n, \forall x \in [0,a], \forall n$
$\langle f_n \rangle = \langle x^n \rangle$ is uniform convergent to $f=0$ on $[0,a]$, $0<a<1$

since, given $\varepsilon > 0$, $|f_n(x) - f(x)| = |x^n - 0| = |x^n|$
By A.P. $\forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ s.t. $x^n < x^{n_0} < \varepsilon$

$\therefore |f_n(x) - f(x)| = x^n < \varepsilon , \forall n > n_0(\varepsilon)$.

(3) Let $f_n : \mathbb{R} \to \mathbb{R}$ defined as $f_n(x) = \frac{nx}{1+n^2x^2}, \forall x \in \mathbb{R}, \forall n$.

f_n is pointwise convergent to $f=0$, since given $\varepsilon > 0$

$|f_n(x) - f(x)| = \left|\frac{nx}{1+n^2x^2} - 0\right| = \frac{n|x|}{n^2x^2} = \frac{1}{n|x|}$

By A.P. $\forall \varepsilon > 0, \exists n_0(x,\varepsilon)$ s.t. $\frac{1}{n_0|x|} < \varepsilon$

$\therefore |f_n(x) - f(x)| = \frac{1}{n|x|} < \frac{1}{n_0|x|} < \varepsilon , \forall n > n_0(x,\varepsilon)$.

But f_n is not uniform convergent to $f=0$ on $\mathbb{R}$, since
given $\varepsilon = \frac{1}{4}$, $x = \frac{1}{n}$

$|f_n(x) - f(x)| = \left|\frac{n \cdot \frac{1}{n}}{1+n^2 \cdot \frac{1}{n^2}} - 0\right| = \frac{1}{2} > \frac{1}{4}$.

If $f_n : (a,\infty) \to \mathbb{R}$, $f_n(x) = \frac{nx}{1+n^2x^2}, \forall x \in (a,\infty), \forall n$

f_n is uniform convergent to $f=0$ on (a,∞)
since, given $\varepsilon > 0$

$|f_n(x) - f(x)| = \left|\frac{nx}{1+n^2x^2} - 0\right| = \left|\frac{nx}{1+n^2x^2}\right| < \left|\frac{nx}{n^2x^2}\right| = \frac{1}{n|x|} < \frac{1}{na}$

By A.P. $\forall \varepsilon > 0, \exists n_0(\varepsilon)$ s.t. $\frac{1}{n_0 a} < \varepsilon$

$\therefore |f_n(x) - f(x)| < \frac{1}{na} < \frac{1}{n_0 a} < \varepsilon , \forall n > n_0(\varepsilon)$.

Theorem:

Let $\langle f_n \rangle$ be a sequence of r.v.fs. on S which is convergent uniformly to f.

1. If $\langle f_n \rangle$ bounded, $\forall n$ then f is bounded.
2. If $\langle f_n \rangle$ continuous, $\forall n$ then f is continuous.

proof: (For (1)):

$\because f_n \xrightarrow[S]{U} f \Rightarrow \forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ s.t. $|f_n(x) - f(x)| < \varepsilon, \forall n > n_0(\varepsilon), \forall x \in S$

$\because f_n$ bounded, $\forall n \Rightarrow \exists M > 0$ $(M \in \mathbb{R})$ s.t. $|f_n(x)| \leq M, \forall x \in S, \forall n \in \mathbb{N}$

$$|f(x)| = |f(x) - f_n(x) + f_n(x)|$$
$$\leq |f_n(x) - f(x)| + |f_n(x)| < \varepsilon + M = M_1, \forall x \in S$$

$\therefore f$ is bounded.

(For (2)):

$\because f_n \xrightarrow[S]{U} f \Rightarrow \forall \frac{\varepsilon}{3} > 0, \exists n_0(\varepsilon) > 0$ s.t. $|f_n(x) - f(x)| < \frac{\varepsilon}{3}, \forall n > n_0(\varepsilon), \forall x \in S$

$\because f_n$ continuous, $\forall n$ i.e. $x_n \to x_0 \Rightarrow f_n(x_n) \to f_n(x_0)$

$\therefore \forall \frac{\varepsilon}{3} > 0, \exists n_1(\varepsilon) > 0$ s.t. $|f_n(x_n) - f_n(x_0)| < \frac{\varepsilon}{3}, \forall n > n_1(\varepsilon)$

Choose $n_2(\varepsilon) = \max[n_0(\varepsilon), n_1(\varepsilon)]$

$$|f(x_n) - f(x_0)| = |f(x_n) - f_n(x_n) + f_n(x_n) - f_n(x_0) + f_n(x_0) - f(x_0)|$$
$$\leq |f_n(x_n) - f(x_n)| + |f_n(x_n) + f_n(x_0)| + |f_n(x_0) - f(x_0)|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \forall n > n_2(\varepsilon)$$

$\therefore f$ is continuous.

Theorem:

Let $\langle f_n \rangle$ be a sequence of real-valued function on S and f_n continuous on S, $\forall n$ such that f_n is pointwise convergent to f and either $f_{n+1}(x) \leq f_n(x)$ or $f_{n+1}(x) \geq f_n(x)$, $\forall n \in \mathbb{N}, \forall x \in S$ and S is compact then f_n is uniform convergent to f on S.

proof:

$\because f_n \xrightarrow[S]{P} f$, $\forall x \in S, \forall \varepsilon > 0, \exists K_x(x, \varepsilon) > 0$ s.t.

$|f_n(x) - f(x)| < \varepsilon, \forall n > K_x(x, \varepsilon)$

Let $f_{n+1}(x) \leq f_n(x), \forall n, \forall x \in S$

and $g_n(x) = f_n(x) - f(x)$, g_n is continuous on S and $|g_n(x)| < \varepsilon/2$

$\because S$ is compact $\Rightarrow \forall$ open cover for S, $\exists$ finite subcover for S $\left(\forall S \subseteq \bigcup_{x \in S} J_x, \exists S \subseteq \bigcup_{i=1}^{n} J_{x_i}\right)$

$\because g_n$ continuous on $S \Rightarrow \exists J_x$ s.t. $x' \in S \cup J_x$

$\therefore g_{K_x}(x') < g_{K_x}(x) + \frac{\varepsilon}{2} = \varepsilon$

$\because g_{n+1}(x) < g_n(x) \Rightarrow g_n(x) < g_{K_x}(x') < \varepsilon, \forall n > K_x$

i.e. $|g_n(x) - 0| < \varepsilon, \forall n > K_x, \forall x \in S$

$\therefore \langle g_n \rangle$ is uniform convergent to $f = 0$ on S

$\therefore \langle f_n \rangle$ is unifor convergent to f.